Cremona Orbits in $\mathbb{P}^4$ and Applications

Olivia Dumitrescu and Rick Miranda

Abstract This article is motivated by the authors' interest in the geometry of the Mori dream space $\mathbb{P}^4$ blown up in 8 general points. In this article, we develop the necessary technique for determining Weyl orbits of linear cycles for the four-dimensional case, by explicit computations in the Chow ring of the resolution of the standard Cremona transformation. In particular, we close this paper with applications to the question of the dimension of the space of global sections of effective divisors having at most 8 base points.

1 Introduction

Let X^n_s be the projective space $\mathbb{P}^n$ blown up at s general points. Motivated by the study of the dimensionality problem for effective divisors on X^n_s, we analyze the standard Cremona action on X^4_8 and give several applications. We first establish the terminology we use throughout the paper. We call a *Weyl line/Cremona line (Weyl hyperplane, respectively)* to be the orbit under the Weyl group action of a line passing through two of the s points (hyperplane passing through n of the points). In dimension two, the *Weyl lines* are also known in the literature as (-1) curves; via a theorem of Nagata [15, Theorem 2a] they can be described via numerical properties as irreducible classes with self-intersection -1 and anticanonical degree 1. In [10], the authors noticed that Nagata's work can be generalized, and similar numerical properties via the Dolgachev–Mukai bilinear form are equivalent to *Weyl divisors*. In dimension three, the Weyl group action on curves was analyzed by Laface and

The first author is supported by NSF grant DMS1802082.

O. Dumitrescu
University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA

Simion Stoilow Institute of Mathematics, Romanian Academy, 010702 Bucharest, Romania

R. Miranda (✉)
Colorado State University, Fort Collins, CO 80523, USA
e-mail: Rick.Miranda@colostate.edu

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023

T. Dedieu et al. (eds.), *The Art of Doing Algebraic Geometry*, Trends in Mathematics,
https://doi.org/10.1007/978-3-031-11938-5_7

Ugaglia in [13]. Finally, in arbitrary dimension, the Weyl group action on curves in X^n_s and their connection to the (-1)-curves introduced by Kontsevich is analyzed by the two authors in a forthcoming paper [11].

In the planar case, the Gimigliano-Harbourne–Hirschowitz conjecture, still open, predicts that the dimension of the space of global sections of an effective divisor depends on the Euler characteristic and the multiplicity of containment of Weyl lines in the base locus of the divisor. In $\mathbb{P}^3$, the conjecture of Laface–Ugaglia [13] predicts that this dimension depends on the multiplicity of containment of Weyl lines, Weyl hyperplanes, and Weyl orbit of the unique quadric in X^3_s passing through nine general points.

In general, for a small number of points, X^n_{n+2}, it was proved that this dimension depends on the Euler characteristic and the multiplicity of containment of *linear cycles* spanned by the fixed points in the base locus of the divisor D as in [3, Theorem 2.3]. Moreover, the birational geometry of the space X^n_{n+3}, studied in several publications (e.g., [1, 2, 4]), namely the effective and movable cone of divisors, their Mori chamber decompositions together with the dimension of space of global sections is determined by secant varieties to the rational normal curve of degree n passing through $n+3$ general points together with their joins. In general, the case X^n_{n+4} seems to be mysterious.

We dedicate this paper to study X^4_8, which is a Mori Dream Space, whose birational geometry is not totally explained in the literature. In this paper, together with [5] we define and classify the varieties that determine combinatorial data describing the geometry of X^4_8.

The two spaces $X_{2,8}$ and $X_{4,8}$ are related by Gale duality as described in [14]. The precise relation between $X_{2,8}$ and $X_{4,8}$ was established in the following theorem of Mukai (semistability refers to semistability in the sense of Gieseker–Maruyama): $X_{4,8}$ *is isomorphic to the moduli space of rank* 2 *torsion free sheaves* F *on* $X_{2,8}$ *for which* $c_1(F) = -K_S$ *and* $c_2(F) = 2$. Via Mukai's correspondence, Casagrande et al. describe in [7] the five types of surfaces in $X_{4,8}$ playing a special role in the Mori program. In this paper, we rediscover these surfaces as *Weyl planes*, defined below analogously to Weyl lines and hyperplanes.

The Weyl group of X^4_8 is generated by the standard Cremona transformations together with permutations of the base points. In order to define and construct Weyl planes, we introduce Y^4_8 to denote the blowup of X^4_8 along all lines joining any two points and the eight rational normal curves of degree 4 passing through 7 points. (These curves are all disjoint in X^4_8.)

Definition 1.1 A *Weyl plane* is the Weyl orbit of the proper transform of a plane through three fixed points under the blowup of the three lines joining any two points in Y^4_8.

It is important to remark that *Weyl planes* live on the space Y^4_8. We emphasize that this orbit is different (in the Chow ring) than the Weyl orbit of *planes through three points*. Moreover, in [5], the authors introduce and classify the notions of *Weyl curves* and *Weyl surfaces* in X^4_8 as the intersection of two distinct *Weyl divisors* that are orthogonal with respect to the Dolgachev–Mukai bilinear pairing. Since the classification of *Weyl* *surfaces* [5] in X_8^4 is the same with the classification of Proposition 7.3, we can deduce that the two definitions of *Weyl planes* (1.1) and *Weyl surfaces* [5] are equivalent in X_8^4. By definition, *Weyl lines* coincide with *Weyl curves* in the projective plane X_s^2, but the explicit relation between the two definitions, in general, will be studied in a different paper.

In this paper, Corollaries 5.3 and 7.2 enable us to determine the Weyl action on
(a) 1-cycles (i.e., curves) on the Chow ring of blowup of X_s^4;
(b) 2-cycles (i.e., surfaces) on the Chow ring of Y_8^4.

As a consequence, Proposition 7.3 determines the complete list of *Weyl planes* and *Weyl divisors* on X_8^4, and it also gives the formulas for all *Weyl lines* on X_s^4, (for arbitrary number of fixed points s). In particular, for X_8^4, *the only Weyl lines are lines through two fixed points and the rational normal curve of degree* 4 *passing through* 7 *of the* 8 *points*. In fact, in a forthcoming paper [11], we prove that this statement holds for all Mori Dream Spaces. Let Q_i denote *Weyl line* of degree 4 (the rational normal quartic) skipping only the ith point. In particular, we prove that on X_8^4, there are 5 types of Weyl planes (modulo permutation of points), matching computations in [7, Theorem 8.7] and [5]:

- The 56 planes $S_1(ijk)$ through three of the eight points (p_i, p_j, p_k); it has multiplicity one along the three lines L_{ij}, L_{ik}, and L_{jk}.
- The 56 cubic surfaces $S_3(i, j)$ triple at p_i, passing through all other points except p_j; it has multiplicity one along the lines L_{ik} for $k \neq i, j$, and along Q_j
- The 56 sextic surfaces $S_6(ijk)$ passing through p_i, p_j, and p_k and triple at the other five points; it has multiplicity one along all lines joining two of the five points, and along Q_i, Q_j, and Q_k
- The 28 surfaces $S_{10}(ij)$ of degree 10 having two points p_i and p_j of multiplicity 6 and triple at the other six points; it has multiplicity 3 along the line L_{ij}, multiplicity one along all lines L_{ik} and L_{jk} for $k \neq i, j$, and multiplicity one along the curves Q_k for $k \neq i, j$
- The 8 surfaces $S_{15}(i)$ of degree 15 having one point p_i with multiplicity 3 and having multiplicity 6 at the other seven points; it has multiplicity one along all lines L_{jk} for $j, k \neq i$, multiplicity one along each Q_j for $j \neq i$, and multiplicity 3 along Q_i.

In addition to the multiplicities at the points p_i, the reader will note that for all of these surfaces we also compute the multiplicities along the lines L_{ij} and along the rational normal quartics (through 7 of the 8 points). This is important for computations in the Chow ring: unless one takes into account that these surfaces have multiplicity along these curves, one does not fully capture the intersection behavior of these surfaces after one blows up the points (and in general, the curves and surfaces that appear as base loci of linear systems of divisors). It is also critical for computations of the dimensions of the linear systems: it is one of the principles of this article that the multiplicities along these curves must be taken into account in determining the difference between the virtual dimension and the actual dimension of linear systems. Indeed, for certain purposes, it is useful to consider not only the blowup X^4_8 of $\mathbb{P}^4$ at the 8 general points but also then the further blowup Y^4_8 of all of the proper transforms of the lines L_{ij} and the rational normal quartics Q_k; these are easily seen to be disjoint in X^4_8 and therefore Y^4_8 is smooth.

Remark 1.2 In paper [5], the authors use a different notation for the Chow ring basis. For example, $\{h, e_i, e_{ij}\}$ and $\{h^1, e^1_i\}$ of [5] represent here $\{S, S_i, G_{ij}\}$ and $\{l, l_i\}$, respectively. In [5], surfaces denoted above by $S_1(ijk)$, $S_3(i, j)$, $S_6(ijk)$, $S_{10}(ij)$, and $S_{15}(i)$ are denoted by H_{ijk}, $S^3_{i,\hat{j}}$, S^6_{ijk}, S^{10}_{ij}, and S^{15}_i, respectively.

We predict that the birational geometry of X^4_8 is determined not only by Weyl hyperplanes but also Weyl lines and Weyl planes classified in Proposition 7.3. Finally, in Sect. 8, we present applications to the vanishing conjecture and dimensionality problem.

2 The Standard Cremona Transformation and Its Resolution

The standard Cremona transformation of $\mathbb{P}^n$ can be elegantly factored into a series of blowups at the proper transforms of the coordinate linear spaces, followed by a series of symmetric blowdowns.

Fix coordinates $[x_0 : x_1 : \cdots : x_n]$ in $\mathbb{P}^n$, and consider the standard Cremona involution

$$[x_0 : x_1 : \cdots : x_n] \longrightarrow [x_0^{-1} : x_1^{-1} : \cdots : x_n^{-1}]$$

which simply inverts all the coordinates. This is well defined on the torus where all coordinates are non-zero, and has a fundamental locus the union of the coordinate hyperplanes. The transformation is relatively straightforward to resolve in a sequence of blowups and blowdowns, as follows.

Let $p_0, p_1, \ldots, p_n$ be the coordinate points of $\mathbb{P}^n$. For an index set $I \subset \{0, 1, \ldots, n\}$, denoted by L_I, the linear span of the coordinate points indexed by I: $L_I = \text{span}\{p_i \mid i \in I\}$. We have that $\dim L_I = |I| - 1$.

We set $\mathbb{X}^n_0 = \mathbb{P}^n$, and define $\pi_j : \mathbb{X}^n_j \to \mathbb{X}^n_{j-1}$ to be the blowup of the proper transforms of all L_I with $|I| = j$. Hence, π_1 is the blowup of all the coordinate points in $\mathbb{P}^n$; π_2 is the blowup of the (proper transforms of the) coordinate lines L_{ij}, etc. Note that the sequence of blowups stops with π_{n-1}, the blowup of the codimension two coordinate linear spaces, creating the space $\mathbb{X}^n_{n-1}$. We will denote by E_I the exceptional divisor created when L_I is blown up. E_I is created on $\mathbb{X}^n_{|I|}$, and we will use the notation E_I for the proper transform on subsequent blowups too. If $|I| = n$, then L_I is a coordinate hyperplane in $\mathbb{P}^n$; we will denote its proper transform in $\mathbb{X}^n_{n-1}$ by E_I as well.

We note that, at this point , on $\mathbb{X}^n_{n-1}$, the nature and configuration of the divisors E_I are completely symmetric, with respect to taking complements; in other words, we have an isomorphism of $\mathbb{X}^n_{n-1}$ that switches the roles of E_I and E_J when I and J

are complementary in $\{0, 1, \ldots, n\}$. Hence, we can reverse the sequence of blowups with the complementary divisors, and blow down to $\mathbb{P}^n$ "the other way": first blow down the E_I with $|I| = 2$, then the E_I with $|I| = 3$, etc., finishing by blowing down the proper transforms of the coordinate hyperplanes $E_{|I|}$ with $|I| = n$. This is the resolution of the birational involution.

We note that:

- On $\mathbb{X}^n_{|I|-1}$ when the L_I are blown up, they are all disjoint.
- Each linear space L_I experiences a sequence of blowups (by the earlier blowups); on $\mathbb{X}^n_{|I|-1}$, the proper transform of each L_I is isomorphic to $\mathbb{X}^{|I|-1}_{|I|-2}$.
- By induction, this proper transform has both the hyperplane divisor class H (the pullback of the hyperplane divisor class on $\mathbb{X}^{|I|-1}_0 = \mathbb{P}^{|I|-1}$) and its Cremona involution image H'.
- On $\mathbb{X}^{|I|-1}_{|I|-2}$, the normal bundle of the proper transform of L_I is isomorphic to

$$\mathcal{O}(-H')^{\oplus n-|I|+1}.$$

- Since the normal bundle of the proper transform of L_I splits as a direct product of identical line bundles, when E_I is created on $\mathbb{X}^n_{|I|}$, it is isomorphic to a product $\mathbb{X}^{|I|-1}_{|I|-2} \times \mathbb{P}^{n-|I|}$.
- E_I experiences further blowups on its way to $\mathbb{X}^n_{n-1}$, and there it is isomorphic to $\mathbb{X}^{|I|-1}_{|I|-2} \times \mathbb{X}^{n-|I|}_{n-|I|-1}$, where it has a normal bundle isomorphic to the tensor product of the anti-Cremona-hyperplane bundles coming from the two factors.

This construction generalizes the familiar construction of the quadratic Cremona transformation of $\mathbb{P}^2$, which is obtained by blowing up the three coordinate points L_0, L_1, and L_2 (obtaining $\mathbb{X}^2_1$) and then blowing down the three coordinate lines L_{01}, L_{02}, and L_{12}.

3 The Case of Three Space

For three space, the sequence of iterated blowups, in this case, involves two sets of blowups:

$$\mathbb{X}^3_2 \xrightarrow{\pi_2} \mathbb{X}^3_1 \xrightarrow{\pi_1} \mathbb{X}^3_0 = \mathbb{P}^3$$

where π_1 blows up the four coordinate points $p_i = L_i$ and π_2 blows up the six proper transforms of the coordinate lines L_{ij}. The exceptional divisors E_i start out as $\mathbb{P}^2$'s in $\mathbb{X}^3_1$, and then are further blown up to become isomorphic to $\mathbb{X}^2_1$'s in $\mathbb{X}^3_2$. The coordinate lines start in $\mathbb{P}^2$ having normal bundle of bidegree $(1, 1)$; after blowing up the two coordinate points on each, the proper transforms have normal bundles with bidegree $(-1, -1)$ in $\mathbb{X}^3_1$. They are then blown up to $E_{ij} \cong \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{X}^3_2$. Finally the coordinate hyperplanes L_{ijk} are each blown up three times by π_1, and then not blown up further by π_2, and so arrive at $\mathbb{X}^3_2$ as surfaces isomorphic to $\mathbb{X}^2_1$.

The blowing down proceeds by blowing down the E_{ij} via the other ruling, which blows down each L_{ijk} to a $\mathbb{P}^2$; one then blows down each of these to points, finishing the process.

If one is interested in intersection phenomena related to these coordinate subspaces, the Chow ring is the appropriate tool; it is useful primarily for recording two different kinds of phenomena. One is *containment* (with multiplicity) by a given subvariety of one of the blowup centers. In $\mathbb{P}^3$, for divisors, this is the multiplicity of the divisor at one of the coordinate points, and the multiplicity of containment along one of the coordinate lines. For curves, this is the multiplicity of the curve at one of the coordinate points. For a divisor written in the form $D = dH - \sum_i m_i E_i - \sum_{ij} n_{ij} E_{ij}$, the coefficient d is the degree; m_i is the multiplicity at the coordinate point L_i; and m_{ij} is the multiplicity along the line L_{ij}.

The other phenomenon which the Chow ring coefficients can record is the higher-dimensional *contact* that the given subvariety may have with one of the blowup centers. (Higher-dimensional contact in the sense of higher than expected dimension.) In $\mathbb{P}^3$, for surfaces, this is not relevant for the coordinate points and lines; higher-dimensional contact is containment with multiplicity. This is also true for curves with respect to the points: the only phenomenon is that of containment. However, with curves, one can have additional contact with the lines, without containment.

The Chow ring of $\mathbb{X}^3_2$ is not difficult to compute; all the relevant tools are presented in [12], Chaps. 9 and 13. The codimension zero classes are one-dimensional, generated by $[\mathbb{X}^3_2]$ itself; the codimension three classes are also one-dimensional, generated by the class $[p]$ of a point. The codimension one classes are freely generated by the pullback H of the hyperplane class, and the exceptional divisors E_i and E_{ij}.

In codimension two, the group $A^2(\mathbb{X}^3_2)$ contains the following elements. The pullback of the general line class in $\mathbb{P}^3$ will be denoted by ℓ. The general line class inside the exceptional divisor E_i will be denoted by ℓ_i. The exceptional divisor E_{ij} is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and contributes a priori two curve classes: the class f_{ij} of the fiber of the blowup π_2, and the class g_{ij} which is the horizontal ruling of E_{ij}. These are not independent though in $A^2(\mathbb{X}^3_2)$; it is an exercise to check that

$$g_{ij} = f_{ij} + \ell - \ell_i - \ell_j$$

and that this is the only relation in A^2.

For a curve class C written as $C = d\ell - \sum_i m_i \ell_i - \sum_{ij} n_{ij} f_{ij}$, the coefficient d is the degree, m_i is the multiplicity of C at the coordinate point L_i, and n_{ij} is the additional contact of C with the coordinate line L_{ij} (over and above the contact implied by the multiplicities at the two coordinate points on L_{ij}).

We have the following, where we use typical δ-notation: $\delta_{I,J} = 1$ if $I \subseteq J$ and 0 otherwise.

Proposition 3.1 *(a) A basis for the Chow ring of X_2^3 is given by*

$$\begin{array}{ll} A^0: & [X_2^3] \\ A^1: & H, E_0, E_1, E_2, E_3, E_{01}, E_{02}, E_{03}, E_{12}, E_{13}, E_{23} \\ A^2: & \ell, \ell_0, \ell_1, \ell_2, \ell_3, f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23} \\ A^3: & p \end{array}$$

(b) Multiplication of these basis elements is given by

$A^1 \cdot A^1$	H	E_i	E_{ij}
H	ℓ	0	f_{ij}
E_k	0	$-\ell_i \delta_{ik}$	$f_{ij}\delta_{k,ij}$
E_{kl}	f_{kl}	$f_{kl}\delta_{i,kl}$	$(-2f_{ij} - \ell + \ell_i + \ell_j)\delta_{ij,kl}$

$A^1 \cdot A^2$	H	E_i	E_{ij}
ℓ	p	0	0
ℓ_k	0	$-p\delta_{i,k}$	0
f_{kl}	0	0	$-p\delta_{ij,kl}$

The Cremona involution extends to an involution ϕ on the Chow ring; we denote the image of the involution using a superscript prime:

- $[\mathbb{X}_2^3] \leftrightarrow [\mathbb{X}_2^3]$
- $H \leftrightarrow H' = 3H - 2\sum_i E_i - \sum_{ij} E_{ij}$
- $E_l \leftrightarrow E_l' = L_{ijk} = H - E_i - E_j - E_k - E_{ij} - E_{ik} - E_{jk}$ for $i, j, k \neq l$
- $E_{ij} \leftrightarrow E_{ij}' = E_{kl}$ for $k, l \neq i, j$.
- $\ell \leftrightarrow \ell' = 3\ell - \sum_i \ell_i$
- $\ell_i \leftrightarrow \ell_i' = 2\ell - \sum_{j\neq i} \ell_j$
- $f_{ij} \leftrightarrow f_{ij}' = g_{kl} = f_{kl} + \ell - \ell_k - \ell_l$ for $k, l \neq i, j$.
- $p \leftrightarrow p$.

We leave it to the reader to check that this is a ring automorphism, and is an involution.

Proposition 3.2 *(a) Let $D = dH - \sum_i m_i E_i - \sum_{ij} n_{ij} E_{ij}$ be a general class in $A^1(\mathbb{X}_2^3)$. Then the Cremona image D' of D under the involution is $D' = d'H - \sum_i m_i' E_i - \sum_{ij} n_{ij}' E_{ij}$ where*

$$d' = D' \cdot \ell = D \cdot \ell' = D \cdot \left(3\ell - \sum_i \ell_i\right) = 3d - \sum_i m_i;$$

$$m'_i = D' \cdot \ell_i = D \cdot \ell'_i = D \cdot \left(2\ell - \sum_{j \neq i} \ell_j\right) = 2d - \sum_{j \neq i} m_j;$$

$$n'_{ij} = D' \cdot f_{ij} = D \cdot f'_{ij} = D \cdot f_{kl} + \ell - \ell_k - \ell_l = d + n_{kl} - m_k - m_l$$
$$\text{for} \quad k, l \neq i, j$$

(b) Let $C = d\ell - \sum_i m_i \ell_i - \sum_{ij} n_{ij} f_{ij}$ *be a general class in* $A^2(\mathbb{X}^3_2)$. *Then the Cremona image* C' *of* C *under the involution is* $C' = d'\ell - \sum_i m'_i \ell_i - \sum_{ij} n'_{ij} f_{ij}$ *where*

$$d' = C' \cdot H = C \cdot H' = C \cdot (3H - 2\sum_i E_i - \sum_{ij} E_{ij}) = 3d - 2\sum_i m_i - \sum_{ij} n_{ij};$$

$$m'_i = C' \cdot E_i = C \cdot E'_i = C \cdot (H - \sum_{j \neq i} E_j - \sum_{j,k \neq i} E_{jk}) = d - \sum_{j \neq i} m_j - \sum_{j,k \neq i} n_{jk};$$

$$n'_{ij} = C' \cdot E_{ij} = C \cdot E'_{ij} = C \cdot E_{kl} = n_{kl} \quad \text{for} \quad k, l \neq i, j$$

(In the computations above, we abuse notation and give the multiplications as integers instead of integer multiples of the point class p.)

If one is in the position of not needing to consider the contact phenomena for curves, one can simplify the formulas as follows.

Corollary 3.3 *The subspace of* $A^2(\mathbb{X}^3_2)$ *spanned by* ℓ *and the* ℓ_i, *is invariant under the Cremona involution. If* $C = d\ell - \sum_i m_i \ell_i$ *is a general class in* $A^2(\mathbb{X}^3_2)$ *in this subspace, then the Cremona image* C' *of* C *under the involution is* $C' = d'\ell - \sum_i m'_i \ell_i$ *where*

$$d' = C' \cdot H = C \cdot H' = C \cdot (3H - 2\sum_i E_i - \sum_{ij} E_{ij}) = 3d - 2\sum_i m_i;$$

$$m'_i = C' \cdot E_i = C \cdot E'_i = C \cdot (H - \sum_{j \neq i} E_j - \sum_{j,k \neq i} E_{jk}) = d - \sum_{j \neq i} m_j;$$

4 The Chow Ring for the Case of $\mathbb{P}^4$

The sequence of iterated blowups in this case involves three sets of blowups:

$$\mathbb{X}^4_3 \xrightarrow{\pi_3} \mathbb{X}^4_2 \xrightarrow{\pi_2} \mathbb{X}^4_1 \xrightarrow{\pi_1} \mathbb{X}^4_0 = \mathbb{P}^4$$

where π_1 blows up the five-coordinate points $p_i = L_i$ to divisors E_i, π_2 blows up the ten proper transforms of the coordinate lines L_{ij} to E_{ij}, and π_3 blows up the ten proper transforms of the coordinate planes L_{ijk} to E_{ijk}.

We denote by H the general hyperplane class in $\mathbb{P}^4$ (and all its pullbacks); let us denote by $S = H^2$ the class of the general 2-plane, and $\ell = H^3$ the class of the general line; the point class will be p as usual.

In this section, we'll present the Chow ring $A^*(\mathbb{X}_3^4)$, proceeding through the sequence of three blowups. In the starting fourfold $\mathbb{X}_0^4 \cong \mathbb{P}^4$, the relevant subvarieties are simply the linear spaces L_I for $I \subset \{0, 1, 2, 3, 4\}$.

After blowing up the points via π_1, we have

- The divisors $E_i \cong \mathbb{P}^3$.
- The proper transforms of the lines $L_{ij} \cong \mathbb{P}^1$.
- The proper transforms of the 2-planes $L_{ijk} \cong \mathbb{X}_1^2$.
- The proper transforms of the hyperplanes $L_{ijk\ell} \cong \mathbb{X}_1^3$.

We now blow up with π_2 the proper transforms of the ten lines L_{ij}, to the exceptional divisors E_{ij}, to obtain $\mathbb{X}_2^4$; there, we have the following descriptions of the relevant subvarieties:

- The divisors $E_i \cong \mathbb{X}_1^3$.
- The exceptional divisors $E_{ij} \cong \mathbb{P}^1 \times \mathbb{P}^2$.
- The 2-planes $L_{ijk} \cong \mathbb{X}_1^2$.
- The hyperplane threefolds $L_{ijk\ell} \cong \mathbb{X}_2^3$.

Finally, we blow up the proper transforms of the ten surfaces L_{ijk}, to the exceptional divisors E_{ijk}, to obtain $\mathbb{X}_3^4$; there, the relevant subvarieties are:

- The divisors $E_i \cong \mathbb{X}_2^3$.
- The divisors $E_{ij} \cong \mathbb{P}^1 \times \mathbb{X}_1^2$.
- The exceptional divisors $E_{ijk} \cong \mathbb{X}_1^2 \times \mathbb{P}^1$.
- The hyperplane threefolds $L_{ijk\ell} \cong \mathbb{X}_2^3$.

The codimension one classes in $A^1(\mathbb{X}_3^4)$ are freely generated by the pullback H of the hyperplane class in $\mathbb{P}^4$ and the exceptional divisors E_i, E_{ij}, and E_{ijk}; there are no relations among these.

In the group $A^2(\mathbb{X}_3^4)$ of codimension two classes, we have the class $S = H^2$ of the pullback of a general 2-plane in $\mathbb{P}^4$. The other classes that will generate A^2 are supported in the exceptional divisors.

In E_i, which starts in $\mathbb{X}_1^4$ as a $\mathbb{P}^3$, we have the general 2-plane; pulled back to $\mathbb{X}_3^4$ this gives a class S_i for each i.

The divisor E_{ij} starts in $\mathbb{X}_2^4$ as isomorphic to the product $\mathbb{P}^1 \times \mathbb{P}^2$. This contributes to two surface classes: the fiber $\{\text{point}\} \times \mathbb{P}^2$ of the blowup, and the product $\mathbb{P}^1 \times \{\text{general line in } \mathbb{P}^2\}$. Denote by F_{ij} the pullback to $\mathbb{X}_3^4$ of the former, the fiber class; and by G_{ij} the pullback to $\mathbb{X}_3^4$ of the latter.

Finally, the divisor E_{ijk} is isomorphic to $\mathbb{X}_1^2 \times \mathbb{P}^1$, and contributes five surface classes. One is $M_{ijk} = \mathbb{X}_1^2 \times \{\text{point}\}$, a cross section of the blowup map. The others come from products of curve classes in $L_{ijk} \cong \mathbb{X}^2_1$ with the fiber $\mathbb{P}^1$. The curve classes in L_{ijk} are generated by the pullback (from $\mathbb{P}^2$) of the general line class ℓ_{ijk} and the three exceptional curves $e_{ijk,i}$, $e_{ijk,j}$, and $e_{ijk,k}$ which are (in $\mathbb{X}^4_2$) the intersection of L_{ijk} with the three divisors E_i, E_j, and E_k respectively. These four classes give classes $H_{ijk} = \ell_{ijk} \times \mathbb{P}^1$ and $V_{ijk,i}$, $V_{ijk,j}$, and $V_{ijk,k}$ where $V_{ijk,i}$ comes from the product of $e_{ijk,i} \times \mathbb{P}^1$ and the same for the other two.

It is useful to introduce two new classes, for notational convenience. These are:

$$P_{ij} = G_{ij} - F_{ij} \quad \text{and} \quad \Lambda_{ijk} = 2H_{ijk} - V_{ijk,i} - V_{ijk,j} - V_{ijk,k}; \tag{4.1}$$

we note that Λ_{ijk} is the pullback of the Cremona image of the line class on the 2-plane L_{ijk}. This will allow us to replace G_{ij} by P_{ij} among the generators for A^2.

There is a single relation among these codimension two classes beyond the definitional ones of (4.1). It is that

$$M_{ijk} = S - S_i - S_j - S_k - P_{ij} - P_{ik} - P_{jk} + \Lambda_{ijk}. \tag{4.2}$$

Finally, we have the classes of the curves, the codimension three classes in $A^3(\mathbb{X}^4_3)$. We again have the pullback ℓ of the general line class in $\mathbb{P}^4$, and the classes ℓ_i of the general lines in the E_i.

The curve classes supported on E_{ij} (which when it is created on $\mathbb{X}^4_2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$) are generated by the class $\ell_{ij} = \{\text{point}\} \times \{\text{general line in} \mathbb{P}^2\}$ and $h_{ij} = \mathbb{P}^1 \times \{\text{point}\}$.

The curve classes coming from E_{ijk} are the 'horizontal' ones living in L_{ijk}, crossed with a point; these we can denote again by ℓ_{ijk} and $e_{ijk,i}$, $e_{ijk,j}$, and $e_{ijk,k}$ as before. The final one is a general fiber of the blowup f_{ijk}.

There are relations among these curve classes also; these are:

$$h_{ij} = \ell_{ij} + \ell - \ell_i - \ell_j; \quad \ell_{ijk} = 2f_{ijk} + \ell - \ell_{ij} - \ell_{ik} - \ell_{jk}; \tag{4.3}$$

$$e_{ijk,i} = f_{ijk} + \ell_i - \ell_{ij} - \ell_{ik}; \quad e_{ijk,j} = f_{ijk} + \ell_j - \ell_{ij} - \ell_{jk}; \quad e_{ijk,k} = f_{ijk} + \ell_k - \ell_{ik} - \ell_{jk}. \tag{4.4}$$

(Hence, we can dispense with these to generate $A^3(\mathbb{X}^4_3)$.)

It is the case that, for a surface class T, one measures multiplicity along the line L_{ij} by the intersection with F_{ij}, and one measures higher-dimensional contact with L_{ij} by the intersection with G_{ij}. Hence, if the coefficients of T include the terms $-mP_{ij} - nF_{ij}$, then m is the multiplicity of T along the line and n is the additional contact of T with the line, so that one can read off these geometric phenomena from the coefficients directly. (P and F are the dual basis to F and G in A^2.)

We can similarly observe that a general surface class T should meet the 2-plane L_{ijk} in a finite number of points. The coefficients of H_{ijk} and $V_{ijk,i}$, $V_{ijk,j}$, and $V_{ijk,k}$ (which generate the Picard group of the blown-up L_{ijk}) record the higher-dimensional contact of a surface with L_{ijk}, namely, contact in a curve class rather than in a finite number of points. Hence, if the coefficients of T include the terms $-\alpha H_{ijk} + \beta_{ijk,i} V_{ijk,i} + \beta_{ijk,j} V_{ijk,j} + \beta_{ijk,k} V_{ijk,k}$ then the higher-dimensional contact of T with

L_{ijk} (away from the coordinate lines) is a curve in the class $\alpha\ell_{ijk} - \beta_{ijk,i}e_{ijk,i} - \beta_{ijk,j}e_{ijk,j} - \beta_{ijk,k}e_{ijk,k}$.

Having described the generators for the Chow ring $A^*(\mathbb{X}_3^4)$, we can now present the ring structure. The computations are relatively straightforward, using, for example, the formulas for the Chow rings of blowups presented in [12], Chap. 13. (The computation is iterative, first computing $A^*(\mathbb{X}_1^4)$, then using that to compute $A^*(\mathbb{X}_2^4)$, and finally $A^*(\mathbb{X}_3^4)$,)

Proposition 4.5 *The Chow ring of $\mathbb{X}_3^4$ can be described as follows.*

(a) A basis for the Chow ring $A(\mathbb{X}_3^4)$ is given by the classes:

$$\begin{array}{rc} A^0: & [\mathbb{X}_3^4] = 1 \\ A^1: & H, E_i, E_{ij}, E_{ijk} \\ A^2: & S, S_i, P_{ij}, F_{ij}, H_{ijk}, V_{ijk,i} \\ A^3: & \ell, \ell_i, \ell_{ij}, f_{ijk} \\ A^4: & p \end{array}$$

(b) Multiplication of basis elements is given in the following tables.

$A^1 \cdot A^1$	H	E_i	E_{ij}	E_{ijk}
H	S	0	F_{ij}	H_{ijk}
E_m	0	$-S_i\delta_{i,m}$	$F_{ij}\delta_{m,ij}$	$V_{ijk,m}\delta_{m,ijk}$
E_{mn}	F_{mn}	$F_{mn}\delta_{i,mn}$	$-(P_{ij} + 2F_{ij})\delta_{ij,mn}$	$(H_{ijk} - V_{ijk,m} - V_{ijk,n})\delta_{mn,ijk}$
E_{mnr}	H_{mnr}	$V_{mnr,i}\delta_{i,mnr}$	$(H_{mnr} - V_{mnr,i} - V_{mnr,j})\delta_{ij,mnr}$	$-(M_{ijk} + \Lambda_{ijk})\delta_{ijk,mnr}$

$A^1 \cdot A^2$	H	E_i	E_{ij}	E_{ijk}
S	ℓ	0	0	f_{ijk}
S_m	0	$-\ell_i\delta_{i,m}$	0	$f_{ijk}\delta_{m,ijk}$
P_{mn}	ℓ_{mn}	$\ell_{mn}\delta_{i,mn}$	$(-\ell_{ij} - \ell + \ell_i + \ell_j)\delta_{ij,mn}$	$-f_{ijk}\delta_{mn,ijk}$
F_{mn}	0	0	$-\ell_{ij}\delta_{ij,mn}$	$f_{ijk}\delta_{mn,ijk}$
G_{mn}	ℓ_{mn}	$\ell_{mn}\delta_{i,mn}$	$(-2\ell_{ij} - \ell + \ell_i + \ell_j)\delta_{ij,mn}$	0
H_{mnr}	f_{ijk}	0	$f_{mnr}\delta_{ij,mnr}$	$(-4f_{ijk} - \ell + \ell_{ij} + \ell_{ik} + \ell_{jk})\delta_{ijk,mnr}$
$V_{mnr,m}$	0	$-f_{mnr}\delta_{i,m}$	$f_{mnr}\delta_{m,ij}$	$(-2f_{mnr} - \ell_m + \ell_{mn} + \ell_{mr})\delta_{ijk,mnr}$

$A^1 \cdot A^3$	H	E_i	E_{ij}	E_{ijk}
ℓ	p	0	0	0
ℓ_m	0	$-p\delta_{i,m}$	0	0
ℓ_{mn}	0	0	$-p\delta_{ij,mn}$	0
f_{mnr}	0	0	0	$-p\delta_{ijk,mnr}$

$A^2 \cdot A^2$	S	S_i	P_{ij}	F_{ij}	G_{ij}	H_{ijk}	$V_{ijk,i}$
S	p	0	0	0	0	0	0
S_m	0	$-p\delta_{i,m}$	0	0	0	0	0
P_{mn}	0	0	$p\delta_{ij,mn}$	$-p\delta_{ij,mn}$	0	0	0
F_{mn}	0	0	$-p\delta_{ij,mn}$	0	$-p\delta_{ij,mn}$	0	
G_{mn}	0	0	0	$-p\delta_{ij,mn}$	$-p\delta_{ij,mn}$	0	0
H_{mnr}	0	0	0	0	0	$-p\delta_{ijk,mnr}$	0
$V_{mnr,m}$	0	0	0	0	0	0	$p\delta_{ijk,mnr}\delta_{i,m}$

5 The Cremona Involution on $\mathbb{P}^4$

Consider now the Cremona involution

$$[x_0 : x_1 : x_2 : x_3 : x_4] \longrightarrow [\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3} : \frac{1}{x_4}]$$
$$= [x_1x_2x_3x_4 : x_0x_2x_3x_4 : x_0x_1x_3x_4 : x_0x_1x_2x_4 : x_0x_1x_2x_3]$$

which lifts to a biregular automorphism of $\mathbb{X}^4_3$. The induced action ϕ on the Chow ring $A(\mathbb{X}^4_3)$ is given as follows.

Proposition 5.1

$$\phi(H) = 4H - 3\sum_i E_i - 2\sum_{ij} E_{ij} - \sum_{ijk} E_{ijk}$$
$$\phi(E_i) = [L_{jkmn\neq i}] = H - \sum_{m\neq i} E_m - \sum_{mn\neq i} E_{mn} - \sum_{mnr\neq i} E_{mnr}$$
$$\phi(E_{ij}) = E_{mnr\neq i,j}$$
$$\phi(E_{ijk}) = E_{mn\neq i,j,k}$$
$$\phi(S) = 6S - 3\sum_i S_i - \sum_{ij} P_{ij}$$
$$\phi(S_m) = 3S - 2\sum_{i\neq m} S_i - \sum_{ij\neq m} P_{ij}$$
$$\phi(F_{mn}) = M_{ijk\neq mn} = S - S_i - S_j - S_k + F_{ij} + F_{ik} + F_{jk} - G_{ij} - G_{ik} - G_{jk} + \Lambda_{ijk}$$
$$= S - S_i - S_j - S_k - P_{ij} - P_{ik} - P_{jk} + 2H_{ijk} - V_{ijk,i} - V_{ijk,j} - V_{ijk,k}$$
$$\phi(G_{mn}) = \Lambda_{ijk\neq mn} = 2H_{ijk} - V_{ijk,i} - V_{ijk,j} - V_{ijk,k}$$
$$\phi(P_{mn}) = -S + S_i + S_j + S_k + P_{ij} + P_{ik} + P_{jk} (ijk \neq mn)$$
$$\phi(H_{mnr}) = 2G_{ij} - (H_{ijm} - V_{ijm,i} - V_{ijm,j}) - (H_{ijn} - V_{ijn,i} - V_{ijn,j}) - (H_{ijr} - V_{ijr,i} - V_{ijr,j})$$
$$= 2P_{ij} + 2F_{ij} - (H_{ijm} - V_{ijm,i} - V_{ijm,j}) - (H_{ijn} - V_{ijn,i} - V_{ijn,j}) - (H_{ijr} - V_{ijr,i} - V_{ijr,j})$$
$$\text{for} \quad i, j \neq m, n, r$$
$$\phi(V_{mnr,m}) = G_{ij} - (H_{ijn} - V_{ijn,i} - V_{ijn,j}) - (H_{ijr} - V_{ijr,i} - V_{ijr,j})$$
$$= P_{ij} + F_{ij} - (H_{ijn} - V_{ijn,i} - V_{ijn,j}) - (H_{ijr} - V_{ijr,i} - V_{ijr,j})$$

$$
\begin{aligned}
&\quad \textit{for} \quad i, j \neq m, n, r \\
\phi(\ell) &= 4\ell - \sum_i \ell_i \\
\phi(\ell_m) &= 3\ell - \sum_{i \neq m} \ell_i \\
\phi(\ell_{mn}) &= 2\ell - \ell_i - \ell_j - \ell_k + f_{ijk} \textit{ for } i, j, k \neq m, n \\
\phi(f_{mnr}) &= h_{ij} = \ell - \ell_i - \ell_j + \ell_{ij} \textit{ for } i, j \neq m, n, r
\end{aligned}
$$

Proposition 5.2 *(a) Let $D = dH - \sum_i m_i E_i - \sum_{ij} m_{ij} E_{ij} - \sum_{ijk} m_{ijk} E_{ijk}$ be a general class in $A^1(\mathbb{X}_3^4)$. Then the Cremona image $\phi(D)$ of D under the involution is*

$$\phi(D) = d'H - \sum_i m'_i E_i - \sum_{ij} m'_{ij} E_{ij} - \sum_{ijk} m'_{ijk} E_{ijk}$$

where

$$
\begin{aligned}
d' &= \phi(D) \cdot \ell = D \cdot \phi(\ell) = D \cdot (4\ell - \sum_r \ell_r) = 4d - \sum_r m_r; \\
m'_i &= \phi(D) \cdot \ell_i = D \cdot \phi(\ell_i) = D \cdot (3\ell - \sum_{r \neq i} \ell_r) = 3d - \sum_{r \neq i} m_r \\
m'_{ij} &= \phi(D) \cdot \ell_{ij} = D \cdot \phi(\ell_{ij}) = D \cdot (2\ell - \sum_{r \neq ij} \ell_r + f_{rst \neq ij}) = 2d - \sum_{r \neq ij} m_r + m_{rst \neq ij} \\
m'_{ijk} &= \phi(D) \cdot f_{ijk} = D \cdot \phi(f_{ijk}) = D \cdot (\ell - \sum_{r \neq ijk} \ell_r + \ell_{rs \neq ijk}) = d - \sum_{r \neq ijk} m_r + m_{rs \neq ijk}
\end{aligned}
$$

(b) Let $T = dS - \sum_i m_i S_i - \sum_{ij} m_{ij} P_{ij} - \sum_{ij} n_{ij} F_{ij} - \sum_{ijk} m_{ijk} H_{ijk} + \sum_{ijk} (n_{ijk,i} V_{ijk,i} + n_{ijk,j} V_{ijk,j} + n_{ijk,k} V_{ijk,k})$ be a general class in $A^2(\mathbb{X}_3^4)$. Then the Cremona image $\phi(T)$ of T under the involution is

$$
\begin{aligned}
\phi(T) = d'S &- \sum_i m'_i S_i - \sum_{ij} m'_{ij} P_{ij} - \sum_{ij} n'_{ij} F_{ij} \\
&- \sum_{ijk} m'_{ijk} H_{ijk} + \sum_{ijk} (n'_{ijk,i} V_{ijk,i} + n'_{ijk,j} V_{ijk,j} + n'_{ijk,k} V_{ijk,k})
\end{aligned}
$$

where

$$
\begin{aligned}
d' &= \phi(T) \cdot S = T \cdot \phi(S) = T \cdot (6S - 3\sum_i S_i - \sum_{ij} P_{ij}) \\
&= 6d - 3\sum_i m_i + \sum_{ij} (m_{ij} - n_{ij}) \\
m'_i &= \phi(T) \cdot S_i = T \cdot \phi(S_i) = T \cdot (3S - 2\sum_{r \neq i} S_r - \sum_{rs \neq i} P_{rs})
\end{aligned}
$$

$$
\begin{aligned}
&= 3d - 2\sum_{r\neq i} m_r + \sum_{rs\neq i}(m_{rs} - n_{rs})\\
m'_{ij} &= \phi(T)\cdot F_{ij} = T\cdot\phi(F_{ij})\\
&= T\cdot(S - S_r - S_s - S_t - P_{rs} - P_{rt} - P_{st} + 2H_{rst} - V_{rst,r} - V_{rst,s} - V_{rst,t})\\
&= d - m_r - m_s - m_t + m_{rs} + m_{rt} + m_{st} - n_{rs} - n_{rt} - n_{st} + 2m_{rst} - n_{rst,r} - n_{rst,s} - n_{rst,t}\\
n'_{ij} &= \phi(T)\cdot G_{ij} = T\cdot\phi(G_{ij}) = T\cdot(\Lambda_{rst\neq ij}) = T\cdot(2H_{rst} - V_{rst,r} - V_{rst,s} - V_{rst,t})\\
&= 2m_{rst} - n_{rst,r} - n_{rst,s} - n_{rst,t}\\
m'_{ijk} &= \phi(T)\cdot H_{ijk} = T\cdot\phi(H_{ijk})\\
&= T\cdot(2G_{rs} - (H_{rsi} - V_{rsi,r} - V_{rsi,s}) - (H_{rsj} - V_{rsj,r} - V_{rsj,s}) - (H_{rsk} - V_{rsk,r} - V_{rsk,s})\\
&\quad \mathit{for}\ \ rs \neq ijk\\
&= 2n_{rs} - (m_{rsi} - n_{rsi,r} - n_{rsi,s}) - (m_{rsj} - n_{rsj,r} - n_{rsj,s}) - (m_{rsk} - n_{rsk,r} - n_{rsk,s}\\
n'_{ijk,i} &= \phi(T)\cdot V_{ijk,i} = T\cdot\phi(V_{ijk,i})\\
&= T\cdot(G_{rs} - (H_{rsj} - V_{rsj,r} - V_{rsj,s}) - (H_{rsk} - V_{rsk,r} - V_{rsk,s}))\\
&= n_{rs} - (m_{rsj} - n_{rsj,r} - n_{rsj,s}) - (m_{rsk} - n_{rsk,r} - n_{rsk,s})
\end{aligned}
$$

(c) *Let* $C = d\ell - \sum_i m_i\ell_i - \sum_{ij} m_{ij}\ell_{ij} - \sum_{ijk} m_{ijk} f_{ijk}$ *be a general class in* $A^3(\mathbb{X}^4_3)$. *Then the Cremona image* $\phi(C)$ *of* C *under the involution is*

$$\phi(C) = d'\ell - \sum_i m'_i\ell_i - \sum_{ij} m'_{ij}\ell_{ij} - \sum_{ijk} m'_{ijk} f_{ijk}$$

where

$$
\begin{aligned}
d' &= \phi(C)\cdot H = C\cdot\phi(H) = C\cdot(4H - 3\sum_i E_i - 2\sum_{ij} E_{ij} - \sum_{ijk} E_{ijk})\\
&= 4d - 3\sum_i m_i - 2\sum_{ij} m_{ij} - \sum_{ijk} m_{ijk};\\
m'_i &= \phi(C)\cdot E_i = C\cdot\phi(E_i) = C\cdot(H - \sum_{r\neq i} E_r - \sum_{rs\neq i} E_{rs} - \sum_{rst\neq i} E_{rst})\\
&= d - \sum_{r\neq i} m_r - \sum_{rs\neq i} m_{rs} - \sum_{rst\neq i} m_{rst};\\
m'_{ij} &= \phi(C)\cdot E_{ij} = C\cdot\phi(E_{ij}) = C\cdot E_{rst\neq ij} = m_{rst\neq ij}\\
m'_{ijk} &= \phi(C)\cdot E_{ijk} = C\cdot\phi(E_{ijk}) = C\cdot E_{rs\neq ijk} = m_{rs\neq ij}
\end{aligned}
$$

We note that, for surface classes in $A^2(\mathbb{X}^4_3)$, higher-dimensional contact is observed by having nonzero coefficients in the F, H, and V basis elements. For curve classes in A^3, this higher-dimensional contact corresponds to nonzero coefficients in the ℓ_{ij} and the f_{ijk} basis elements (corresponding to a curve meeting a coordinate line or a coordinate plane). The formulas above show that a similar phenomenon happens as in the $\mathbb{P}^3$ case: if these are all zero, that is preserved under the involution.

Corollary 5.3 *(a) The subspace of $A^2(\mathbb{X}_3^4)$ spanned by S, the S_i, and the P_{ij} is invariant under the Cremona involution ϕ. If $T = dS - \sum_i m_i S_i - \sum_{ij} m_{ij} P_{ij}$ is an element in this subspace, then $\phi(T) = d'S - \sum_i m_i' S_i - \sum_{ij} m_{ij}' P_{ij}$ where*

$$d' = 6d - 3\sum_i m_i + \sum_{ij} m_{ij}$$

$$m_i' = 3d - 2\sum_{r \neq i} m_r + \sum_{rs \neq i} m_{rs}$$

$$m_{ij}' = d - m_r - m_s - m_t + m_{rs} + m_{rt} + m_{st} \quad \textit{for} \quad r, s, t \neq i, j$$

(b) The subspace of $A^3(\mathbb{X}_3^4)$ spanned by ℓ, ℓ_i is invariant under the Cremona involution ϕ. If $C = d\ell - \sum_i m_i \ell_i$ is an element in this subspace, then $\phi(C) = d'\ell - \sum_i m_i' \ell_i$ where

$$d' = 4d - 3\sum_i m_i$$

$$m_i' = d - \sum_{r \neq i} m_r$$

For divisors, the natural subspace invariant under the involution is the one generated by the E_{ij}'s and E_{ijk}'s. If we are only interested in the multiplicity conditions at the points, we can therefore mod out by this subspace of A^1, and obtain the following.

Corollary 5.4 *The subspace of $A^1(\mathbb{X}_3^4)$ spanned by the E_{ij}'s and E_{ijk}'s is invariant under the Cremona involution ϕ. Denote by $\bar{A}^1$ the quotient of A^1 by this subspace; the involution ϕ descends to an involution of $\bar{A}^1$. If $\bar{D} = dH - \sum_i m_i E_i$ represents a coset in this subspace, then $\phi(\bar{D}) = d'H - \sum_i m_i' E_i$ where*

$$d' = 4d - \sum_i m_i \quad \textit{and} \quad m_i' = 3d - \sum_{r \neq i} m_r.$$

6 Six and Seven Points in $\mathbb{P}^4$

The formulas for how degrees and multiplicities change for curves, surfaces, and divisors in $\mathbb{P}^4$ under the standard Cremona transformation can be used to analyze compositions of such Cremona transformations based at more than five points. We will present the orbits of the linear subspaces spanned by subsets of the points in this section.

If we first consider six general points in $\mathbb{P}^4$, it is easy to see using the formulas above that any line through 2 of the six points, 2-plane through 3 of them, or a hyperplane through 4, is either contracted by the Cremona transformation or is sent to itself.

The case of seven general points in $\mathbb{P}^4$ is one step more interesting. In this case, for a line through two of the seven points, it is either contracted by the Cremona transformation based at five of the points (if the two points are a subset of the five), is sent to itself (if one of the two is a subset of the five) or is sent to the rational normal quartic (RNQ) through all seven points (if neither of the two is among the five).

The iteration of Cremona now leads us to consider the transformation of the RNQ; applying Cremona at any five yields back the line joining the other two (since the Cremona is an involution).

Hence the Cremona orbit of the line through two points is the collection of all of the 21 lines, plus the rational normal quartic through all seven points.

Now consider the 2-plane spanned by three of the 7 points. Performing a Cremona transformation at 5 of the 7 points, we see that if all three points are among the 5, the plane is contracted as part of the fundamental locus. If two of the three points are among the five, the plane is sent to itself. If only one of the three points is among the five, then the Cremona image is a surface of degree three, with a point of multiplicity 3 at that one point, and multiplicity 1 at the other six points. It contains the line joining that one point to the other six, with a multiplicity of one each, and no other lines joining the points. It also contains the RNQ with multiplicity one. This cubic surface is a cone over a twisted cubic in $\mathbb{P}^3$.

Iterating the Cremona by applying it to this cone, we see that if the five points contain the vertex, it will be transformed back into the 2-plane. If it does not, it is preserved.

Hence, the Cremona orbit of the 2-plane through 3 points in $\mathbb{P}^4$ consists of the 35 planes and the 7 cubic cones.

For the hyperplanes through 4 of the seven points, there are four cases to consider. We choose five of the seven to perform the Cremona transformation at. If all 4 of the hyperplane points are among the five, then the hyperplane is contracted to a point. If 3 of the hyperplane points are among the five, then the hyperplane is transformed to another hyperplane. If 2 of the hyperplane points are among the five, it is transformed into a quadric double cone: a cone over a smooth conic with vertex a line (the line corresponding to the two points). To be explicit, take the line joining the two points, and a complementary plane; projection from the line to the plane sends the other five points to five general points in the plane, and there is a unique conic in that plane through those five points. The threefold is obtained as the cone over the conic with vertex the line. The surfaces contain all the lines joining the two points with the other five, as well as containing the RNQ too.

If we apply a second Cremona transformation to this quadric, we either return to the hyperplane, preserve the quadric, or (if we use as the base points the five points not on the vertex line) we obtain a cubic surface double at all seven points. It is also double all along the RNQ; this cubic surface is the secant variety to the RNQ, in fact.

Further applications of Cremona to this cubic surface lower the degree and return us to the quadric double cone; we see then that the orbit of the hyperplane consists of the set of 35 hyperplanes, the 21 quadric double cones, and the cubic secant variety to the RNQ.

It is interesting that the two special linear systems with irreducible members in $\mathbb{P}^4$ imposing only double points appear here: the quadrics double at two points and the cubics double at 7.

7 Eight Points in $\mathbb{P}^4$

We now consider the case of Cremona transformations based at 8 general points $p_1, \ldots, p_8$ in $\mathbb{P}^4$. Denote by L_{ij} the line joining p_i and p_j as usual. Denote by Q_i the rational normal quartic curve passing through all eight points except p_i (i.e., passing through the other 7).

It is easy to see, with a parallel computation as that done above for seven points, that the orbit of a line through two points, say L_{12}, consists of all 28 such lines L_{ij}, and all 8 of the RNQ's Q_k.

We can now take up the case of surfaces, which is more involved. We will record the data for a surface of degree d, having multiplicity m_i at p_i, multiplicity n_i along Q_i, and multiplicity m_{ij} along L_{ij}, by the triangular array of numbers:

$$\begin{array}{ccccccccc} d & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 & m_8 \\ & n_1 & n_2 & n_3 & n_4 & n_5 & n_6 & n_7 & n_8 \\ & & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} & m_{17} & m_{18} \\ & & & m_{23} & m_{24} & m_{25} & m_{26} & m_{27} & m_{28} \\ & & & & m_{34} & m_{35} & m_{36} & m_{37} & m_{38} \\ & & & & & m_{45} & m_{46} & m_{47} & m_{48} \\ & & & & & & m_{56} & m_{57} & m_{58} \\ & & & & & & & m_{67} & m_{68} \\ & & & & & & & & m_{78} \end{array} \tag{7.1}$$

Suppose we perform the five-point Cremona on the first five points 1, 2, 3, 4, 5. Then the degree d, the multiplicities m_i for $i \le 5$, and the m_{ij} for $i, j \le 5$, are transformed as indicated in Corollary 5.3(a).

For multiplicity m'_{ij} with $i \le 5$ and $j \ge 6$, we note that this line L_{ij} is left invariant under the Cremona, so that $m'_{ij} = m_{ij}$ for these indices.

For multiplicities m_{ij} with both $i, j \ge 6$, we note that this L_{ij} is the image of Q_k where $\{i, j, k\} = \{6, 7, 8\}$; k is the third index. Hence $m'_{ij} = n_k$ for $k = \{6, 7, 8\} - \{i, j\}$.

For the n'_k with $k \ge 6$, conversely we have $n'_k = m_{ij}$ where $i, j = \{6, 7, 8\} - \{k\}$. For n'_k with $k \le 5$, since such a Q_k is fixed, we have $n'_k = n_k$. This gives the following:

Corollary 7.2 *The surface with degree and multiplicities indicated by (7.1) is transformed, under the Cremona involution based at the first five points p_1, p_2, p_3, p_4, p_5, into the surface with degree and multiplicities recorded by:*

$$\begin{array}{ccccccccc}
d' & m_1' & m_2' & m_3' & m_4' & m_5' & m_6 & m_7 & m_8 \\
 & n_1 & n_2 & n_3 & n_4 & n_5 & m_{78} & m_{68} & m_{67} \\
 & & m_{12}' & m_{13}' & m_{14}' & m_{15}' & m_{16} & m_{17} & m_{18} \\
 & & & m_{23}' & m_{24}' & m_{25}' & m_{26} & m_{27} & m_{28} \\
 & & & & m_{34}' & m_{35}' & m_{36} & m_{37} & m_{38} \\
 & & & & & m_{45}' & m_{46} & m_{47} & m_{48} \\
 & & & & & & m_{56} & m_{57} & m_{58} \\
 & & & & & & & n_8 & n_7 \\
 & & & & & & & & n_6
\end{array}$$

where

$$d' = 6d - 3\sum_{i=1}^{5} m_i + \sum_{1\le i<j\le 5} m_{ij}$$

$$m_i' = 3d - 2\sum_{r\le 5; r\ne i} m_r + \sum_{r,s\le 5; r,s\ne i} m_{rs} \ \ \text{for } i \le 5$$

$$m_{ij}' = d - m_r - m_s - m_t + m_{rs} + m_{rt} + m_{st} \ \ \text{for } i, j \le 5 \text{ and } r, s, t = \{1, 2, 3, 4, 5\} - \{i, j\}$$

The Proposition below presents the orbit of L_{123}, a 2-plane through three of the points, in (b). For notational consistency with the other surfaces in this orbit, we will also denote L_{ijk} by $S_1(ijk)$. We have included in (a) the remarks above about the orbit of the line L_{12}. In (c), we present the orbit of a hyperplane; the reader can verify the computations as an exercise.

Proposition 7.3 *Fix 8 general points in $\mathbb{P}^4$, and consider Cremona transformations based at 5 of the 8, in series.*

(a) The orbit of a line through two of the 8 points consists of the 28 lines L_{ij} $(1 \le i < j \le 8)$ through two (p_i and p_j) of the 8 points, and the 8 rational normal quartics Q_k ($1 \le k \le 8$ through 7 of the 8 points (through all seven except p_k).

(b) The orbit of a plane through three of the 8 points consists of:

(b1) the 56 planes $L_{ijk} = S_1(1jk)$ through three of the 8 points (namely p_i, p_j, and p_k); the plane $L_{123} = S_1(123)$ is recorded as

$$\begin{array}{r} 1\;1\;1\;1\;0\;0\;0\;0\;0 \\ 0\;0\;0\;0\;0\;0\;0\;0 \\ 1\;1\;0\;0\;0\;0\;0 \\ 1\;0\;0\;0\;0\;0 \\ 0\;0\;0\;0\;0 \\ 0\;0\;0\;0 \\ 0\;0\;0 \\ 0\;0 \\ 0 \end{array}$$

(b2) the 56 *surfaces* $S_3(i, j)$ *of degree* 3 *with one point* p_i *of multiplicity* 3*,* 6 *points of multiplicity one, and one point* p_j *of multiplicity* 0*. It contains the lines joining the triple point* p_i *to all other multiplicity one points* p_k $(k \neq j)$ *and no other lines; it contains the rational normal quartic* Q_j *through the triple point and the six multiplicity one points. For example,* $S_3(8, 1)$ *is recorded as:*

$$\begin{array}{r} 3\;0\;1\;1\;1\;1\;1\;1\;3 \\ 1\;0\;0\;0\;0\;0\;0\;0 \\ 0\;0\;0\;0\;0\;0\;0 \\ 0\;0\;0\;0\;0\;1 \\ 0\;0\;0\;0\;1 \\ 0\;0\;0\;1 \\ 0\;0\;1 \\ 0\;1 \\ 1 \end{array}$$

(b3) the 56 *sextic surfaces* $S_6(ijk)$ *of degree* 6 *with three points* (p_i, p_j, p_k) *of multiplicity one, and the other* 5 *points of multiplicity* 3*. It contains the lines joining any two of the multiplicity* 3 *points and no other lines; It contains the rational normal quartics through the five multiplicity* 3 *points and any two of the three multiplicity one points. For example,* $S_6(678)$ *is recorded as:*

$$\begin{array}{r} 6\;3\;3\;3\;3\;3\;1\;1\;1 \\ 0\;0\;0\;0\;0\;1\;1\;1 \\ 1\;1\;1\;1\;0\;0\;0 \\ 1\;1\;1\;0\;0\;0 \\ 1\;1\;0\;0\;0 \\ 1\;0\;0\;0 \\ 0\;0\;0 \\ 0\;0 \\ 0 \end{array}$$

(b4) the 28 *surfaces* $S_{10}(ij)$ *of degree* 10 *with two points (*p_i *and* p_j*) of multiplicity* 6 *and the other* 6 *points of multiplicity* 3*. It contains the lines joining the multiplicity one points to the multiplicity six points (each with multiplicity*

one) and the line joining the two multiplicity 6 *points with multiplicity* 3*. It contains the* 6 *rational normal quartics that pass through the two multiplicity* 6 *points and five of the six multiplicity one points. For example,* $S_{10}(78)$ *is recorded as:*

$$\begin{array}{ccccccccc}
10 & 3 & 3 & 3 & 3 & 3 & 3 & 6 & 6 \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 & & & 0 & 0 & 0 & 0 & 1 & 1 \\
 & & & & 0 & 0 & 0 & 1 & 1 \\
 & & & & & 0 & 0 & 1 & 1 \\
 & & & & & & 0 & 1 & 1 \\
 & & & & & & & 1 & 1 \\
 & & & & & & & & 3
\end{array}$$

(b5) the 8 *surfaces* $S_{15}(i)$ *of degree* 15 *with one point* (p_i) *of multiplicity* 3 *and the other seven points of multiplicity* 6*. It contains the joining any two points of multiplicity* 6*, and no other lines. It contains all* 8 *of the rational normal quartics; the one through the seven multiplicity* 6 *points with multiplicity three, and all others with multiplicity one. For example,* $S_{15}(1)$ *is recorded as:*

$$\begin{array}{ccccccccc}
15 & 3 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & 1 & 1 & 1 & 1 & 1 \\
 & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & 1 & 1 & 1 \\
 & & & & & & & 1 & 1 \\
 & & & & & & & & 1
\end{array}$$

(c) We use the notation that $(d; m_1 m_2 \cdots m_8)$ *represents a hyperplane of degree* d *having multiplicity* m_i *at* p_i*. The orbit of the hyperplane through the first four points (represented by* (1; 11110000) *consists of the following divisors, and all related divisors obtained by permutations of the eight points:*

(1; 11110000)	(2; 22111110)	(3; 22222220)	(3; 32222111)
(4; 33332221)	(4; 43222222)	(5; 44333322)	(6; 44444432)
(6; 54443333)	(7; 55544443)	(7; 64444444)	(8; 65555544)
(9; 66665555)	(10; 76666666)		

8 Applications

Proposition 8.1 *Let R and T be two Weyl planes on $X_{4,8}$. Then $R \cdot T \in \{0, 1, 3\}$.*

Proof If we choose an element w of the Weyl group that sends Weyl plane R to the actual plane $S_1(123)$, then since the intersection form is preserved we have $R \cdot T = S_1(123) \cdot w(T)$. Hence it suffices to show that the intersection of $S_1(123)$ with any Weyl plane is in $\{0, 1, 3\}$. This one can check by hand for all of the cases.

Even easier would be to notice that, if ϕ is the Cremona transformation centered at the first five points, then by Corollary 7.2 we have $\phi(S_1(123)) = -P_{45}$ in the Chow ring. Hence it also suffices to show that $-P_{45} \cdot T \in \{0, 1, 3\}$ for all Weyl planes T. By Proposition 4.5, intersecting with $-P_{45}$ picks out exactly the multiplicity m_{45} for the Weyl plane. Hence it suffices, after taking account of permutations, to observe that for all Weyl planes, all m_{ij} are in $\{0, 1, 3\}$. □

Proposition 8.2 *Let R and T be any Weyl planes on X_8^4. If $R \cdot T \neq 3$, then there exists w in the Weyl group of X_8^4 and $i \in \{1, 4\}$ such that $w(R) = H_{123}$ and $w(T) = H_{i56}$.*

Proof It is enough to prove the statement for $R \neq T$. One can use the same technique as in Proposition 8.1 and reduce one Weyl surface to $-P_{45}$ and select Weyl surfaces from the list of Proposition 7.3(b) that have $m_{45} \in \{0, 1\}$. Then applying the Cremona transformation ϕ centered at the first five points, we have the first Weyl surface being $S_1(123)$ and the other on the following lists (up to permutations that fix $\{1, 2, 3\}$):

1. Case $S_1(123) \cdot T = 1$:

   (a) $S_1(123) \cdot S_1(456) = 1$
   (b) $S_1(123) \cdot S_3(4, 1) = 1$
   (c) $S_1(123) \cdot S_6(126) = 1$
   (d) $S_1(123) \cdot S_{10}(45) = 1$
   (e) $S_1(123) \cdot S_{15}(1) = 1$

   We are done in the first case of course. In the other cases it suffices to find five indices, two of them among $\{1, 2, 3\}$, so that the corresponding Cremona transformation reduces the degree of the second surface; such a Cremona will fix $S_1(123)$ and we proceed then by induction on the degree.
   To reduce the cubic surface, $\{2, 3, 4, 7, 8\}$ will work; for the sextic, $\{1, 3, 5, 7, 8\}$ works. For the surface of degree 10, $\{1, 2, 4, 5, 6\}$ suffices; finally for the last surface of degree 15, $\{2, 3, 6, 7, 8\}$ works.
2. Case $S_1(123) \cdot T = 0$: In this case a similar approach yields the following lists to analyze:

   (a) $S_1(123) \cdot S_1(145) = 0$ or $S_1(123) \cdot S_1(124) = 0$
   (b) $S_1(123) \cdot S_3(1, 2) = 0$ or $S_1(123) \cdot S_3(1, 4) = 0$ or $S_1(123) \cdot S_3(4, 5) = 0$
   (c) $S_1(123) \cdot S_6(145) = 0$ or $S_1(123) \cdot S_6(456) = 0$
   (d) $S_1(123) \cdot S_{10}(12) = 0$ or $S_1(123) \cdot S_{10}(15) = 0$

(e) $S_1(123) \cdot S_{15}(4) = 0$

The same proof as in the prior case works; in each situation one finds five indices, two among {1, 2, 3}, that reduce the degree of the second surface. For example, {2, 3, 6, 7, 8} works for the degree 15 surface. We leave the details of the other cases to the reader.

□

We remark that Weyl planes that intersect in three points (modulo permutations of points) are

$$S_1(123) \cdot S_6(123) = S_3(1, 8) \cdot S_3(8, 1) = 3.$$

Corollary 8.3 *Assume R and T are Weyl planes in the base locus of the linear system $|D|$ for an effective divisor $D = dH - \sum_{i=1}^{8} m_i E_i$ on X_8^4. Then $R \cdot T = 0$.*

Proof We argue by contradiction. Assume first that $R \cdot T = 1$. By Proposition 8.2, we can apply a series of Cremona transformations, which do not change the hypothesis on the base locus, and assume that $R = S_1(123)$ and $T = S_1(456)$. It follows from the results of [6], Sect. 4, and [8], Proposition 4.2, that we therefore have

$$m_1 + m_2 + m_3 - 2d > 0 \text{ and } m_4 + m_5 + m_6 - 2d > 0.$$

Hence, the system of rational normal curves of degree 4 passing through the first 6 points must be in the base locus of $|D|$; since this family of curves covers $\mathbb{P}^4$, we conclude $|D|$ is empty, a contradiction.

If the two Weyl planes intersect in three points, then they are either $S_3(1, 8)$ and $S_3(8, 1)$ or $S_1(123)$ and $S_6(123)$ (up to permutations). We will analyze the first case; the other is handled by a similar argument. Assume by contradiction that both such Weyl planes are in the base locus of the linear system $|D|$ of an effective divisor D. By Proposition 3 of [5], the multiplicity of containment of the surface $S_3(1, 8)$ in the base locus of a divisor D is $2m_1 + m_2 + \ldots + m_7 - 5d < 0$; therefore since both $S_3(1, 8)$ and $S_3(8, 1)$ are in the base locus we obtain $2(m_1 + \ldots + m_8) - 10d < 0$. This contradicts the effectivity of the divisor D because $2(m_1 + \ldots + m_6) + m_7 + 3m_8 - 10d \leq 2(m_1 + \ldots + m_8) - 10d < 0$; therefore a family of curves of degree 10 with six double points, one simple point, and one triple point meets D negatively, and so is part of the base locus also. Corollary 5.3 implies that these curves are in the Weyl orbit of a line through a point, and therefore again cover the projective space, a contradiction. The remaining case can be handled by the same argument. □

Remark 8.4 In fact, the linear equations of pencils of curves in the base locus of the linear system of an effective divisor D, that in this case are equivalent to two Weyl planes that meet in the base locus of $|D|$, give the *faces of the cone of effective divisors*. We will prove this theorem in the case of a Mori Dream Space in arbitrary dimension in [11].

Remark 8.5 In [11], we prove that a Weyl curve and a Weyl divisor that meet can not be simultaneously in the base locus of the linear system of an effective divisor D.

For any effective divisor $D \in Pic(X^4_8)$, define $\widetilde{D} \in Pic(\widehat{X^4_8})$ to be the proper transform of D after blowing up all the Weyl lines and Weyl planes in the base locus of $|D|$ to obtain $\widehat{X^4_8}$. Corollary 8.3 proves that the space $\widehat{X^4_8}$ is smooth.

We remark first that the Weyl line C has normal bundle $\oplus\mathcal{O}(-1)^3$. If $D \cdot C < 0$ then the Weyl line C is in the base locus of the linear system $|D|$. Let $D_{(1)}$ denote the proper transform of D under the blowup Y of all fixed Weyl lines in X^4_s. For each Weyl line C, define $k_C = -D \cdot C$.

Proposition 8.6 *If D be an effective divisor on X^4_8, then*

$$h^1(X^4_s, \mathcal{O}_{X^4_s}(D)) = \sum_C \binom{2+k_C}{4} + h^1(Y, \mathcal{O}_Y(D_{(1)}) - h^2(Y, \mathcal{O}_Y(D_{(1)}).$$

A general form of Proposition 8.6 for (-1)-curves in arbitrary dimension will be given in [11]. We conclude that if $k_C \geq 2$ then $h^1(X^4_s, \mathcal{O}_{X^4_s}(D)) \geq 1 + h^1(Y, \mathcal{O}_Y(D_{(1)}) - h^2(Y, \mathcal{O}_Y(D_{(1)})$.

Conjecture 8.7 Let D be an effective divisor on X^4_s, with $H^1(X^4_s, \mathcal{O}_{X^4_s}(D)) = 0$. Then $D \cdot C \geq -1$ for any Weyl line C.

Remark 8.8 For arbitrary number of points s, the converse of Conjecture (8.7) is not true. Indeed, take $D := 4H - 2\sum_{i=1}^{14} E_i \in Pic(X^4_{14})$. We can see that $D \cdot C \geq 0$ for any Weyl line C; however, the Alexander Hirschowitz Theorem implies that

$$h^1(X^4_s, \mathcal{O}_{X^4_s}(D)) = 1.$$

For every r-subset $I(r)$ of the indices $\{1, \ldots, 8\}$, let $L_{I(r)}$ be the linear span of the corresponding points. Let $k_{w(L_{I(r)})}$ be the multiplicity of containment of the Weyl cycle $w(L_{I(r)})$ in the base locus of D, for a Weyl group element w. In [5], the *Weyl expected dimension* for an effective divisor D was introduced as

$$wdim(D) := \chi(D) + \sum_{r=1}^{3} \sum_{I(r)\in\{1,\ldots,8\}} \sum_{w\in W} (-1)^{r+1} \binom{4 + k_{w(L_{I(r)})} - r - 1}{4}.$$

Moreover, in [5] it was conjectured that for every effective divisor D on $\widehat{X^4_8}$, the dimension of space of global sections of D equals the Weyl expected dimension.

Conjecture 8.9 Let D be an effective divisor on $\widehat{X^4_8}$.

1. If $D \cdot C \geq -1$ for all Weyl curves C then

$$H^1(X^4_s, \mathcal{O}_{X^4_s}(D)) = 0.$$

2. $h^0(D) = wdim(D) + \sum_{r=1}^{3}(-1)^{r+1}h^r(\widetilde{D})$.
3. For every $r \geq 1$, $h^r(\widetilde{D}) = 0$.
4. Moreover, $\widetilde{D}$ is globally generated on $\widehat{X^4_8}$.

We remark that Conjecture 8.9 part (2) implies $wdim(D) = \chi(\widetilde{D})$, while part (3) implies that conjecture of [5] regarding dimension $h^0(D)$ is true.

Remark 8.10 We remark that Conjecture 8.9 holds for effective divisors on X^n_{n+2} [3, 8, 9]; therefore it holds for X^4_6. Notice that $\widehat{X^4_9}$ is not a Mori Dream Space and in fact, there are infinitely many Weyl lines. The authors believe that Conjecture 8.9 also holds for $\widehat{X^4_9}$ with a similar construction for the Weyl planes as the one presented here.

Remark 8.11 Conjecture 8.9 fails in $\widehat{X^4_{10}}$, because for arbitrary number of points, in non Mori-dream spaces Weyl cycles are not the only obstructions. Indeed, consider the divisor

$$D := 4H - 4E_1 - 2\sum_{i=2}^{10} E_i.$$

We remark that D contains in the base locus of its linear system just double lines $k_{L_{1i}} = 2$; therefore its proper transform under the blowup of all its Weyl base locus (i.e. only lines) is

$$\widehat{D} := 4H - 4E_1 - 2\sum_{i=2}^{10} E_i - 2\sum_{i=2}^{10} E_{1i}$$

Moreover, since $k_{L_{1i}} = 2$ we have

$$\chi(D) = \binom{4+4}{4} - \binom{4+4-1}{4} - 9\binom{4+2-1}{4} = 70 - 35 - 45 = -10$$

$$wdim(D) = \chi(\widehat{D}) = \chi(D) + 9\binom{2+2}{4} = -1$$

However, this divisor is effective, and, in fact, the Alexander–Hirschowitz theorem implies that it is unique in its linear system. We conclude that $h^0(D) = 1 \neq 0 = wdim(D)$, therefore $h^1(\widehat{D}) = 1$.

Acknowledgements The collaboration was partially supported by NSF grant DMS1802082.

References

1. C. Araujo, C. Casagrande, On the Fano variety of linear spaces contained in two odd-dimensional quadrics. Geom. Topol. **21**(5), 3009–3045 (2017)
2. C. Araujo, A. Massarenti, Explicit log Fano structures on blow-ups of projective spaces. Proc. Lond. Math. Soc. (3)113, no. 4, 445–473 (2016)
3. M.C. Brambilla, O. Dumitrescu, E. Postinghel, On a notion of speciality of linear systems in $\mathbb{P}^n$. Trans. Am. Math. Soc. **367**(8), 5447–5473 (2015)
4. M.C. Brambilla, O. Dumitrescu, E. Postinghel, On the effective cone of $\mathbb{P}^n$ blown-up at $n+3$ points. Exp. Math. **25**(4), 452–465 (2016)
5. M. C. Brambilla, O. Dumitrescu, E. Postinghel, Weyl cycles on the blow-up of $\mathbb{P}^4$ at eight points, to appear in *The Art of Doing Algebraic Geometry* (Springer)
6. S. Cacciola, M. Donten-Bury, O. Dumitrescu, A. Lo Giudice, J. Park, Cones of divisors of blow-ups of projective spaces. Le Matematiche LXVI—Fasc. II, 153–187 (2011)
7. C. Casagrande, G. Codogni, A. Fanelli, The blow-up of $\mathbb{P}^4$ at 8 points and its Fano model, via vector bundles on a del Pezzo surface. Revista Matemática Complutense **32**, 475–529 (2019)
8. O. Dumitrescu, E. Postinghel, Vanishing theorems for linearly obstructed divisors. J. Algebra **477**, 312–359 (2017)
9. O. Dumitrescu, E. Postinghel, Positivity of divisors on blown-up projective spaces II. J. Algebra **529**, 226–267 (2019)
10. O. Dumitrescu, N. Priddis, *On* (-1) *classes*. https://arxiv.org/pdf/1905.00074.pdf
11. O. Dumitrescu, R. Miranda *On* (-1) *Curves in* $\mathbb{P}^r$, in preparation
12. D. Eisenbud, J. Harris, *3264 and all that: a Second Course in Algebraic Geometry* (Cambridge University Press, 2016)
13. A. Laface, L. Ugaglia, On a class of special linear systems on $\mathbb{P}^3$. Trans. Am. Math. Soc. **358**(12), 5485–5500 (2006) (electronic)
14. S. Mukai, *Finite Generation of the Nagata Invariant Rings in A-D-E cases*. RIMS Preprint n. 1502, Kyoto (2005)
15. M. Nagata, On rational surfaces, II. Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **33**, 271–293 (1960)