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Quadric cones on the boundary of the Mori cone for very general blowups of the plane

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Abstract
In this paper we show the existence of cones over a 8-dimensional rational sphere at the boundary of the Mori cone of the blow-up of the plane at $s \geq 13$ very general points. This gives evidence for De Fernex's strong Δ-conjecture, which is known to imply Nagata's conjecture. This also implies the existence of a multitude of good and wonderful rays as defined in Ciliberto et al. (Clay Math Proc 18:185–203, 2013).

Keywords Linear systems · Mori cone · Nagata's conjecture · Nef rays

Introduction

Fix a non-negative integer s (usually we will assume that $s \geq 10$). We denote by X_s the blow-up of the complex projective plane at s very general points; let H be the divisor class of a general line, let E_i be the class of the exceptional divisor over the i-th point, and let $N_s = \mathrm{Pic}(X_s) \otimes_{\mathbb{Z}} \mathbb{R}$, where $\mathrm{Pic}(X_s)$ is the Picard group. Observe that N_s is a real vector space of dimension $s+1$ (with basis $\{H, E_1, \ldots, E_s\}$). The divisor class $L = dH - \sum_i m_i E_i$ represents the plane curves of degree d having multiplicity at least m_i at the i-th point.

The set of non-negative real multiples of a nonzero vector $L \in N_s$ is called a *ray*, which we denote by $R = \langle L \rangle$. A ray is *rational* if it contains an integral vector (in the H, E_i basis). A ray is *effective* if it contains a (necessarily integral) vector representing an effective divisor class. Neither the *degree* of a ray (the coefficient of H) nor the intersection number of two rays are well-defined; however the sign of these quantities are. Therefore it makes sense for example to ask that a ray R satisfies $\deg(R) > 0$ or $R^2 > 0$; for example, any effective ray must have non-negative degree. By the Riemann-Roch Theorem, if a rational ray has

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positive degree and self-intersection, then it is effective. A divisor or ray is *nef* if it intersects all effective divisors non-negatively.

There are three cones in N_s that are of interest for us. The *effective cone* is the cone generated by effective rays: this is the cone of all finite linear combinations of effective divisor classes with non-negative real coefficients. The effective cone is, in general, not closed; its closure is called the *Mori* cone. The dual of the Mori cone is the *nef* cone (consisting of nef rays), which is a closed cone as well.

Closed cones are defined by their extremal rays, and so the identification of extremal rays for the Mori and nef cones is of fundamental importance. Prior results constructing extremal rays of selfintersection zero for these cones gave discrete examples (see [4, 6]) or families of rays contained in the hyperplane orthogonal to $K_{X_s} = -3H + E_1 + \cdots + E_s$ (unpublished work by T. de Fernex for $s = 10$ and by J. C. Ottem for $s = 12$). In this paper we will construct 9-dimensional subsets (which are quadratic cones) at the boundary of the Mori cone for all $s \geq 10$, and having positive intersection with K_{X_s} if $s \geq 13$; this illustrates the existence of both rational and irrational extremal rays.

A key lemma that produces extremal rays is provided by [4, Lemma 3.8], which states that if a ray R is rational, non-effective, and satisfies $\deg(R) \geq 0$ and $R^2 = 0$, then R is nef, and is extremal for the Mori cone and the nef cone. Such a ray is called *good* in [4]; in that paper we also defined a *wonderful* ray, as one which is irrational and nef, with self-intersection zero. Some wonderful rays are known (an early example of wonderful ray is contained in [11]); the aforementioned examples of De Fernex and Ottem provide wonderful rays for $s = 10$ and $s = 12$, and the results of [6] imply the existence of wonderful rays for every $s \geq 10$. Wonderful rays, being irrational and extremal, prove that the Mori and nef cones are not rational polyhedral, and provide evidence for stronger conjectures that we now describe.

Recall the canonical divisor $K_s = -3H + \sum_i E_i$ of X_s. Define the *De Fernex ray* F_s to be the ray generated by $\sqrt{s-1}H - \sum_{i=1}^s E_i$. A ray R is said to be *De Fernex positive, negative or orthogonal* according to $R \cdot F_s$ being positive, negative or null; this terminology is parallel to a ray being K_s-positive, negative, or orthogonal. The *Strong* Δ*-Conjecture* (see [4, Conjecture 3.10]) states that if $s \geq 11$, and R is a rational De Fernex non-positive ray of self-intersection zero, then R is not effective, and therefore is a good ray. (See [8]; there is a refinement for the $s = 10$ case.) Note that $R \cdot F_s \leq 0$ implies $R \cdot K_s > 0$.

The Strong Δ-Conjecture implies the Nagata Conjecture [12], since it would imply that the *Nagata ray* $\langle \sqrt{s}H - \sum_i E_i \rangle$ would be wonderful if s is not a square. It would also imply that, for De Fernex non-positive classes, the boundary of the Mori cone is given by the classes with self-intersection zero. Hence finding large subsets of the boundary of the Mori cone given by such classes is a strong measure of non-polyhedrality for the Mori cone and provides intriguing evidence for the Strong Δ-Conjecture.

De Fernex's result in [8] that for $s = 10$, all rays R of self-intersection zero with $R \cdot K_{10} = 0$ are nef means that this hyperplane section of the Mori cone does have a boundary given by the quadratic equation $R^2 = 0$. The boundary of the Mori cone of X_{10} is so far unknown, but it is constrained by the nefness of all classes L with $L^2 = L \cdot K_{10} = 0$, namely a cone over a 8-dimensional sphere.

In this paper we provide further 9-dimensional quadratic cones on the Mori cone boundary of X_s for all $s \geq 13$, including some that constrain the De Fernex-negative area of the cone. We also provide a complete determination of the boundary classes orthogonal to K_s.

For clarity, because the points are assumed to be very general, specifying the E_i is irrelevant; we will therefore use the notation $L_d(m_1, \ldots, m_s) = dH - m_1E_1 - \cdots - m_sE_s$ for divisor classes in $\mathrm{Pic}(X_s)$, and we will use exponential notation for repeated multiplicities. Thus for example the canonical divisor can be written as $K_s = -L_3(1^s)$. The corresponding linear system of singular curves in $\mathbb{P}^2$, sometimes identified with the projective space $\mathbb{P}(H^0(X_s, \mathcal{O}_{X_s}(L)))$, will be denoted by $\mathcal{L} = \mathcal{L}_d(m_1, \dots, m_s)$. Observe that L is an effective class if and only if the (projective) dimension of $\mathcal{L}$ is non-negative.

Now we state the main results of this article.

Theorem 1 *Let $s \geq 10$, and let L be a class in* $\mathrm{Pic}(X_s)$ *with $L^2 = K_s \cdot L = 0$. The following are equivalent:*

(1) L is nef.
(2) There exist $s - 10$ disjoint (-1)-curves $C_1, \dots, C_{s-10}$ such that $C_i \cdot L = 0$ for each i.
(3) L is equivalent, by the action of the Cremona–Kantor group (see §1), to a multiple of $L_3(1^9, 0^{s-9})$.

Moreover, for each fixed collection of $s - 10$ disjoint (-1)-curves $C_1, \dots, C_{s-10}$, the rays spanned by all classes L with $L^2 = K \cdot L = C_i \cdot L = 0$ for all i, cover all rational points of a cone over a 8-dimensional rational sphere.

Here a cone over a 8-dimensional rational sphere means a rational quadric of rank 10 and signature $(1, 9)$ contained in a 10-dimensional rational linear subspace Π of $N_s \cong \mathbb{R}^{s+1}$ and having rational points. In practice, the quadric will always be cut out on Π by the equation $L^2 = 0$.

Note that the second condition is empty for $s = 10$, so in that case our statement is equivalent to De Fernex's result mentioned above.

Theorem 2 *Let $s \geq 13$. There exist 10-dimensional linear subspaces $\Pi \subset N_s$ such that the intersection of the Mori cone with Π consists of classes $L \in \Pi$ such that $L^2 \geq 0$ and $K_s \cdot L > 0$, namely*

$$\Pi \cap \overline{NE}(X_s) = \{L \in \Pi \mid L^2 \geq 0, K_s \cdot L > 0\}.$$

Theorem 3 *Let $s = k^2 + 4$ for some integer $k \geq 3$. There exist 10-dimensional linear subspaces $\Pi \subset N_s$ such that the intersection of the Mori cone with Π consists of classes $L \in \Pi$ such that $L^2 \geq 0$ and has non-empty intersection with the F_s-negative half space. In particular, the conditions $L \in \Pi$, $L^2 \geq 0$ and $F_s \cdot L < 0$ define an open subset (in the Euclidean topology) of an 8-dimensional sphere such that the cone over it is contained in the boundary of the Mori cone.*

Our starting point is De Fernex's result on extremal rays of the Mori cone orthogonal to K_{10} (that we reprove in a different way in Sect. 1 using Cremona maps), and then we proceed with the uncollision techniques developed in [6] (see Sect. 2) to produce new families of good and wonderful rays, in Sect. 3.

1 Nefness on $K^\perp$

In this section we study classes $L = L_d(m_1, \dots, m_s)$ such that $L^2 = L \cdot K_s = 0$ with respect to nefness. We recover by elementary methods De Fernex's result for $s = 10$ that every class with $L^2 = L \cdot K_{10} = 0$ is nef, and we extend it to describe the locus Nef^0 of all nef classes with $L^2 = L \cdot K_s = 0$ for $s > 10$.

Recall that, due to the Index theorem and the fact that $K_s^2 > 0$ for $s < 9$, $K_s^2 = 0$ for $s = 9$ and $K_s^2 < 0$ for $s > 9$, the intersection form on the space $K_s^\perp$ of classes orthogonal to K_s is negative definite for $s < 9$, it is negative semidefinite for $s = 9$, and it has signature

$(1, s-1)$ for $s > 9$. Therefore, as is well known, there are in N_s nonzero classes with $L^2 = K_s \cdot L = 0$ only for $s \geq 9$; for $s = 9$ these are exactly the multiples of K_9, and the ones with non-negative degree form the ray $\langle -K_9 \rangle$; and for $s \geq 10$ they form a rational quadratic cone of dimension $s-1$ in $N_s \cong \mathbb{R}^{s+1}$.

For $s \geq 9$, let $\mathcal{Q}_s^\perp \subset N_s$ be the locus of all classes with $L^2 = K_s \cdot L = 0$ and $H \cdot L \geq 0$ (since we are interested in effectivity and nefness, only classes meeting the class H nonnegatively are relevant). By the previous paragraph, $\mathcal{Q}_9^\perp$ consists of the single ray spanned by $-K_9$, whereas for $s \geq 10$ it is the H-nonnegative half of a rational quadratic cone, the boundary of a convex "round" cone in the hyperplane $K^\perp$ of classes L with $K_s \cdot L = 0$.

The Cremona–Kantor action

Consider the group $\mathcal{CK}_s$ generated by quadratic birational transformations of $\mathbb{P}^2$ based at subsets of 3 points among the s very general points and by permutations of these points (see [1, 9, 10]). The Cremona–Kantor group acts on the set of divisor classes $L_d(m_1, \ldots, m_s)$ preserving all numerical and cohomological properties (nefness, effectiveness, dimension of the associated linear system $\mathcal{L}_d(m_1, \ldots, m_s)$, etc.). A divisor class L is *Cremona reduced* if it has minimal degree in its $\mathcal{CK}_s$-orbit, and for effective classes this is the case if and only if the degree is greater or equal to the sum of the three largest multiplicities (see [7, p. 402–403, Thms 8–10]; a modern and more precise treatment may be found in [2]).

The canonical class $K_s = -3H + \sum E_i$ is fixed under the action of $\mathcal{CK}_s$, so $\mathcal{CK}_s$ preserves (and acts upon) the set $\mathcal{Q}_s^\perp$ of all classes of selfintersection zero orthogonal to K_s.

Remark 4 Let E be the class of a (-1)-curve in $\mathrm{Pic}(X_s)$, hence $E^2 = K_s \cdot E = -1$. It is well known that for $s \geq 3$ all (-1)-curves are $\mathcal{CK}_s$-equivalent (here we use the fact that the points blown up to construct X_s are very general).

Fix $s \geq 9$. Each hyperplane in N_s either meets the cone $\mathcal{Q}_s^\perp$ only at the origin, is tangent to it along a ray, or cuts it in two regions (which can only happen for $s \geq 10$). Let us consider the case of $E^\perp$, the hyperplane of classes orthogonal to E; because all classes of (-1)-curves are $\mathcal{CK}_s$-equivalent, which kind of intersection there is between $E^\perp$ and $\mathcal{Q}_s^\perp$ depends only on s.

Assume for simplicity that $E = E_s$ is the exceptional curve of the last blowup. In that case, a class belongs to $E^\perp$ if and only if it is the pullback to X_s of a class in X_{s-1} (via $X_s \to X_{s-1}$, the blowing-up of the last point). Since pullback preserves intersection multiplicities, $\mathcal{Q}_s^\perp \cap E^\perp \cong \mathcal{Q}_{s-1}^\perp$. Therefore $E^\perp$ meets $\mathcal{Q}_s^\perp$ at the origin for $s = 9$, they are tangent along a single ray $\langle -K_9 \rangle$ for $s = 10$, and they intersect along a lower dimensional irreducible quadric for $s > 10$, cutting $\mathcal{Q}_s^\perp$ in two regions.

Lemma 5 *Let $L = L_d(m_1, \ldots, m_s)$ be a class satisfying $L^2 = K_s \cdot L = 0$ such that d is greater or equal to the sum of the three largest multiplicities. Then $s \geq 9$, and up to reordering of the divisors E_i, L is a multiple of $L_3(1^9, 0^{s-9})$.*

Proof We will prove that every solution $(d, m_1, \ldots, m_s) \in \mathbb{R}^{s+1}$ to the following set of equations and inequalities:

- $d^2 = \sum_{i=1}^s m_i^2$
- $3d = \sum_{i=1}^s m_i$
- $m_1 \geq m_2 \geq \cdots \geq m_s \geq 0$
- $d \geq m_1 + m_2 + m_3$

is of the form $(d, m_1, \ldots, m_s) = (3m, m^9, 0^{s-9}) \in \mathbb{R}^{s+1}$. Since $K_s = -3H + \sum_{i=1}^{s} E_i$, the two equations are equivalent to $L^2 = 0$ and $K_s \cdot L = 0$ respectively.

All conditions and the statement are homogeneous in the parameters, so we may assume additionally that $d = 1$, and we shall prove that $(d, m_1, \ldots, m_s) = (1, (1/3)^9, 0^{s-9})$. We will need the following auxiliary statement:

Claim 6 Let $A = m_1 + m_2 + m_3$. For any $a \in [0, m_3]$,

$$m_1^2 + m_2^2 + m_3^2 \leq A^2 - 4aA + 6a^2.$$

Moreover equality holds if and only if $m_1 = m_2 = m_3 = a = A/3$.

Proof Define $s_2 = m_1m_2 + m_1m_3 + m_2m_3$, the second symmetric function. Expanding A^2 on the right hand side and cancelling the squares, the claimed inequality is equivalent to

$$2s_2 - 4Aa + 6a^2 \geq 0 \text{ for all } a \in [0, m_3]. \tag{1}$$

For fixed A, the quantity s_2 is minimized when the m's are all equal, i.e., $m_1 = m_2 = m_3 = A/3$; in that case $s_2 = A^2/3$. Since $m_3 \leq A/3$, (1) is implied by the statement that

$$2A^2/3 - 4Aa + 6a^2 \geq 0 \text{ for all a } \in [0, A/3]. \tag{2}$$

This is clear, since the left hand side factors as $(2/3)(A - 3a)^2$, and is always non-negative.

If equality holds, then s_2 must achieve its minimum, all the m's are equal to $A/3$, and the final inequality must also be an equality, forcing $a = A/3$ as well. □

Now we can complete the proof of Lemma 5. Since $m_4 \leq m_3$, the claim applies with $a = m_4$ and we conclude that

$$m_1^2 + m_2^2 + m_3^2 \leq A^2 - 4Am_4 + 6m_4^2 \tag{3}$$

with equality holding if and only if $m_1 = m_2 = m_3 = m_4 = A/3$.

By the second equation in the hypotheses, $\sum_{i=4}^{s} m_i = 3 - A$. Therefore since the m_i's descend, we have

$$\sum_{i=4}^{s} m_i^2 \leq m_4 \sum_{i=4}^{s} m_i = (3 - A)m_4,$$

so that using the first equation in the hypotheses gives

$$1 = \sum_{i=1}^{s} m_i^2 \leq A^2 - 4Am_4 + 6m_4^2 + (3 - A)m_4 = A^2 - (5A - 3)m_4 + 6m_4^2. \tag{4}$$

This gives a quadratic inequality for m_4 that implies that m_4 must lie outside the open interval

$$I = \left(\frac{5A - 3 - \sqrt{A^2 - 30A + 33}}{12}, \frac{5A - 3 + \sqrt{A^2 - 30A + 33}}{12} \right).$$

However m_4 must be at most $A/3$, and at least $\sum_4^s m_i/(s - 3) = (3 - A)/(s - 3)$, and therefore lies in the interval $[(3 - A)/(s - 3), A/3]$. Now suppose that $A \leq 1$. Then a simple calculation shows that the left endpoint of I is at most zero; since m_4 cannot be zero in this case, we must have that m_4 is greater than or equal to the right endpoint of I. However if $A \leq 1$, another calculation shows that this right endpoint is at least $A/3$. This forces $m_4 = A/3$, which forces $m_1 = m_2 = m_3 = m_4 = A/3$ as well. It also implies that the right endpoint of I is equal to $A/3$; this implies that $A = 1$. Therefore the first four multiplicities are equal to $1/3$. Now since $\sum_{i=4}^{s} m_i = 2$ and $\sum_{i=4}^{s} m_i^2 = 2/3$, the only way this works is to have 6 of the m_i's for $i \geq 4$ equal to $1/3$ and the rest equal to zero. □

Proposition 7 *Let $L = L_d(m_1, \ldots, m_s)$ be a class with $d, m_i \in \mathbb{Z}_{\geq 0}$ satisfying $L^2 = K_s \cdot L = 0$. The following are equivalent:*

- *L is nef.*
- *L is equivalent to a multiple of $L_3(1^9, 0^{s-9})$ under the action of the Cremona–Kantor group $\mathcal{CK}_s$.*
- *There is no (-1)-curve E with $E \cdot L < 0$.*

Proof The proof is algorithmic.

If $d = 0$, then L is nef if and only if $L = 0$, so we may assume that $d > 0$.

If d is not smaller than the sum of the three largest m_i, then the previous lemma shows that L is a permutation of a multiple of $L_3(1^9, 0^{s-9})$ (and in particular it is nef).

Alternatively, d is less than the sum of the three largest m_i. Then we may perform a quadratic Cremona map based at the three points of largest multiplicity and the resulting class L' is $\mathcal{CK}_s$-equivalent to L and has a smaller degree. As long as the degree and multiplicities stay non-negative and d is less than the sum of the three largest multiplicities, we can replace L by $\mathcal{CK}_s$-equivalent classes $L^{(k)} = L_{d^{(k)}}(m_1^{(k)}, \ldots, m_2^{(k)})$ with smaller degree. The process finishes in one of the following ways:

- One or more of the multiplicities $m_i^{(k)}$ is negative. Then $L^{(k)}$ is not nef, as it intersects the corresponding E_i negatively. The original L is therefore not nef (and the application to E_i of the same quadratic Cremona maps in the reverse order produces a (-1)-curve meeting L negatively).
- $d^{(k)} < 0$. This obviously is also non-nef, and the equality $3d^{(k)} = \sum m_i^{(k)}$ that follows from $L \cdot K_s = 0$ implies that one or more of the multiplicities $m_i^{(k)}$ is negative, so the previous description applies: there is a (-1) curve meeting L negatively.
- $d^{(k)}$ is not less than the sum of the three largest multiplicities $m_i^{(k)}$. Then by Lemma 5, $L^{(k)}$ is a permutation of a multiple of $L_3(1^9, 0^{s-9})$. In particular L is equivalent to a multiple of $L_3(1^9, 0^{s-9})$ under the action of the Cremona–Kantor group $\mathcal{CK}_s$, and it is nef. □

We now consider the non-nef cases, i.e., classes $L = L_d(m_1, \ldots, m_s)$ with $L^2 = K_s \cdot L = 0$ such that there is a (-1)-curve E with $L \cdot E < 0$.

Proposition 8 [De Fernex [8]] *Let $s = 10$. Every class $L = L_d(m_1, \ldots, m_s)$ with $d, m_i \geq 0$ satisfying $L^2 = K_s \cdot L = 0$ is nef.*

Moreover, there is a bijection ϕ between the set of rays spanned by such integral classes L and the set of (-1)-curves on X_{10}, such that $\phi(L)$ is the unique (-1)-curve E with $L \cdot E = 0$.

We give a proof based on the previous results which seems to us more elementary than De Fernex's, cf. [8, Corollary 4.3].

Proof By Remark 4, for every (-1)-curve E the hyperplane $E^{\perp}$ is tangent to $\mathcal{Q}_{10}^{\perp}$ along a single ray. Therefore, the whole $\mathcal{Q}_{10}^{\perp}$ is contained in the half-space E^{+} of classes L with $L \cdot E \geq 0$. Thus no class in $\mathcal{Q}_{10}^{\perp}$ meets any class of a (-1)-curve negatively and hence, by Proposition 7, all classes on $\mathcal{Q}_{10}^{\perp}$ are nef.

It remains to give the bijection ϕ. We have just seen that every integral class L in $\mathcal{Q}_{10}^{\perp}$ is nef and so, by Proposition 7, for every such L there is an element σ of the Cremona–Kantor group mapping L to a multiple of $L_3(1^9, 0)$. E_{10} is the only (-1)-curve orthogonal to $L_3(1^9, 0)$; indeed, every (-1)-curve E satisfies $-1 = E \cdot K_{10} = -E \cdot L_3(1^{10})$, so in order to be orthogonal to $L_3(1^9, 0)$ it must satisfy $E \cdot E_{10} = -1$, which forces $E = E_{10}$. Therefore $\sigma^{-1}(E_{10})$ is the only (-1)-curve orthogonal to L. □

Proposition 9 *Let $s > 10$ and let $\mathcal{W}_s$ be the pullback to X_s of all classes in $\mathcal{Q}_{10}^{\perp}$ via the blowing-up $X_s \to X_{10}$ of the last $s - 10$ points.*

For each (-1)-curve E on X_s, let $\mathcal{Q}_E$ be the subset of $\mathcal{Q}_s^{\perp}$ formed by all classes L satisfying the inequality $L \cdot E \le 0$. Then $\mathcal{Q}_s^{\perp}$ is covered by the subcones $\mathcal{Q}_E$ (indexed by all (-1)-curves E on X_s) and each $\mathcal{Q}_E$ satisfies:

(1) Every class in the interior of $\mathcal{Q}_E$ intersects E negatively and is non-nef.
(2) If $s = 11$, all classes on the boundary of $\mathcal{Q}_E$ are nef.
(3) If $s > 11$, the boundary of $\mathcal{Q}_E$, namely $\mathcal{Q}_s^{\perp} \cap E^{\perp} \cong \mathcal{Q}_{s-1}^{\perp}$, is covered by subcones of smaller dimension, on which nefness is determined by recursively applying this proposition with $s' = s - 1$.

The nef locus Nef^0 on $\mathcal{Q}_s^{\perp}$ is the topological closure of all $\mathcal{CK}_s$-translates of the ray $\langle L_3(1^9, 0^{s-9})\rangle$, and it coincides with the union of all $\mathcal{CK}_s$-translates of $\mathcal{W}_s$, which are 9-dimensional quadratic cones. At each rational ray belonging to Nef^0, exactly $s-9$ translates of $\mathcal{W}_s$ meet.

Recall that $\mathcal{Q}_s^{\perp}$ is a $(s - 1)$-dimensional quadratic cone, and each $\mathcal{Q}_E$ is the cone over a ball whose boundary is a $(s - 2)$-dimensional quadratic cone.

Proof It is clear that every class in the interior of $\mathcal{Q}_E$ intersects E negatively, and by Remark 4, the boundary is isomorphic to the pullback of $\mathcal{Q}_{s-1}^{\perp}$.

The fact that such cones cover $\mathcal{Q}_s^{\perp}$ follows from Proposition 7: every ray on $\mathcal{Q}_s^{\perp}$ is a limit of rational rays, and for every integral $L \in \mathcal{Q}_s^{\perp}$ either there is a (-1)-curve E such that L belongs to the interior of $\mathcal{Q}_E$ or it belongs to the orbit of a multiple of $3H - \sum_{i=1}^{9} E_i$, in which case there are (-1)-curves E to which it is orthogonal, and so it belongs to the boundary of $\mathcal{Q}_E$.

Finally, the ray spanned by $L_3(1^9, 0^{s-9})$ is orthogonal to exactly $s - 9$ (-1)-curves, namely $E_{10}, \dots, E_s$, and so it belongs to $s - 9$ translates of $\mathcal{W}_s$, which we call $\mathcal{W}_s^{(1)}$, $\mathcal{W}_s^{(2)}$, …, $\mathcal{W}_s^{(s-9)}$. Since every $L \in \mathrm{Nef}^0$ is the translate of a multiple of $L_3(1^9, 0^{s-9})$ by some $\sigma \in \mathcal{CK}_s$, it belongs to $s-9$ distinct translates of $\mathcal{W}_s$, namely $\sigma^{-1}(\mathcal{W}_s^{(1)}), \dots, \sigma^{-1}(\mathcal{W}_s^{(s-9)})$. □

Proof of Theorem 1 If $s = 10$, the claims are equivalent to Proposition 8 and also follow from Proposition 7. If $s > 10$, we use Proposition 9 and induction on s. Indeed, if L belongs to the interior of some cone $\mathcal{Q}_E$ then it clearly does not satisfy conditions (1) and (3) in the statement of the Theorem. It does not satisfy (2) either: each collection of $s - 10$ disjoint (-1)-curves $C_1, \dots, C_{s-10}$ can be blown down $X_s \to X'$ to a surface X' isomorphic to X_{10} (because it is a generic rational surface with Picard number 10, see also [8, Corollary 2.5]) and L being orthogonal to them would imply that L is the pullback to X_s of a well-defined class in $\mathcal{Q}_{10}^{\perp}$, therefore nef by Proposition 8. Thus we may assume that L belongs to the boundary of $\mathcal{Q}_E$ for some E, in which case we may blow down E; the result is isomorphic to X_{s-1}, and there is a class L_{s-1} on X_{s-1} whose pullback to X_s is L. It is easy to see that $L_{s-1}^2 = L_{s-1} \cdot K_{s-1} = 0$ and the equivalence of (1)-(3) follows by induction.

Finally, since each collection of $s-10$ disjoint (-1)-curves $C_1, \ldots, C_{s-10}$ can be blown down to $X' \cong X_{10}$, and the classes L with $C_i \cdot L = 0$ for $i = 1, \ldots, s-10$ cover the image M of the pullback map $N(X') \to N(X_s)$, which is a linear isomorphism onto its image, it follows that the subset cut out on M by the additional equations $L^2 = K \cdot L = 0$ is isomorphic to $\mathcal{Q}_{10}^{\perp}$, namely a 9-dimensional rational quadratic cone. □

2 Collision of r^2 points

Our method to construct round parts of the boundary of the Mori cone which are not orthogonal to the anticanonical divisor relies on a degeneration of X_s where r^2 of the s blown-up points, of equal multiplicity m in certain divisor class

$$L = L_d(m^{r^2}, m_{r^2+1}, \ldots, m_s),$$

come together. This kind of *collision* was explained in detail in [6] using the technique of [5], and we give a brief sketch of what we need here.

We consider a trivial family $\mathcal{X} = X_{s-r^2} \times \Delta$ over a disc Δ, and blow up a general point in the central fiber over $0 \in \Delta$ to obtain the threefold $\mathcal{X}'$. This produces a degeneration of X_{s-r^2} to a union of two surfaces, a plane (the exceptional divisor for the blowup) and the proper transform F of the original X_{s-r^2} fiber, which is now isomorphic to X_{s-r^2+1}. These two surfaces intersect transversely along a smooth rational curve R which is a line in the plane and a (-1)-curve in F.

We now choose r^2 general points on the plane; extend these r^2 general points to the general fiber using r^2 sections of the projection of $\mathcal{X}'$ to Δ, and blow up those r^2 sections to ruled surfaces $\mathcal{E}_1, \ldots, \mathcal{E}_{r^2}$. This then produces a threefold $\mathcal{Y}$ which is a degeneration of X_s, to a union of a surface $P \cong X_{r^2}$ and $F \cong X_{s-r^2+1}$, intersecting transversely along the double curve R. This smooth rational curve R is the pullback of a general line in the surface P and remains a (-1)-curve in the surface F.

We have the line bundle $\mathcal{O}_{X_{s-r^2}}(L')$ corresponding to $L' = L_d(m_{r^2+1}, \ldots, m_s)$ on X_{s-r^2}, and can extend it trivially to $\mathcal{X}$. If we pull that back to the first blowup $\mathcal{X}'$, we see that this restricts to the bundle corresponding to $L_d(m_{r^2+1}, \ldots, m_s, 0)$ on the surface $F \cong X_{s-r^2+1}$, and to the trivial bundle on the plane. We then pull that back to the second blowup $\mathcal{Y}$, and tensor by $\mathcal{O}_{\mathcal{Y}}(-tP - m\sum_{i=1}^{r^2} \mathcal{E}_i)$, with t a non-negative integer (called the *twisting parameter*). This produces a line bundle $\mathcal{M}$ on $\mathcal{Y}$, which restricts to the general fiber as the original bundle $\mathcal{O}_{X_s}(L)$. The principle of semicontinuity guarantees that the dimension of the general linear system $\mathcal{L}_d(m^{r^2}, m_{r^2+1}, \ldots, m_s)$ is at most equal to the dimension of the linear system on the reducible surface $P \cup F$, and in particular L is not effective as soon as $\mathcal{M}|_{P\cup F}$ is not effective, i.e., it has no nonzero sections, for some choice of twisting parameter t.

Fix $t = mr$ as twisting parameter. The restrictions of $\mathcal{M}$ to $P \cong X_{r^2}$ and to $F \cong X_{s-r^2+1}$ are

$$\begin{aligned}
\mathcal{M}|_P &= \mathcal{O}_{X_{r^2}}(L_{mr}(m^{r^2})),\\
\mathcal{M}|_F &= \mathcal{O}_{X_{s-r^2+1}}(L_d(rm, m_{r^2+1}, \ldots, m_s)),
\end{aligned}$$

respectively. The space of global sections of $\mathcal{M}|_{P\cup F}$ is the fiber product of the space of sections on P with the space of sections on F, fibered over the restriction to the space of sections on R. Therefore to prove that L is not effective it suffices to prove that this fibre product is zero. As an application, we have:

Lemma 10 *Fix $r \geq 2$ and multiplicities $m, m_{r^2+1}, \ldots, m_s$. If either*

(a) $r = 2$ and $h^0(\mathcal{O}_{X_{s-3}}(L_d(2m, m_5, \ldots, m_s)) \leq m$, or
(b) $r \geq 3$ and $h^0(\mathcal{O}_{X_{s-r^2+1}}(L_d(rm, m_{r^2+1}, \ldots, m_s))) \leq 1$,

then $L_d(m^{r^2}, m_{r^2+1}, \ldots, m_s)$ is non-effective.

Proof Cases $r = 2, 3$ are parts (a) and (b) from [6, Lemma 2]. For $r \geq 4$, recall that, as explained in [6, end of Sect. 1], Nagata's theorem guarantees that if $r \geq 4$ then all classes of the form $L_{rm}(m^{r^2})$ are non-effective; thus the bundle on P has no nonzero sections, and therefore the fibre product corresponds to the subsystem $\mathcal{L}_d(rm+1, m_{r^2+1}, \ldots, m_s)$ on F. So we have to prove that this is empty. Now, by [3, Proposition 2.3], this is a proper subsystem of $L_d(rm, m_{r^2+1}, \ldots, m_s)$ and therefore it is empty (indeed, if $\mathcal{L}_d(rm, m_{r^2+1}, \ldots, m_s)$ is non-empty, its general member has the first point of multiplicity exactly rm). □

We note that the divisors before and after the collision have the same self-intersection. In particular, if one is zero, so is the other; this will be important in our application.

Given a divisor class $L \in \mathrm{Pic}(X_{s+1})$, and an index i denoting one of the multiplicities, in [6] we defined the *uncollision* $\mathrm{Uncoll}_r(L, i)$ as the class $\mathrm{Uncoll}_r(L, i) \in \mathrm{Pic}(X_{s+r^2})$ obtained replacing the i-th multiplicity m_i by r^2 points of multiplicity m_i/r. This makes sense at the level of divisor classes if m_i is divisible by r, but it also makes sense as a map $N(X_{s+1}) \to N(X_{s+r^2})$. Observe as well that the process of considering an uncollision behaves linearly with respect to multiplicities and degrees, and the corresponding linear map is injective. This will be key in our application, and additionally it means that uncollision makes sense applied to rational rays in $N(X_{s+1})$.

3 Quadratic sections of $\partial\overline{NE}$ in K_s^+

By the Cone Theorem, the shape of the Mori cone on the half-space K_s^- of classes which intersect the canonical divisor negatively is governed by the rays generated by (-1)-curves. On the orthogonal hyperplane $K_s^\perp$ this is no longer quite the case, but we saw in the previous section that nefness is still characterized by intersection with (-1)-rays, and there are no good rays on $K_s^\perp$.

In this section we show how to exploit uncollisions to build 9-dimensional quadric cones in the boundary of the Mori cone consisting entirely of good and wonderful rays. Since the collision/uncollision analysis and construction is only available for divisor classes of rational rays but not for irrational rays, we work on rational rays to obtain good rays and then use the closed convex nature of the nef cone to obtain families of good and wonderful rays.

Remark 11 If D is an $\mathbb{R}$-divisor class with $D \cdot K_s = 0$ and D' is obtained from D by uncolliding a point of multiplicity $rm > 0$ to $r^2 \geq 4$ points of multiplicity m, then $D' \cdot K_s > 0$. Indeed, writing $D = dL - \sum m_i E_i$ we have $D \cdot K_s = \sum m_i - 3d$ and

$$D' \cdot K_s = \sum m_i - rm + r^2 m - 3d = D \cdot K_s + (r^2 - r)m > D \cdot K_s.$$

Proposition 12 *Let L be a nef class in $\mathcal{Q}_s^\perp$. For every $i = 1, \ldots, s$ such that $m_i \neq 0$ and every $r \geq 2$ the ray $\langle \mathrm{Uncoll}_r(L, i) \rangle$ is good.*

Proof By Proposition 9, L is $\mathcal{CK}_s$-equivalent to $L_{3a}(a^9, 0^{s-9})$ for some integer $a > 0$, and therefore its space of global sections has dimension 1. Therefore by Lemma 10 $\mathrm{Uncoll}_r(L, i)$

is not effective, and the same holds for its multiples, which are uncollisions of multiples of L, which themselves have 1-dimensional global sections. □

Proof of Theorem 2 Let $s' = s - 3 \geq 10$. By Theorem 1, each collection of $s' - 10$ disjoint (-1)-curves $C_1, \dots, C_{s'-10}$ determines a 10-dimensional linear subspace $\Pi' \subset N_{s'}$ (of those classes orthogonal to $K_{s'}$ and to the C_i) such that each $D \in \Pi'$ with $D^2 = 0$ is nef, and the rays spanned by such D cover a 9-dimensional rational quadratic cone. By Proposition 12, for each $i = 1, \dots, s'$, every such class D gives rise by uncollision to a good ray $\langle \mathrm{Uncoll}_2(D, i) \rangle$ in N_s of self-intersection zero. Consider $\mathrm{Uncoll}_2(\cdot, i)$ as a linear map $\Pi' \to N_s$ and let Π be its image. By linearity and injectivity, the good rays obtained as images of the nef rays $\langle D \rangle$ with $D^2 = 0$ cover the cone $\{L \in \Pi \mid L^2 = 0, H \cdot L \geq 0\}$, and therefore $\Pi \cap \overline{NE}(X_s) = \{L \in \Pi \mid L^2 \geq 0, H \cdot L \geq 0\}$.

By Remark 11, all rational classes L on the cone $\mathcal{C} = \{L \in \Pi \mid L^2 = 0\}$ satisfy $L \cdot K_s > 0$. Now $\mathcal{C}^\perp := \mathcal{C} \cap K_s^\perp$ is the intersection of a rational quadratic cone with a rational hyperplane, thus either it is a rational quadratic cone, or it consists only of the single point at the origin. However, we know that $\mathcal{C}^\perp$ contains no rational ray, so it must be reduced to a point, and we conclude that $L \cdot K_s > 0$ for every nonzero $L \in \mathcal{C}$. □

The 9-dimensional quadratic cones of good and wonderful rays in $K_s^\perp$ we just constructed do not in general consist of De Fernex negative classes. However, the wonderful De Fernex negative rays constructed in [6] are uncollisions of nef rays in $K_s^\perp$, so they do belong to some of these 9-dimensional cones, which is the basis for the proof of Theorem 3.

Proof of Theorem 3 Proposition 17 in [6] exhibits De Fernex negative wonderful rays $\langle D \rangle$ on X_{k^2+4} for every $k \geq 3$ by uncolliding classes L on $\mathcal{Q}_{2k+4}^\perp$ (use $n = k - 2$ in [6, Proposition 17]). By Proposition 9, for every such L there is a rational 9-dimensional quadratic cone $\mathcal{C}$ of nef classes in $\mathcal{Q}_{2k+4}^\perp$ containg L. Therefore, by Proposition 12, the uncollision $\mathrm{Uncoll}_{k-1}(L, 1)$ spans a good ray for every L in $\mathcal{C}$; these rays cover the 9-dimensional cone over a 8-dimensional sphere and at least one such ray is De Fernex negative. Since being De Fernex negative is an open condition, the claim follows. □

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