

Variations on Nagata's Conjecture

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ABSTRACT. Here we discuss some variations of Nagata's conjecture on linear systems of plane curves. The most relevant concerns non-effectivity (hence nefness) of certain rays, which we call *good rays*, in the Mori cone of the blow-up X_n of the plane at $n \geq 10$ general points. Nagata's original result was the existence of a good ray for X_n with $n \geq 16$ a square number. Using degenerations, we give examples of good rays for X_n for all $n \geq 10$. As with Nagata's original result, this implies the existence of counterexamples to Hilbert's XIV problem. Finally we show that Nagata's conjecture for $n \leq 89$ combined with a stronger conjecture for $n = 10$ implies Nagata's conjecture for $n \geq 90$.

CONTENTS

Introduction

A fundamental problem in algebraic geometry is understanding which divisor classes on a given variety have effective representatives. One of the simplest contexts for this problem is that of curves in the plane, and here already it is of substantial interest, and not only in algebraic geometry. For example, given n sufficiently general points $x_1, \dots, x_n$ in the complex plane $\mathbb{C}^2$, nonnegative integers $m_1, \dots, m_n$ and an integer d , when is there a polynomial $f \in \mathbb{C}[x, y]$ of degree d vanishing to order at least m_i at each point x_i ? Although there is a conjectural answer to this question (the (SHGH) Conjecture; see Conjecture ?? and also [?]), the conjecture remains open after more than a half century of attention by many researchers.

This problem is closely related to the question of what self-intersections occur for reduced irreducible curves on the surface X_n obtained by blowing up the projective plane at the n points x_i . Blowing up the points introduces rational curves (infinitely many, in fact, when $n > 8$) of self-intersection -1 . Each curve C on X_n corresponds to a projective plane curve D_C of some degree d vanishing to orders

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m_i at the points x_i ; the self-intersection C^2 is $d^2 - m_1^2 - \cdots - m_n^2$. An example of a curve D_C corresponding to a curve C of self-intersection -1 on X_n is the line through two of the points x_i , say x_1 and x_2 ; in this case, $d = 1$, $m_1 = m_2 = 1$ and $m_i = 0$ for $i > 2$, so we have $d^2 - m_1^2 - \cdots - m_n^2 = -1$. According to the (SHGH) Conjecture, these (-1) -curves should be the only reduced irreducible curves of negative self-intersection (see Conjecture ??) but proving that there are no others turns out to be itself very hard and is still open.

One could hope that a weaker version of this problem might satisfy the criterion Hilbert stated in his address to the International Congress in Paris in 1900, of being difficult enough “to entice us, yet not completely inaccessible so as not to mock our efforts” (“uns reizt, und dennoch nicht völlig unzugänglich, damit es unserer Anstrengung nicht spottet”). In fact, Nagata, in connection with his negative solution of the 14-th of the problems Hilbert posed in his address, made such a conjecture, Conjecture ??. It is weaker than Conjecture ?? yet still open for every non-square $n \geq 10$. Nagata’s conjecture does not rule out the occurrence of curves of self-intersection less than -1 , but it does rule out the worst of them. In particular, Nagata’s conjecture asserts that if there is a curve of degree d with $n \geq 10$ sufficiently general points of multiplicities $m_1, \dots, m_n$, then $d^2 \geq nm^2$ must hold, where $m = (m_1 + \cdots + m_n)/n$. Thus perhaps there are curves with $d^2 - m_1^2 - \cdots - m_n^2 < 0$, such as the (-1) -curves mentioned above, but $d^2 - m_1^2 - \cdots - m_n^2$ is (conjecturally) only as negative as is allowed by the condition that after averaging the multiplicities m_i for $n \geq 10$ one must have $d^2 - nm^2 \geq 0$.

What our results here show is that in order to prove Nagata’s Conjecture for all $n \geq 10$ it is enough to prove it only for $n < 90$, if one can verify a slightly stronger conjecture for $n = 10$. But what we hope is to persuade the reader that it satisfies Hilbert’s criteria of being both enticing and challenging, and at least not *completely* inaccessible!

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1. Linear systems on general blow-ups of the plane

1.1. Generalities. Fix n points $x_1, \dots, x_n$ in the complex projective plane $\mathbb{P}^2$ (which will be often assumed to be in *very general* position and called *general*) and nonnegative integers $d, m_1, \dots, m_n$. We denote by $\mathcal{L}(d; m_1, \dots, m_n)$, or simply by $(d; m_1, \dots, m_n)$, the linear system of plane curves of degree d having *multiplicity* at least m_i at the *base point* x_i , for $1 \leq i \leq n$. Often we will use exponents to denote repetition of multiplicities. Sometimes, we may simply denote $\mathcal{L}(d; m_1, \dots, m_n)$ by $\mathcal{L}$.

The linear system $(d; m_1, \dots, m_n)$ is the projective space corresponding to a vector subspace $\mathfrak{a}_d \subset H^0(\mathcal{O}_{\mathbb{P}^2}(d))$, and $\mathfrak{a} = \bigoplus_{d=0}^n \mathfrak{a}_d$ is the homogeneous ideal, in the coordinate ring $S = \bigoplus_{d=0}^n H^0(\mathcal{O}_{\mathbb{P}^2}(d))$ of $\mathbb{P}^2$, of the scheme $\text{Proj}(S/\mathfrak{a})$ denoted by $\sum_{i=1}^n m_i x_i$, usually called a *fat points scheme*.

The *expected dimension* of $(d; m_1, \dots, m_n)$ is defined to be

$$e(d; m_1, \dots, m_n) = \max \{-1, v(d; m_1, \dots, m_n)\}$$

where

$$v(d; m_1, \dots, m_n) = \frac{d(d+3)}{2} - \sum_{i=1}^n \frac{m_i(m_i+1)}{2}$$

is the *virtual dimension* of the system. The system is said to be *special* if

$$h(d; m_1, \dots, m_n) > e(d; m_1, \dots, m_n)$$

where $h(d; m_1, \dots, m_n) := \dim(\mathcal{L}(d; m_1, \dots, m_n))$ is its *true dimension*. In particular, an empty linear system is never special.

We record the following definitions:

- $(d; m_1, \dots, m_n)$ is *asymptotically non-special* (ANS) if there is an integer y such that for all nonnegative integers $x \geq y$ the system $(xd; xm_1, \dots, xm_n)$ is non-special;
- the *multiplicities vector* $(m_1, \dots, m_n)$ of nonnegative integers is *stably non-special* (SNS) if the linear system $(d; xm_1, \dots, xm_n)$ is non-special for all positive integers d, x .

Consider the *Cremona–Kantor (CK) group* $\mathcal{G}_n$ generated by quadratic transformations based at n general points $x_1, \dots, x_n$ of the plane and by permutations of these points (see [?]). The group $\mathcal{G}_n$ acts on the set of linear systems of the type $(d; m_1, \dots, m_n)$. All systems in the same *(CK)-orbit* (or *(CK)-equivalent*) have the same expected, virtual and true dimension. A linear system $(d; m_1, \dots, m_n)$ is *Cremona reduced* if it has minimal degree in its (CK)-orbit. We note that (CK)-orbits need not contain a Cremona reduced element if they are orbits of empty linear systems, but orbits of non-empty linear systems always contain Cremona reduced members. It is a classical result, which goes back to Max Noether (see, e.g., [?]), that a non-empty system $(d; m_1, \dots, m_n)$ with general base points is Cremona reduced if and only if the sum of any pair or triple of distinct multiplicities does not exceed d . In this case the system is called *standard* and we may assume $m_1 \geq \dots \geq m_n$.

1.2. General rational surfaces. Consider the blow-up $f : X_n \rightarrow \mathbb{P}^2$ of the plane at $x_1, \dots, x_n$, which we call a *general rational surface*. The *Picard group* $\text{Pic}(X_n)$ is the abelian group freely generated by:

- the *line class*, i.e., total transform $L = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$;
- the classes of the *exceptional divisors* $E_1, \dots, E_n$ which are contracted to $x_1, \dots, x_n$.

More generally we may work in the $\mathbb{R}$ -vector space $N_1(X_n) = \text{Pic}(X_n) \otimes_{\mathbb{Z}} \mathbb{R}$.

We will often abuse notation, identifying divisors on X_n with the corresponding line bundles and their classes in $\text{Pic}(X_n)$, thus passing from additive to multiplicative notation. We will use the same notation for a planar linear system $\mathcal{L} = (d; m_1, \dots, m_n)$ and its *proper transform*

$$\mathcal{L} = dL - \sum_{i=1}^n m_i E_i$$

on X_n . With this convention the integers $d, m_1, \dots, m_n$ are the components with respect to the ordered basis $(L, -E_1, \dots, -E_n)$ of $N_1(X_n)$. The *canonical divisor* on X_n is $K_n = (-3; -1^n)$ (denoted by K if there is no danger of confusion) and, if $(d; m_1, \dots, m_n)$ is an ample line bundle on X_n , then $(d; m_1, \dots, m_n)$ is ANS.

Using the intersection form on $N_1(X_n)$, one can *intersect* and *self-intersect* linear systems $(d; m_1, \dots, m_n)$. Given a linear system $\mathcal{L} = (d; m_1, \dots, m_n)$, one has

$$v(d; m_1, \dots, m_n) = \frac{\mathcal{L}^2 - \mathcal{L} \cdot K}{2}$$

and, if $d \geq 0$, Riemann-Roch's theorem says that

$$(1.1) \quad \mathcal{L} \text{ is special if and only if } h^0(\mathcal{L}) \cdot h^1(\mathcal{L}) > 0.$$

1.3. Special effects. Though (??) says that speciality is a cohomological property, the only known reason for speciality comes from geometry in the following way.

Assume we have an *effective* linear system $\mathcal{L}$, i.e. $h^0(\mathcal{L}) > 0$, and suppose there is an irreducible curve C of arithmetic genus g on X_n such that:

- $h^2(\mathcal{L}(-C)) = 0$, e.g. $h^0(\mathcal{L}(-C)) > 0$;
- $h^1(\mathcal{L}|_C) > 0$, e.g. $\mathcal{L} \cdot C \leq g - 1$ if $g \geq 2$ and $\mathcal{L} \cdot C \leq g - 2$ if $g \leq 1$.

Then the *restriction exact sequence*

$$0 \rightarrow \mathcal{L}(-C) \rightarrow \mathcal{L} \rightarrow \mathcal{L}|_C \rightarrow 0$$

implies that

$$h^1(\mathcal{L}) \geq h^1(\mathcal{L}|_C) > 0$$

hence $\mathcal{L}$ is special. In this case C is called a *special effect curve* for $\mathcal{L}$ (see [?]). For example, C is a special effect curve for $\mathcal{L}$ if $g = 0$ and $\mathcal{L} \cdot C \leq -2$, in which case C sits in the base locus of $\mathcal{L}$. But then $C^2 < 0$ and therefore, if $x_1, \dots, x_n$ are general points, one has $C^2 = -1$, i.e. C is a (-1) -curve (see [?]). In this case $\mathcal{L}$ is said to be (-1) -special and contains $-(\mathcal{L} \cdot C)C$ in its fixed part. Thus, any (-1) -curve C gives rise to infinitely many (-1) -special linear systems, e.g. all systems of the type $|nC|$, with n a positive integer. Note that for $1 \leq n \leq 8$ there are only finitely many (-1) -curves on X_n , whereas if $n \geq 9$ there are infinitely many (-1) -curves on X_n .

1.4. The Segre–Harbourne–Gimigliano–Hirschowitz Conjecture. The only known examples of special linear systems on a general rational surface X_n are (-1) -special. This motivates the conjecture (see [?, ?, ?, ?, ?], quoted in chronological order):

CONJECTURE 1.1 (Segre–Harbourne–Gimigliano–Hirschowitz (SHGH)). *A linear system $\mathcal{L}$ on X_n is special if and only if it is (-1) -special.*

It goes back to Castelnuovo that Conjecture ?? holds if $n \leq 9$ (see [?]; more recent treatments can be found in [?, ?, ?, ?]). The general conjecture remains open.

Since standard linear systems are not (-1) -special (see [?, ?]), an equivalent formulation of the (SHGH) conjecture is: *a standard system of plane curves with general base points is not special.*

Recall that a linear system $\mathcal{L}$ is *nef* if $\mathcal{L} \cdot \mathcal{L}' \geq 0$ for all effective $\mathcal{L}'$. Since the (CK)-orbit of a nef divisor always contains a Cremona reduced element and hence a standard element (see [?]), the (SHGH) conjecture implies the following conjecture, which we regard as a weak form of (SHGH), a point-of-view justified by Proposition ??(ii) below.

CONJECTURE 1.2. *A nef linear system $\mathcal{L}$ on X_n is not special.*

This conjecture is also open in general. The notion of nefness extends to elements in $N_1(X_n)$ and $\xi \in N_1(X_n)$ is nef if and only if $\lambda\xi$ is nef for all $\lambda > 0$. Given a nonzero $\xi \in N_1(X_n)$, the set $[\xi] = \{\lambda\xi : \lambda > 0\}$ is called the *ray* generated by ξ . Thus it makes sense to talk of *nef rays*.

2. Hilbert's 14-th problem and Nagata's conjecture

2.1. Hilbert's 14-th problem. Let k be a field, let $t_1, \dots, t_n$ be indeterminates over k and let $\mathbb{K}$ be an *intermediate field* between k and $k(t_1, \dots, t_n)$, i.e.

$$k \subseteq \mathbb{K} \subseteq k(t_1, \dots, t_n).$$

Hilbert's 14-th problem asks: *is $\mathbb{K} \cap k[t_1, \dots, t_n]$ a finitely generated k -algebra?* Hilbert had in mind the following situation coming from invariant theory. Let G be a subgroup of the *affine group* or of $\mathrm{GL}(n, k)$. Then G acts as a set of automorphisms of the k -algebra $k[t_1, \dots, t_n]$, hence on $k(t_1, \dots, t_n)$, and we let $\mathbb{K} = k(t_1, \dots, t_n)^G$ be the field of G -invariant elements. Then the question is: *is*

$$k[t_1, \dots, t_n]^G = \mathbb{K} \cap k[t_1, \dots, t_n]$$

a finitely generated k -algebra?

In [?], Nagata provided counterexamples to the latter formulation of Hilbert's problem. To do this he used the nefness of a certain line bundle of the form $(d; m_1, \dots, m_n)$ (see §?? below).

Hilbert's problem has trivially an affirmative answer in the case $n = 1$. The answer is also affirmative for $n = 2$, as proved by Zariski in [?]. Nagata's *minimal counterexample* has $n = 32$ and $\dim(G) = 13$. Several other counterexamples have been given by various authors, too long a story to be reported on here (see, e.g., [?, ?, ?]).

2.2. Nagata's Conjecture. In his work on Hilbert's 14-th problem, Nagata made the following conjecture:

CONJECTURE 2.1 (Nagata's Conjecture (N)). *If $n \geq 9$ and $(d; m_1, \dots, m_n)$ is an effective linear system on X_n , then*

$$(2.1) \quad \sqrt{n} \cdot d \geq m_1 + \dots + m_n$$

and strict inequality holds if $n \geq 10$.

Using a degeneration argument, Nagata proved the following result, on which his counterexamples to Hilbert's 14-th problem rely:

PROPOSITION 2.2. *(N) holds if $n = k^2$, with $k \geq 3$.*

Taking this into account, it is clear that (N) is equivalent to saying that the Nagata class $N_n = (\sqrt{n}, 1^n)$, or the Nagata ray $\nu_n = [\sqrt{n}, 1^n]$ it generates, is nef if $n \geq 9$. Note that it suffices to verify (N) for linear systems containing prime divisors.

Let C be an irreducible curve of genus g on X_n . If (SHGH) holds, then the virtual dimension of $\mathcal{O}_{X_n}(C)$ is nonnegative, which reads

$$(2.2) \quad C^2 \geq g - 1.$$

In particular, (SHGH) implies the following conjecture:

CONJECTURE 2.3. *If C is a prime divisor on X_n , then $C^2 \geq -1$ with $g = 0$ when $C^2 = -1$.*

LEMMA 2.4. *Conjecture ?? implies (N).*

PROOF. Suppose C is a prime divisor in $(d; m_1, \dots, m_n)$ violating (N), i.e., $n \geq 9$ and $\sqrt{n} \cdot d < m_1 + \dots + m_n$. Then, for $m = (m_1 + \dots + m_n)/n$ and using Cauchy–Schwartz inequality $m^2 \leq (m_1^2 + \dots + m_n^2)/n$, we have

$$d^2 < n \frac{(m_1 + \dots + m_n)^2}{n^2} = nm^2 \leq m_1^2 + \dots + m_n^2.$$

Thus $C^2 < 0$ and therefore $C^2 = -1$ and C has genus 0 by Conjecture ???. But now we have the contradiction $1 = -C \cdot K_n = 3d - (m_1 + \dots + m_n) \leq \sqrt{n}d - (m_1 + \dots + m_n) < 0$. $\square$

Hence Conjecture ??? can be regarded as a strong form of (N). The aforementioned result in [?] (that $C^2 \geq -1$ if $C \subset X_n$ is irreducible and rational) yields (??) if $g = 0$. If $g = 1$ and $C^2 = 0$, then C is (CK)-equivalent to $(3; 1^9, 0^{n-9})$ (see [?]). Thus the following conjecture (see [?, Conjecture 3.6]) is at least plausible.

CONJECTURE 2.5 (Strong Nagata’s Conjecture (SN)). *If C is an irreducible curve of genus $g > 0$ on X_n , then $C^2 > 0$ unless $n \geq 9$, $g = 1$ and C is (CK)-equivalent to $(3; 1^9, 0^{n-9})$, in which case $C^2 = 0$.*

PROPOSITION 2.6. *We have the following:*

- (i) *(SHGH) implies (SN) (in particular (SN) holds for $n \leq 9$);*
- (ii) *(SHGH) holds if and only if (SN) and Conjecture ??? both hold;*
- (iii) *(SN) implies (N).*

PROOF. Part (i), hence also the forward implication of (ii), is clear. As for the reverse implication in (ii), note that (SN) implies Conjecture ???, and Conjecture ??? together with Conjecture ??? is the formulation of (SHGH) given in [?]. Finally we prove part (iii), hence we assume $n \geq 9$. Let C be an irreducible curve in $(d; m_1, \dots, m_n)$ on X_n . If $C^2 \geq 0$ then $d^2 \geq m_1^2 + \dots + m_n^2$. By the Cauchy–Schwartz inequality this implies (??). Equality holds if and only if $m_1 = \dots = m_n = m$ and $d = m\sqrt{n}$, hence n is a square, which is only possible if $n = 9$ by Proposition ???. If $C^2 < 0$, then $g = 0$ and $C^2 = -1$, so that

$$C \cdot N_n = C \cdot (N_n - K_n) - C \cdot K_n = (\sqrt{n} - 3)C \cdot L + 1 \geq 1.$$

$\square$

3. The Mori cone viewpoint

3.1. Generalities. A class $\xi \in N_1(X_n)$ is *integral* [resp. *rational*] if it sits in $\text{Pic}(X_n)$ [resp. in $\text{Pic}(X_n) \otimes_{\mathbb{Z}} \mathbb{Q}$]. A ray in $N_1(X_n)$ is *rational* if it is generated by a rational class. A rational ray in $N_1(X_n)$ is *effective* if it is generated by an effective class.

The *Mori cone* $\overline{\text{NE}}(X_n)$ is the closure in $N_1(X_n)$ of the set $\text{NE}(X_n)$ of all effective rays, and it is the dual of the *nef cone* $\text{Nef}(X_n)$ which is the closed cone described by all nef rays.

A *(-1)-ray* in $N_1(X_n)$ is a ray generated by a *(-1)-curve*, i.e., a smooth, irreducible, rational curve C with $C^2 = -1$ (hence $C \cdot K_n = -1$).

Mori’s Cone Theorem says that

$$\overline{\text{NE}}(X_n) = \overline{\text{NE}}(X_n)^{\succ} + R_n$$

where $\overline{\text{NE}}(X_n)^\succ$ [resp. $\overline{\text{NE}}(X_n)^\prec$] is the subset of $\overline{\text{NE}}(X_n)$ described by rays generated by nonzero classes ξ such that $\xi \cdot K_n \geq 0$ [resp. $\xi \cdot K_n \leq 0$] and

$$R_n = \sum_{\rho \text{ a } (-1)\text{-ray}} \rho \subseteq \overline{\text{NE}}(X_n)^\prec.$$

We will denote by $\overline{\text{NE}}(X_n)^\succ$ [resp. $\overline{\text{NE}}(X_n)^\prec$] the interior of $\overline{\text{NE}}(X_n)^\succ$ [resp. $\overline{\text{NE}}(X_n)^\prec$].

Concerning (SHGH), the situation is well understood for classes in R_n , in view of this result (see [?]):

THEOREM 3.1. *A nef linear system on X_n with class in R_n is non-special.*

The *nonnegative cone* $\mathcal{Q}_n$ in $N_1(X_n)$ is the cone of classes ξ such that $\xi \cdot L \geq 0$ and $\xi^2 \geq 0$, whose boundary, which is a quadric cone, we denote by $\partial\mathcal{Q}_n$. By Riemann-Roch's theorem one has $\mathcal{Q}_n \subseteq \overline{\text{NE}}(X_n)$. We will use the obvious notation $\mathcal{Q}_n^\succ, \mathcal{Q}_n^\prec, \mathcal{Q}_n^\succ, \mathcal{Q}_n^\prec$ to denote the intersection of $\mathcal{Q}_n$ with $\overline{\text{NE}}(X_n)^\succ$ etc., and similarly for $\partial\mathcal{Q}_n$.

The situation is quite different according to the values of n :

- (i) The *Del Pezzo case* $n \leq 8$. Here $-K_n$ is ample, hence $\overline{\text{NE}}(X_n) = R_n$. There are only finitely many (-1) -curves on X_n , hence $\overline{\text{NE}}(X_n)$ is polyhedral and $\overline{\text{NE}}(X_n) \subseteq \overline{\text{NE}}(X_n)^\prec$. If $\kappa_n = [3, 1^n]$ is the *anticanonical ray*, then κ_n is in the interior of $\mathcal{Q}_n$.
- (ii) The *quasi Del Pezzo case* $n = 9$. Here $-K_9$ is an irreducible curve with self-intersection 0. Hence κ_9 is nef, sits on $\partial\mathcal{Q}_9$, and the tangent hyperplane to $\partial\mathcal{Q}_9$ at κ_9 is the hyperplane $\kappa_9^\perp$ of classes ξ such that $\xi \cdot K_9 = 0$. Then $\overline{\text{NE}}(X_9)^\succ = \kappa_9$ and $\overline{\text{NE}}(X_9) = \kappa_9 + R_9 \subseteq \overline{\text{NE}}(X_9)^\prec$. There are infinitely many (-1) -curves on X_9 , and κ_9 is the only limit ray of (-1) -rays. The anticanonical ray κ_9 coincides with the Nagata ray ν_9 .
- (iii) The *general case* $n \geq 10$. Here $-K_n$ is not effective, and has negative self-intersection $9 - n$. Hence κ_n lies off $\mathcal{Q}_n$, which in turn has non-empty intersection with both $\overline{\text{NE}}(X_n)^\succ$ and $\overline{\text{NE}}(X_n)^\prec$. There are infinitely many (-1) -curves on X_n , whose rays lie in $\overline{\text{NE}}(X_n)^\prec$ and their limit rays lie at the intersection of $\partial\mathcal{Q}_n$ with the hyperplane $\kappa_n^\perp$. The Nagata ray ν_n sits on $\partial\mathcal{Q}_n^\succ$. The plane joining the rays κ_n and ν_n is the *homogeneous slice*, formed by the classes of *homogeneous linear systems* of the form $(d; m^n)$, with $d \geq 0$.

For information on the homogeneous slice, and relations between (N) and (SHGH) there, see [?].

3.2. More conjectures. The following conjecture is in [?]. Taking into account the aforementioned result in [?], it would be a consequence of (SN).

CONJECTURE 3.2. *If $n \geq 10$, then*

$$(3.1) \quad \overline{\text{NE}}(X_n) = \mathcal{Q}_n^\succ + R_n.$$

Let $D_n = (\sqrt{n-1}, 1^n) \in N_1(X_n)$ be the *de Fernex* point and $\delta_n = [\sqrt{n-1}, 1^n]$ the corresponding ray. One has $D_n^2 = -1$, $D_n \cdot K_n = n - 3\sqrt{n-1} = \frac{n^2-9n+9}{n+3\sqrt{n-1}} > 0$ for $n \geq 8$ and, if $n = 10$, $D_n = -K_n$. We will denote by $\Delta_n^\succ$ [resp. $\Delta_n^\prec$] the set of classes $\xi \in N_1(X_n)$ such that $\xi \cdot D_n \geq 0$ [resp. $\xi \cdot D_n \leq 0$].

One has (see [?]):

THEOREM 3.3. *If $n \geq 10$ one has:*

- (i) *all (-1) -rays lie in the cone $\mathcal{D}_n := \mathcal{Q}_n - \delta_n$;*
- (ii) *if $n = 10$, all (-1) -rays lie on the boundary of the cone $\mathcal{D}_n$;*
- (iii) *if $n > 10$, all (-1) -rays lie in the complement of the cone $\mathcal{K}_n := \mathcal{Q}_n - \kappa_n$;*
- (iv) $\overline{\text{NE}}(X_n) \subseteq \mathcal{K}_n + R$;
- (v) *if Conjecture ?? holds, then*

$$(3.2) \quad \overline{\text{NE}}(X_n) \cap \Delta_n^{\leq} = \mathcal{Q}_n \cap \Delta_n^{\leq}.$$

REMARK 3.4. As noted in [?], Conjecture ?? does not imply that $\overline{\text{NE}}(X_n)^{\geq} = \mathcal{Q}_n^{\geq}$, unless $n = 10$, in which case this is exactly what it says (see Theorem ??(v)). Conjecture ?? implies (N) but not (SN).

Consider the following:

CONJECTURE 3.5 (The Δ -conjecture (ΔC)). *If $n \geq 10$ one has*

$$(3.3) \quad \partial \mathcal{Q}_n \cap \Delta_n^{\leq} \subset \text{Nef}(X_n).$$

PROPOSITION 3.6. *If (ΔC) holds, then*

$$(3.4) \quad \overline{\text{NE}}(X_n) \cap \Delta_n^{\leq} = \text{Nef}(X_n) \cap \Delta_n^{\leq} = \mathcal{Q}_n \cap \Delta_n^{\leq}.$$

PROOF. By (??) and by the convexity of $\text{Nef}(X_n)$ one has

$$\mathcal{Q}_n \cap \Delta_n^{\leq} \subseteq \text{Nef}(X_n) \cap \Delta_n^{\leq}.$$

Moreover $\text{Nef}(X_n) \cap \Delta_n^{\leq} \subseteq \overline{\text{NE}}(X_n) \cap \Delta_n^{\leq}$. Finally (??) implies (??) because $\overline{\text{NE}}(X_n)$ is dual to $\text{Nef}(X_n)$. $\square$

The following proposition indicates that Nagata-type conjectures we are discussing here can be interpreted as asymptotic forms of the (SHGH) conjecture.

PROPOSITION 3.7. *Let $n \geq 10$.*

- (i) *If (ΔC) holds, then all classes in $\mathcal{Q}_n \cap \Delta_n^{\leq} - \partial \mathcal{Q}_n \cap \Delta_n^{\leq}$ are ample and therefore, if integral, they are (ASN);*
- (ii) *If (SN) holds, then a rational class in $\mathcal{Q}_n^{\geq} - \partial \mathcal{Q}_n^{\geq}$ is (ASN) unless it has negative intersection with some (-1) -curve.*

PROOF. Part (i) follows from Proposition ?? and the fact that the ample cone is the interior of the nef cone (by Kleiman's theorem, see [?]).

As for part (ii), if $\xi \in \mathcal{Q}_n^{\geq} - \partial \mathcal{Q}_n^{\geq}$ is nef, then it is also big. If C is an irreducible curve such that $\xi \cdot C = 0$, then $C^2 < 0$ by the index theorem, hence C is a (-1) -curve. Contract it, go to X_{n-1} and take the class $\xi_1 \in N_1(X_{n-1})$ which pulls back to ξ . Repeat the argument on ξ_1 , and go on. At the end we find a class $\xi_i \in N_1(X_{n-i})$ for some $i \leq n$, which is ample by Nakai-Moishezon criterion, and the (ASN) follows for ξ .

If ξ is not nef and C is an irreducible curve such that $\xi \cdot C < 0$, then $C^2 < 0$ hence C is a (-1) -curve. $\square$

One can give a stronger form of (ΔC).

LEMMA 3.8. *Any rational, non-effective ray in $\partial \mathcal{Q}_n$ is nef and it is extremal for both $\overline{\text{NE}}(X_n)$ and $\text{Nef}(X_n)$. Moreover it lies in $\partial \mathcal{Q}_n^{\geq}$.*

PROOF. Let ξ be a generator of the ray and let $\xi = P + N$ be the Zariski decomposition of ξ . Since the ray is not effective, one has $P^2 = 0$. Since $\xi^2 = 0$, then $N^2 = 0$, hence $N = 0$, proving that ξ is nef.

Suppose that $\xi = \alpha + \beta$, with $\alpha, \beta \in \overline{\text{NE}}(X_n)$. Then $\xi^2 = 0$, $\xi \cdot \alpha \geq 0$ and $\xi \cdot \beta \geq 0$, imply $\alpha^2 = -\alpha \cdot \beta = \beta^2$ which yields that α and β are proportional. This shows that the ray is extremal for $\overline{\text{NE}}(X_n)$. The same proof shows that it is extremal also for $\text{Nef}(X_n)$.

The final assertion follows by the Mori's Cone theorem. $\square$

A rational, non-effective ray in $\partial\mathcal{Q}_n$ will be called a *good ray*. An irrational, nef ray in $\partial\mathcal{Q}_n$ will be called a *wonderful ray*. No wonderful ray has been detected so far. The following is clear:

LEMMA 3.9. *Suppose that $(\delta; m_1, \dots, m_n)$ generates either a good or wonderful ray. If $(d; m_1, \dots, m_n)$ is an effective linear system then*

$$d > \delta = \sqrt{\sum_{i=1}^n m_i^2}.$$

The following conjecture implies (ΔC) .

CONJECTURE 3.10 (The strong Δ -conjecture $(S\Delta C)$). *If $n > 10$, all rational rays in $\partial\mathcal{Q}_n \cap \Delta_n^{\leq}$ are non-effective. If $n = 10$, a rational ray in $\mathcal{Q}_{10} \cap \Delta_{10}^{\leq} = \mathcal{Q}_{10}^{\leq}$ is non-effective, unless it is generated by a curve (CK) -equivalent to $(3; 1^9, 0)$.*

PROPOSITION 3.11. *For $n = 10$, $(S\Delta C)$ is equivalent to (SN) .*

PROOF. If $(S\Delta C)$ holds then clearly (SN) holds. Conversely, assume (SN) holds, consider a rational effective ray in $\partial\mathcal{Q}_{10}^{\leq}$ and let C be an effective divisor in the ray. Then $C = n_1 C_1 + \dots + n_h C_h$, with $C_1, \dots, C_h$ distinct irreducible curves and $n_1, \dots, n_h$ positive integers. One has $C_i \cdot C_j \geq 0$, hence $C_i \cdot C_j = 0$ for all $1 \leq i < j \leq h$. This clearly implies $h = 1$, hence the assertion. $\square$

By the proof of Proposition ??, any good ray gives a constraint on $\overline{\text{NE}}(X_n)$, so it is useful to find good rays. Even better would be to find wonderful rays. We will soon give more reasons for searching for such rays (see §??).

EXAMPLE 3.12. Consider the family of linear systems

$$\mathcal{B} = \{B_{q,p} := (9q^2 + p^2; 9q^2 - p^2, (2qp)^9) : (q, p) \in \mathbb{N}^2, q \leq p\}$$

generating rays in $\partial\mathcal{Q}_{10}^{\leq}$. Take a sequence $\{(q_n, p_n)\}_{n \in \mathbb{N}}$ such that $\lim_n \frac{p_n + q_n}{p_n} = \sqrt{10}$. For instance take $\frac{p_n + q_n}{p_n}$ to be the convergents of the periodic continued fraction expansion of $\sqrt{10} = [3; \overline{6}]$, so that

$$p_1 = 2, p_2 = 13, p_3 = 80, \dots \quad q_1 = 1, q_2 = 6, q_3 = 37, \dots$$

The sequence of rays $\{[B_{q_n, p_n}]\}_{n \in \mathbb{N}}$ converges to the Nagata ray ν_{10} . If we knew that the rays of this sequence are good, this would imply (N) for $n = 10$.

A way of searching for good rays is the following (see §??). Let $(m_1, \dots, m_n)$ be a (SNS) multiplicity vector, with $d = \sqrt{\sum_{i=1}^n m_i^2}$ an integer such that $3d < \sum_{i=1}^n m_i$. Then $[d; m_1, \dots, m_n]$ is a good ray. We will apply this idea in §??.

4. Good rays and counterexamples to Hilbert's 14-th problem

In this section we show that any good or wonderful ray for $n \geq 10$ provides a counterexample to Hilbert's 14-th problem. The proof follows Nagata's original argument in [?], which we briefly recall.

Let $\mathbb{F}$ be a field. Let $\mathbf{X} = (x_{ij})_{1 \leq i \leq 3; 1 \leq j \leq n}$ be a matrix of indeterminates over $\mathbb{F}$ and consider the field $k = \mathbb{F}[\mathbf{X}] := \mathbb{F}[x_{ij}]_{1 \leq i \leq 3; 1 \leq j \leq n}$ (we use similar vector notation later). The points $x_j = [x_{1j}, x_{2j}, x_{3j}] \in \mathbb{P}_k^2$, $1 \leq j \leq n$, may be seen as n general points of $\mathbb{P}_{\mathbb{F}}^2$. The subspace $V \subset k^n$ formed by all vectors $\mathbf{b} = (b_1, \dots, b_n)$ such that $\mathbf{X} \cdot \mathbf{b}^t = \mathbf{0}$, is said to be *associated* to $x_1, \dots, x_n$.

Fix a multiplicities vector $\mathbf{m} = (m_1, \dots, m_n)$ of positive integers and consider the subgroup H of the multiplicative group $(k^*)^n$ formed by all vectors $\mathbf{c} = (c_1, \dots, c_n)$ such that $\mathbf{c}^{\mathbf{m}} := c_1^{m_1} \cdots c_n^{m_n} = 1$. We set $\delta = \sqrt{\sum_{i=1}^n m_i^2}$.

Let $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ be vectors of indeterminates over k , and consider $k[\mathbf{u}, \mathbf{v}]$. The group $G = H \times V$ acts on the k -algebra $k[\mathbf{u}, \mathbf{v}]$ in the following way: if $\sigma = (\mathbf{c}, \mathbf{b})$ and $c = c_1 \cdots c_n$, then

$$\sigma(u_i) = \frac{c_i}{c}(u_i + b_i v_i), \quad \sigma(v_i) = c_i v_i \text{ for } 1 \leq i \leq n.$$

THEOREM 4.1. *If $(\delta; m_1, \dots, m_n)$ generates a good or a wonderful ray, then the k -algebra $A = k[\mathbf{u}, \mathbf{v}]^G$ is not finitely generated.*

PROOF. The elements $t := \mathbf{v}^{\mathbf{m}}$ and

$$w_i = \sum_{j=1}^n x_{ij}(v_1 \cdots v_{j-1} u_j v_{j+1} \cdots v_n), \text{ for } 1 \leq i \leq 3$$

are in A . Set $\mathbf{w} = (w_1, w_2, w_3)$, which is a vector of indeterminates on k . Then $S := k[\mathbf{w}]$ is the homogeneous coordinate ring of $\mathbb{P}_k^2$. Imitating the argument in [?, Lemma 2], one proves that $A = k[\mathbf{u}, \mathbf{v}] \cap k(\mathbf{w}, t)$ and, as a consequence (see [?, Lemma 3]), that A consists of all sums $\sum_{i \in \mathbb{Z}} a_i t^{-i}$, such that $a_i \neq 0$ for finitely many $i \in \mathbb{Z}$, $a_i \in S$ for all $i \in \mathbb{Z}$, and $a_i \in \mathfrak{b}_i := \bigcap_{j=1}^n \mathfrak{p}_j^{i m_j}$ where $\mathfrak{p}_j$ is the homogeneous ideal of the point x_j .

By [?, Lemma 3], to prove that A is not finitely generated it suffices to show that

$$(4.1) \quad \text{for all positive } m \in \mathbb{Z}, \text{ there is a positive } \ell \in \mathbb{Z} \text{ such that } \mathfrak{b}_m^\ell \neq \mathfrak{b}_{m\ell}.$$

This is proved as in [?, Lemma 1]. Indeed, let $\alpha(\mathfrak{q})$ be the minimum degree of a polynomial in a homogeneous ideal $\mathfrak{q}$ of S . Since

$$v(d; mm_1, \dots, mn_n) = \frac{d^2 - m^2 \delta^2}{2} + \dots,$$

where $\dots$ denote lower degree terms, we have $\lim_{m \rightarrow \infty} \frac{\alpha(\mathfrak{b}_m)}{m} \leq \delta$. But because $(\delta; m_1, \dots, m_n)$ is nef, we have $\frac{\alpha(\mathfrak{b}_m)}{m} \geq \delta$ for all positive integers m . Hence $\lim_{m \rightarrow \infty} \frac{\alpha(\mathfrak{b}_m)}{m} = \delta$. By Lemma ??, one has

$$\frac{\alpha(\mathfrak{b}_m^\ell)}{m\ell} = \frac{\alpha(\mathfrak{b}_m)}{m} > \delta$$

from which (??) follows. $\square$

5. Existence of good rays

5.1. The existence theorem.

THEOREM 5.1.1 (Existence Theorem (ET)). *For every $n \geq 10$, there are good rays $[d; m_1, \dots, m_n] \in \overline{\text{NE}}(X_n)$, with $m_i > 0$ for all $1 \leq i \leq n$.*

The proof goes by induction on n (see §??). The *induction step* is based on the following proposition:

PROPOSITION 5.1.2. *Set $n = s + t - 1$, with s, t positive integers. Assume $D = (\delta; \mu_1, \dots, \mu_s) \in \text{Nef}(X_s)$ with $\mu_1, \dots, \mu_s$ nonnegative, rational numbers. Let $\nu_1, \dots, \nu_s$ be nonnegative rational numbers such that $m_1 = \sum_{i=1}^s \mu_i \nu_i$ is an integer, and let $d, m_2, \dots, m_t$ be nonnegative rational numbers such that $C = (d; m_1, \dots, m_t)$ generates a non-effective ray in $N_1(X_t)$. Then for every rational number $\eta \geq \delta$, $C_\eta = (d; \eta \nu_1, \dots, \eta \nu_s, m_2, \dots, m_t)$ generates a non-effective ray in $N_1(X_n)$.*

The proof of Proposition ?? relies on a degeneration argument introduced in [?] (see also [?, ?]), which is reviewed in §??. The next corollary shows how Proposition ?? may be applied to inductively prove Theorem ??.

COROLLARY 5.1.3. *Same setting as in Proposition ??. Assume that:*

- (i) C generates a good ray in $N_1(X_t)$;
- (ii) $D \in \text{Nef}(X_s)$ and $D^2 = 0$.

Then $C_\delta \in \partial \mathcal{Q}_n^\succ$ is nef.

PROOF. We use the vector notation $\mu = (\mu_1, \dots, \mu_s)$, $\nu = (\nu_1, \dots, \nu_s)$, and denote by $\|\cdot\|_p$ the ℓ^p norm of vectors. For $\eta \geq \delta$, one has

$$\begin{aligned} C_\eta \cdot K_n - C \cdot K_t &= \eta \|\nu\|_1 - m_1 = \eta \|\nu\|_1 - \mu \cdot \nu \geq \eta \|\nu\|_1 - \|\nu\|_2 \|\mu\|_2 = \\ &= \eta \|\nu\|_1 - \delta \|\nu\|_2 \geq \delta (\|\nu\|_1 - \|\nu\|_2) \geq 0. \end{aligned}$$

Since $C \cdot K_t \geq 0$, then also $C_\eta \cdot K_n \geq 0$, hence $C_\delta \cdot K_n \geq 0$. Moreover

$$\begin{aligned} C_\eta^2 &= C^2 + m_1^2 - \eta^2 \sum_{i=1}^s \nu_i^2 = C^2 + \left(\left(\sum_{i=1}^s \mu_i^2 \right) - \eta^2 \right) \sum_{i=1}^s \nu_i^2 \\ &= C^2 + (\delta^2 - \eta^2) \sum_{i=1}^s \nu_i^2 \leq C^2 = 0, \end{aligned}$$

in particular $C_\delta^2 = 0$.

Assume C_δ is not nef, hence there is an irreducible curve E such that $C_\delta \cdot E < 0$ and $E^2 < 0$. Take $\eta \geq \delta$ close to δ and rational. Set $E_\epsilon = \epsilon E + C_\eta$, with $\epsilon \in \mathbb{R}$. One has $E_\epsilon^2 = \epsilon^2 E^2 + 2\epsilon(C_\eta \cdot E) + C_\eta^2$ and $(C_\eta \cdot E)^2 - C_\eta^2 \cdot E^2 > 0$ because it is close to $(C_\delta \cdot E)^2 > 0$. Then

$$\tau = \frac{-(C_\eta \cdot E) - \sqrt{(C_\eta \cdot E)^2 - C_\eta^2 \cdot E^2}}{E^2}$$

is negative, close to 0 and such that $E_\tau^2 = 0$, and $E_\epsilon^2 > 0$ for $\epsilon < \tau$ and close to τ . Then for these values of ϵ the class $C_\eta = E_\epsilon - \epsilon E$ would generate an effective ray, a contradiction. $\square$

REMARK 5.1.4. We will typically apply Proposition ?? and Corollary ?? with $\mu_1 = \dots = \mu_s = 1$, $\delta = \sqrt{s}$ and $\nu_1 = \dots = \nu_s = \frac{m_1}{s}$. If either $s = 4, 9$ or $s \geq 10$ and (N) holds, then hypothesis (ii) of Corollary ?? holds. Hence, if $C = (d; m_1, \dots, m_t)$ generates a good ray in $N_1(X_t)$, then $C_{\frac{s}{\eta}} = (d; (\frac{m_1}{\eta})^s, m_2, \dots, m_t)$ generates a noneffective ray for all rational numbers $\eta \leq \sqrt{s}$, therefore the ray $[d; (\frac{m_1}{\sqrt{s}})^s, m_2, \dots, m_t]$ is either good or wonderful, in particular it is nef.

In this situation, if C is standard and $s \geq 9$, then C_δ is also standard. The same holds for $s = 4$ if $2d \geq 3m_1$. This will be the case for the examples we will provide to prove Theorem ??, so all of them will be standard.

The *base of the induction*, consists in exhibiting SNS multiplicity vectors for $10 \leq n \leq 12$, giving rise to good rays as indicated at the end of § ???. They will provide the starting points of the induction for proving Theorem ?? (see §??). This step is based on a slight improvement of the same degeneration technique used to prove Proposition ?? (see §??).

REMARK 5.1.5. To the best of our knowledge, it is only for a square number of points that SNS multiplicity vectors and good rays were known so far: i.e., $[d; 1^{d^2}]$ is a good ray (see [?]) and (1^{d^2}) is an SNS multiplicity vector for $d \geq 4$ (see [?, ?, ?]).

EXAMPLE 5.1.6. Using the goodness of $[d; 1^{d^2}]$ and applying Corollary ??, we see that all rays of the form $[dh; h^{d^2-\ell}, 1^{\ell h^2}]$, with $d \geq 4$, $h \geq 1$ and $0 \leq \ell \leq d^2$ integers, are good.

5.2. The basic degeneration. We briefly recall the degeneration we use to prove Theorem ?? (see [?, ?, ?] for details).

Consider $Y \rightarrow \mathbb{D}$ the family obtained by blowing up the trivial family $\mathbb{D} \times \mathbb{P}^2 \rightarrow \mathbb{D}$ over a disc $\mathbb{D}$ at a point in the central fiber. The general fibre Y_u for $u \neq 0$ is a $\mathbb{P}^2$, and the central fibre Y_0 is the union of two surfaces $V \cup Z$, where $V \cong \mathbb{P}^2$ is the exceptional divisor and $Z \cong \mathbb{F}_1$ is the original central fibre blown up at a point. The surfaces V and Z meet transversally along a rational curve E which is the negative section on Z and a line on V .

Choose s general points on V and $t - 1$ general points on Z . Consider these $n = s + t - 1$ points as limits of n general points in the general fibre Y_u and blow these points up in the family Y , getting a new family. We will abuse notation and still denote by Y this new family. The blow-up creates n exceptional surfaces R_i , $1 \leq i \leq n$, whose intersection with each fiber Y_u is a (-1) -curve, the exceptional curve for the blow-up of that point in the family. The general fibre Y_u of the new family is an X_n . The central fibre Y_0 is the union of V blown-up at s general points, and Z blown-up at $t - 1$ general points. We will abuse notation and still denote by V and Z the blown-up surfaces which are now isomorphic to X_s, X_t respectively.

Let $\mathcal{O}_Y(1)$ be the pullback on Y of $\mathcal{O}_{\mathbb{P}^2}(1)$. Given a vector $(m_1, \dots, m_n)$ of multiplicities, a degree d and a *twisting integer* a , consider the line bundle

$$\mathcal{L}(a) = \mathcal{O}_Y(d) \otimes \mathcal{O}_Y(-m_1 R_1) \otimes \dots \otimes \mathcal{O}_Y(-m_n R_n) \otimes \mathcal{O}_Y(-aV).$$

Its restriction to Y_u for $u \neq 0$ is $(d; m_1, \dots, m_n)$. Its restrictions to V and Z are $\mathcal{L}_V = (a; m_1, \dots, m_s)$, $\mathcal{L}_Z = (d; a, m_{s+1}, \dots, m_n)$ respectively. Every *limit line bundle* of $(d; m_1, \dots, m_n)$ on Y_t is the restriction to $Y_0 = V \cup Z$ of $\mathcal{L}(a)$ for an integer a .

We will say that a line bundle $\mathcal{L}(a)$ is *centrally effective* if its restriction to both V and Z is effective.

THEOREM 5.2.7 (The Basic Non-Effectivity Criterion (BNC), (see [?])). *If there is no twisting integer a such that $\mathcal{L}(a)$ is centrally effective, then $(d; m_1, \dots, m_n)$ is non-effective.*

5.3. The proof of the induction step. In this section we use (BNC) to give the:

PROOF OF PROPOSITION ??. . We need to prove that

$$(xd; x\eta\nu_1, \dots, x\eta\nu_s, xm_2, \dots, xm_t)$$

is not effective for all positive integers x .

In the setting of §??, fix the multiplicities on V to be $x\eta\nu_1, \dots, x\eta\nu_s$ and on Z to be $xm_2, \dots, xm_t$. We argue by contradiction, and assume there is a central effective $\mathcal{L}(a)$. Then $\mathcal{L}_V = (a; x\eta\nu_1, \dots, x\eta\nu_s)$ is effective hence $D \cdot \mathcal{L}_V \geq 0$, i.e. $a\delta \geq x\eta m_1$, therefore $a \geq xm_1$. Since $\mathcal{L}_Z = (xd; a, xm_2, \dots, xm_t)$ is effective, so is $(xd; xm_1, xm_2, \dots, xm_t)$ which contradicts $C = (d; m_1, \dots, m_t)$ not being effective. $\square$

5.4. 2-throws. To deal with the base of the induction, we need to analyse the *matching* of the sections of the bundles $\mathcal{L}_V$ and $\mathcal{L}_Z$ on the double curve. For this we need a modification of the basic degeneration, based on the concept of a *2-throw*, described in [?], which we will briefly recall now. In doing this we will often abuse notation, which we hope will create no problems for the reader.

Consider a degeneration of surfaces over a disc, with central fibre containing two components X_1 and X_2 meeting transversally along a double curve R . Let E be a (-1) -curve on X_1 that intersects R transversally at two points. Blow it up in the threefold total space of the degeneration. The exceptional divisor $T \cong \mathbb{F}_1$ meets X_1 along E , which is the negative section of T . The surface X_2 is blown up twice, with two exceptional divisors G_1 and G_2 .

Now blow-up E again, creating a double component $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ of the central fibre that meets X_1 along E and T along its negative section. The blow-up affects X_2 , by creating two more exceptional divisors F_1 and F_2 which are (-1) curves on X_2 , while G_1 and G_2 become (-2) -curves. Blowing S down by the other ruling contracts E on the surface X_1 . The curve R becomes nodal, and T changes into a $\mathbb{P}^2$. The surface X_2 becomes non-normal, singular along the identified (-1) -curves F_1, F_2 .

On X_2 we introduced two pairs of *infinitely near points* corresponding to the (-1) -curves F_i and $F_i + G_i$, which is also a curve with self-intersection -1 , and we call F_i and $F_i + G_i$ a pair of *infinitely near* (-1) -curves, with $1 \leq i \leq 2$. We denote the assignment of multiplicities to a pair of infinitely near points as above by $[a, b]$, indicating a multiple point a and an infinitely near multiple point b , namely $-a(F_i + G_i) - bF_i$.

5.5. The base of the induction. The above discussion is general. In order to deal with the base of the induction, we will now apply it to the degeneration $V \cup Z$ described in section ??, with $n = 10$ (for the cases $11 \leq n \leq 12$ the basic degeneration, plus some more care on the matching, suffices). The proofs here are quite similar to the ones in [?, ?], hence we will be brief.

5.5.1. *The $n = 10$ case.*

PROPOSITION 5.5.8. *The multiplicity vector $(5, 4^9)$ is (SNS). In particular $B_{1,2} = (13; 5, 4^9)$ generates a good ray.*

PROOF. It suffices to prove that, for every positive integer x , $(13x; 5x, (4x)^9)$ is non-effective and $(13x + 1; 5x, (4x)^9)$ is non-special.

We will show that $(15x; 6x, (4x)^3, (5x)^5, 4x)$ ((CK)-equivalent to $(13x; 5x, (4x)^9)$) is not effective. We assume by contradiction that the linear system is effective for some x .

Consider first the basic degeneration with $s = 4$, $t = 7$, endowed with the line bundles $\mathcal{L}(a)$ as in §?? (whose restrictions to V and Z are $\mathcal{L}_V = (a; 6x, (4x)^3)$, $\mathcal{L}_Z = (15x; a, (5x)^5, 4x)$). Then perform the 2-throw of the (-1) -curve $E = (3; 2, 1^6)$ on Z (see §??). The normalization of V is a 8-fold blow up of $\mathbb{P}^2$, two of the exceptional divisors being identified in V . More precisely, the normalization of V is the blow-up of the plane at 8 points: 4 of them are in general position, 4 lie on a line, and two of them are infinitely near. It is better to look at the surface Z before blowing down E . Then $Z \cong X_7$. Finally, by executing the 2-throw we introduce a plane T .

We record that the pencil $P_V = (5; 3, 2^3, [1, 1]^2)$ on the normalization of V and the pencil $P_Z = (3; 2, 1^5, 0)$ on Z are nef.

We abuse notation and still denote by $\mathcal{L}(a)$ the pullback of this bundle to the total space of the family obtained by the double blow-up of E (see §??). For each triple of integers (a, b_1, b_2) , we can consider the bundle $\mathcal{L}(a, b_1, b_2) = \mathcal{L}(a) \otimes \mathcal{O}_Y(-b_1T - (b_1 + b_2)S)$. We will still denote by $\mathcal{L}(a, b_1, b_2)$ the pushout of this bundle to the total space of the 2-throw family. Every limit line bundle of $(15x; 6x, (5x)^5, (4x)^4)$ has the form $\mathcal{L}(a, b_1, b_2)$.

We are interested in those $\mathcal{L}(a, b_1, b_2)$ which are centrally effective. Such systems were studied in [?, section 2], whose notation had $\mu = a, y = u = b_1, y' = v = b_2, h = \sigma/2 = -\mathcal{L}(a) \cdot E/2 = -8x + a$ and several other parameters x_i, z_i corresponding to 6 additional curves that were thrown, which we can safely equate to 0. The effectivity of the restriction to S and T implies that $b_1 \geq -8x + a$, $b_1 + b_2 \geq -16x + 2a$, and the effectivity of the restriction to V , that $-8x + a \geq x > 0$. Then the computation of [?] shows that for every b_1, b_2 satisfying the preceding inequalities, the restriction of $\mathcal{L}(a, b_1, b_2)$ to Z and V are subsystems of the restriction of $\mathcal{L}(a, -8x + a, -8x + a)$, namely

$$\mathcal{L}_Z = (63x - 6a; 32x - 3a, (21x - 2a)^5, 20x - 2a)$$

and

$$\mathcal{L}_V = (a; 6x, (4x)^3, [-8x + a, -8x + a]^2).$$

It suffices to see that there is no value of a which makes both $\mathcal{L}_Z$ and $\mathcal{L}_V$ effective, and for which there are divisors in these two systems which agree on the double curve.

For $\mathcal{L}(a, b_1, b_2)$ to be centrally effective one needs

$$\mathcal{L}_Z \cdot P_Z = 20x - 2a \geq 0, \quad \mathcal{L}_V \cdot P_V = a - 10x \geq 0.$$

This forces $a = 10x$ and the restriction of $\mathcal{L}(a, b_1, b_2)$ to Z and V are equal to

$$\mathcal{L}_Z = (3x; 2x, x^5, 0), \quad \mathcal{L}_V = (10x; 6x, (4x)^3, [2x, 2x]^2),$$

which means $b_1 = b_2 = -8x + a = 2x$, hence the restriction of the line bundle to T is trivial, and the systems $\mathcal{L}_Z$ and $\mathcal{L}_V$ are composed with the pencils P_V and P_Z , thus $\dim(\mathcal{L}_Z) = x$, $\dim(\mathcal{L}_V) = 2x$.

Focusing on $\mathcal{L}_V$, we only need to consider the subspace of sections that match along F_1 and F_2 . Since the identification $F_1 = F_2$ is done via a sufficiently general projectivity, this vector space has dimension 1 (see [?, §8] for details). Then, by transversality on $Z \cap V = R$ (see [?, §3]), and since $\mathcal{L}_{Z|R}$ has dimension x and degree $2x$, no section on Z matches the one on V to create a section on X_0 .

Now consider $(13x+1; 5x, (4x)^9)$ and its (CK)-equivalent system $(15x+3; 6x+2, (4x)^3, 5x, (5x+1)^4, 4x)$. A similar analysis as before, using $a = 10x+1$, leads to the following limit systems on Z and V (trivial on T)

$$\mathcal{L}_Z = (3x+12; 2x+7, (x+4)^4, x+3, 3), \quad \mathcal{L}_V = (10x+1; 6x+2, (4x)^3, [2x-1, 2x-2]^2).$$

The system $\mathcal{L}_Z$ is (CK)-equivalent to $(x+5; x+2, 2, 1^4)$ so it is nef, non-empty of the expected dimension (see [?]). The system $\mathcal{L}_V$ is also non-empty of the expected dimension: it consists of three lines plus a residual system (CK)-equivalent to the nef system $(2x+3; 3, 2, 1^2, 2x-2)$. Thus to compute the dimension of the limit system as in [?], it remains to analyse the restrictions to R (or rather, the *kernel systems* $\hat{\mathcal{L}}_Z, \hat{\mathcal{L}}_V$ of such restrictions). Since both surfaces are anticanonical, this can be done quite easily, showing that they are non-special with $\dim(\hat{\mathcal{L}}_Z) = 2x+4$ and $\dim(\hat{\mathcal{L}}_V) = 10x-7$. Thus [?, 3.4, (b)], applies and non-speciality of $(15x+3; 6x+2, (4x)^3, 5x, (5x+1)^4, 4x)$ follows. $\square$

5.5.2. The $n = 11$ case.

PROPOSITION 5.5.9. *The multiplicity vector $(3, 2^{10})$ is (SNS). In particular $(7; 3, 2^{10})$ generates a good ray.*

PROOF. We prove that $(7x+\delta; (2x)^{10}, 3x)$ is non-effective for all x and $\delta = 0$ and non-special for $\delta = 1$.

Consider the basic degeneration as in §??, with $s = 4$, $t = 8$. Then $\mathcal{L}(a)$ restrict as

$$\mathcal{L}_V = (a; (2x)^4), \quad \mathcal{L}_Z = (7x+\delta; a, (2x)^6, 3x).$$

Look at the case $\delta = 0$, where we want to prove non-effectivity. (BNC) does not suffice for this, so we will compute the dimension of a limit system as in [?]. To do this, pick $a = 4x$. The systems $\mathcal{L}_V$ and $\mathcal{L}_Z$ are composed with the pencils $P_V = (2; 1^4)$ and $P_Z = (7; 4, (2)^6, 3)$ respectively (note that $(7; 4, (2)^6, 3)$ is (CK)-equivalent to a pencil of lines), and $\dim(\mathcal{L}_V) = 2x$, $\dim(\mathcal{L}_Z) = x$. The restriction to R has degree $4x \geq 2x + x + 1$, so by transversality of the restricted systems [?, §3], the limit linear system consists of the kernel systems $\hat{\mathcal{L}}_V = (4x-1; (2x)^4)$ and $\hat{\mathcal{L}}_Z = (7x; 4x+1, (2x)^6, 3x)$. These are non-effective, because they meet negatively P_V and P_Z respectively. So $(7x; (2x)^{10}, 3x)$ is non-effective.

For $\delta = 1$ pick again $a = 4x$. Then $\mathcal{L}_V$ is the same, $\mathcal{L}_Z = (7x+1; 4x, (2x)^6, 3x)$ and the kernel systems are both nef, hence they are non-special by [?]. Moreover the restriction of $\mathcal{L}_Z$ to R is the complete series of degree $4x$. Again by transversality as in [?, §3], the claim follows. $\square$

5.5.3. The $n = 12$ case.

PROPOSITION 5.5.10. *The multiplicity vector $(2^8, 1^4)$ is (SNS). In particular $(6; 2^8, 1^4)$ generates a good ray.*

PROOF. We prove that $(6x + \delta; x^4, (2x)^8)$ is empty for all x and $\delta = 0$ and non-special for $\delta = 1$.

Consider the degeneration of §??, with $s = 4$, $t = 9$. Then $\mathcal{L}(a)$ restrict as

$$\mathcal{L}_V = (a; (x)^4), \quad \mathcal{L}_Z = (6x; a, (2x)^8).$$

Let us analyze the case $\delta = 0$ for $a = 2x$. The system $\mathcal{L}_Z$ consists of $2x$ times the unique cubic E through the 9 points and $\mathcal{L}_V$ is composed with the pencil $P_V = (2, 1^4)$, and its restriction to R is composed with a general pencil of degree 2. By transversality it does not match the divisor cut out by $2xE$ on R and the limit system is formed by the kernel systems. An elementary computation shows that they are not effective.

For $\delta = 1$ and $a = 2x$, $\mathcal{L}_V$ is the same, whereas $\mathcal{L}_Z = (6x + 1; (2x)^9)$ and the kernel $\hat{\mathcal{L}}_Z = (6x + 1; 2x + 1, (2x)^8)$ are both nef, hence they are non-special by [?]. As before, the restriction of $\mathcal{L}_Z$ to R is the complete series of degree $2x$, and transversality gives the claim. $\square$

REMARK 5.5.11. (i) In Example ?? we saw that $[dh; h^{d^2-\ell}, 1^{\ell h^2}]$, with $0 \leq \ell \leq h$, and $d \geq 4$, $h \geq 1$, is a good ray. Proposition ?? shows that if $d = 3$, $h = 2$, $\ell = 1$, the ray is still good. By Corollary ??, this implies that if $d = 3$, $h = 2$, $1 \leq \ell \leq 9$, the ray is still good. With a similar argument, one sees that all cases $d = 3$, $h \geq 1$, also give rise to good rays. We leave this to the reader.

(ii) For any integer $d \geq 6$, take positive integers r, s such that $d^2 = 4s + r$. The ray generated by $(d; 2^s, 1^r)$ on X_n is nef. Indeed, one has $(d-1)(d-2) \geq 2s$, so there exists an irreducible curve C of degree d with exactly s nodes $p_1, \dots, p_s$ ([?], Anhang F). On the blow-up of the s nodes and r other points $q_1, \dots, q_r$ of the curve, the proper transform of the curve is a prime divisor of selfintersection zero, thus nef.

If $d = 2k$ is even then $r = 4k^2 - 4s$ is a multiple of four and $(d; 2^s, 1^r) = (2k; 2^{k^2-\ell}, 1^{4\ell})$ generates a good ray with $\ell = k^2 - s$ by example ?? (because of (i) we may assume $d > 6$). This suggests that that the ray $[d; 2^s, 1^r]$ may always be good. To prove it, taking into account Corollary ??, it would suffice to show that $[2k + 1; 2^{k^2+k}, 1]$ is good for all $k \geq 3$.

5.6. The proof of the ET. For $10 \leq n \leq 12$ the problem is settled by Propositions ??, ?? and ??. To cover all $n \geq 13$, we apply Corollary ?? with $s = 4$, $D = (2; 1^4)$ and $\nu_1 = \dots = \nu_4 = \frac{m_1}{4}$ (see Remark ??). For instance one finds the good rays

$$\begin{aligned} & [13 \cdot 2^h; 5^4, (5 \cdot 2)^3, \dots, (5 \cdot 2^{h-1})^3, (2^{h+2})^9] & \text{if } n = 10 + 3h, \text{ for } h \geq 1, \\ & [7 \cdot 2^h; 3^4, (3 \cdot 2)^3, \dots, (3 \cdot 2^{h-1})^3, (2^{h+1})^{10}] & \text{if } n = 11 + 3h, \text{ for } h \geq 1, \\ & [6 \cdot 2^{h-1}; (2)^3, \dots, (2^{h-1})^3, (2^h)^7, 1^8] & \text{if } n = 12 + 3h, \text{ for } h \geq 1. \end{aligned}$$

6. An application

PROPOSITION 6.1. *If (SN) holds for $n = 10$ and (N) holds for all $n \leq 89$ then (N) holds for all $n \geq 90$.*

The proof is based on the following:

LEMMA 6.2. Assume (SN) holds for $n = 10$. Let $n = s_1 + \dots + s_{10}$, where $s_1, \dots, s_{10}$ are positive integers such that the Nagata ray ν_{s_i} is nef for $1 \leq i \leq 10$ and

$$(6.1) \quad 3\sqrt{n} \leq \sum_{i=1}^{10} \sqrt{s_i}.$$

Then ν_n is nef.

PROOF. Consider the ray $[\sqrt{n}, \sqrt{s_1}, \dots, \sqrt{s_{10}}]$ which, by the hypotheses, is in $\partial \mathcal{Q}_{10}^\succ$. We can approximate it by a sequence $\{[d_h, m_{1,h}, \dots, m_{10,h}]\}_{h \in \mathbb{N}}$ of good rays (see Proposition ??). Make now an obvious multiple application of Corollary ?? with $C = [d_h, m_{1,h}, \dots, m_{10,h}]$ and D each of the Nagata points ν_{s_i} , for $1 \leq i \leq 10$ (see Remark ??). We obtain that $[d_h, (\frac{m_{1,h}}{\sqrt{s_1}})^{s_1}, \dots, (\frac{m_{10,h}}{\sqrt{s_{10}}})^{s_{10}}]$ is nef for all $h \in \mathbb{N}$. Since this ray tends to ν_n for $h \rightarrow \infty$, the assertion follows. $\square$

PROOF OF PROPOSITION ?. We argue by induction. Let $n \geq 90$, and write $n = 9h + k$, with $9 \leq k \leq 17$ and $h \geq 9$. By induction both ν_h and ν_k are nef. Moreover (??) is in this case $3\sqrt{9h+k} \leq 9\sqrt{h} + \sqrt{k}$, which reads $h \geq \frac{16}{81}k$, which is verified because $k \leq 17$ and $h \geq 9$. Then ν_n is nef by Lemma ?. $\square$

REMARK 6.3. Lemma ? is reminiscent of the results in [?] and [?].

The hypotheses in Proposition ? can be weakened. For instance, Lemma ? implies that, if (SN) holds for $n = 10$, then ν_{13} is nef. Actually, it suffices to know that $[\sqrt{13}; 2, 1^9]$ is nef. As in Example ??, we may take a sequence $\{(q_n, p_n)\}_{n \in \mathbb{N}}$ such that $\frac{p_n + 2q_n}{p_n}$ are the convergents of the periodic continued fraction expansion of $\sqrt{13} = [3; 1^3, 6]$, so that

$$p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 20, \dots \quad q_1 = 1, q_2 = 2, q_3 = 3, q_4 = 13 \dots$$

The sequence of rays $\{[B_{q_n, p_n}]\}_{n \in \mathbb{N}}$ converges to ν_{13} . If we knew that the rays of the sequence are good, this would imply (N) for $n = 13$. Note that $B_{q_1, p_1} = (13; 5, 4^9)$ generates a good ray by Proposition ?.

Similarly, if (SN) holds for $n = 10$, then ν_n is nef for $n = 10h^2$, etc. We do not dwell on these improvements here.

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