Homework 5 Solutions M331 Fall 2002

Average: 11.9/15

Problem 4.1. (2 pts). See section 4.1 for definitions of linear, homogenous and order.

a $x_{n+1}^2 + x_n = 1$: nonlinear, nonhomogenous, order 1
b $x_{n+1} = x_{n-1} + 2$: linear, nonhomogenous, order 2
c $x_{n+1} = sin(x_{n-1})$: nonlinear, homogenous, order 2
d $x_{n+3} = x_{n+1} + x_{n-3} + n^2$: linear, nonhomogenous, order 6

Problem 4.2. (1 pt.) Determine particular solutions...

a $x_{n+1} = x_n + 1$: assume $p_n = An + B$. Plug in p_n to equation and solve for A and B. Find $A = 1$ and $B = \textit{anything}$. Let $B = 0$. Then solution is: $p_n = n$. Plug back in to be sure p_n satisfies difference equation.
b $x_{n+1} = 5x_{n-1} + n^2$: assume $p_n = An^2 + Bn + C$. Plug in p_n to equation and solve for A, B and C. Find $A = \frac{-1}{4}$, $B = \frac{-1}{8}$ and $C = \frac{-3}{32}$. Then solution is: $p_n = \frac{-1}{4}n^2 - \frac{1}{8}n - \frac{3}{32}$. Plug back in to be sure p_n satisfies difference equation.
c $x_{n+1} = \frac{x_n}{2} + 6^n$ assume $p_n = A6^n$. Plug in p_n to equation and solve for A. Find $A = \frac{2}{11}$. Then solution is: $p_n = \frac{2}{11}6^n$. Plug back in to be sure p_n satisfies difference equation.

Problem 4.3. (3 pts.) Determine general solutions... recall this means find homogeneous solution and particular solution and add them to get your general solution. Recall also from notes that the problem $x_{n+1} = \alpha x_n$ has the solution $x_n = c\alpha^n$.

a $x_{n+1} = x_n + 1$: Homogeneous solution is: $h_n = c1^n = c$ From prob 4.2, particular solution is: $p_n = n$. So general solution is: $x_n = h_n + p_n = c + n$ Plug back in to be sure x_n satisfies difference equation.
b $x_{n+1} = 5x_{n-1} + n^2$: Homogeneous solution is: $h_n = c5^n$ From prob 4.2, particular solution is: $p_n = \frac{-1}{4}n^2 - \frac{1}{8}n - \frac{3}{32}$. So general solution is: $x_n = h_n + p_n = c5^n - \frac{1}{4}n^2 - \frac{1}{8}n - \frac{3}{32}$. Plug back in to be sure x_n satisfies difference equation.
c $x_{n+1} = \frac{x_n}{2} + 6^n$ Homogeneous solution is: $h_n = c\frac{1}{2}^n = \frac{c}{2^n}$ From prob 4.2, particular solution is: $p_n = \frac{2}{11}6^n$. So general solution is: $x_n = h_n + p_n = \frac{c}{2^n} + \frac{2}{11}6^n$. Plug back in to be sure x_n satisfies difference equation.

d $x_{n+1} = \frac{x_n}{2} + 4n^2 + 2n + 1$ Homogeneous solution is: $h_n = c\frac{1}{2}^n = \frac{c}{2^n}$ Assume $p_n = An^2 + Bn + C$. Plug in p_n to equation and solve for A, B and C. Find $A = 8$, $B = -28$ and $C = 42$. Then particular solution is: $p_n = 8n^2 - 28n + 42$. So general solution is: $x_n = h_n + p_n = \frac{c}{2^n} + 8n^2 - 28n + 42$. Plug back in to be sure x_n satisfies difference equation.

Problem 4.4. (3 pts) This problem is set up and solved in a manner similar to the forensics problem in the notes. Let T_n be the temperature of the roast at hour n, and $\Delta T_n = T_{n+1} - T_n$ Then $\Delta T_n \propto 400 - T_n$ where 400 is the temperature of the oven. So, $\Delta T_n = k(400 - T_n) \Rightarrow T_{n+1} = (1-k)T_n + 400k$. The homogenous solution is: $h_n = c(1-k)^n$. Assume $p_n = A$. Then plugging in and solving we find $A = 400$. So the general solution is $T_n = h_n + p_n = c(1-k)^n + 400$ where we need to solve for c and k. To do this we will use the given information that the roast initally is 50 degrees F, i.e. $T_0 = 50$ and after an hour, the roast is 90 degrees, i.e. $T_1 = 90$.

So if $n = 0$ we have $T_0 = 50 = c(1-k)^0 + 400 \Rightarrow c = -350$ which gives us that now $T_n = 400 - 350(1-k)^n$ we still need to find k. If $n = 1$ we have $T_1 = 90 = 400 - 350(1-k)^1 \Rightarrow k = \frac{4}{35}$

Thus $T_n = 400 - 350(\frac{31}{35})^n$. We want to know what time (n) the roast will reach 166 degrees F. We must solve $166 = 400 - 350(\frac{31}{35})^n$ for n. So, $n = \frac{\ln(\frac{117}{175})}{\ln(\frac{31}{35})} = 3.317$ or set the table about 3 hours and 19 minutes after you start the roast (assuming that you can set the table while the roast cools before you cut it)!

Problem 4.5. (1 pt.) Note that a stable equilibrium point can be thought of as an attractor. An equilibrium point is unstable if initial conditions that start nearby end up somewhere other than the equilibrium point that was started next to. $p_{n+1} = p_n + \alpha p_n(1 - p_n)$ has equilibrium solutions when $\bar{p} = p_{n+1} = p_n$ or solve $\bar{p} = \bar{p} + \alpha\bar{p}(1 - \bar{p})$ for $\bar{p}$ and get $\bar{p} = 0$ and $\bar{p} = 1$ are the equilibrium solutions.

Find that for all given p_0 and α, $\bar{p} = 0$ is unstable and $\bar{p} = 1$ is stable.

Matlab code:

```
alpha = 0.1

p(1) = 0
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i));
end
```

```
a0=p;
p(1) = 0.0001
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i));
end
a1=p;
p(1) = 2
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i));
end
a2=p;

x=1:size(a2,2);
subplot(3,1,1);
plot(x,a0,'o',x,a1,'rx',x,a2,'g^')
legend('p0=0','p0=0.0001','p0=2',0);
title('p_{n+1} = p_n + \alpha p_n (1-p_n), \alpha=0.1');

alpha = 0.7

p(1) = 0
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i));
end
a0=p;
p(1) = 0.0001
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i));
end
a1=p;
p(1) = 2
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i));
end
a2=p;

x=1:size(a2,2);
subplot(3,1,2);
plot(x,a0,'o',x,a1,'rx',x,a2,'g^')
legend('p0=0','p0=0.0001','p0=2',0);
title('p_{n+1} = p_n + \alpha p_n (1-p_n), \alpha=0.7');
```

```
alpha = 1.2

p(1) = 0
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i));
end
a0=p;
p(1) = 0.0001
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i));
end
a1=p;
p(1) = 2
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i));
end
a2=p;

x=1:size(a2,2);
subplot(3,2,5);
plot(x,a0,'o',x,a1,'rx',x,a2,'g^')
legend('p0=0','p0=0.0001','p0=2',0);
title('p_{n+1} = p_n + \alpha p_n (1-p_n), \alpha=1.2');
subplot(3,2,6);
plot(x,a0,'o',x,a1,'rx')
legend('p0=0','p0=0.0001',0);
```

Problem 4.6. (3 pts). $p_{n+1} = p_n + \alpha p_n(1-p_n)(2-p_n)$ has equilibrium solutions when $\bar{p} = p_{n+1} = p_n$ or solve $\bar{p} = \bar{p} + \alpha\bar{p}(1-\bar{p})(2-\bar{p})$ for $\bar{p}$ and get $\bar{p} = 0, \bar{p} = 1$ and $\bar{p} = 2$ are the equilibrium solutions.

Find that for all given p_0 and α, $\bar{p} = 0$and$\bar{p} = 2$ are unstable and $\bar{p} = 1$ is stable.

Matlab code:

```
alpha = 0.1

p(1) = 0
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i))*(2-p(i));
end
a0=p;
p(1) = 0.0001
for i = 1:100
```

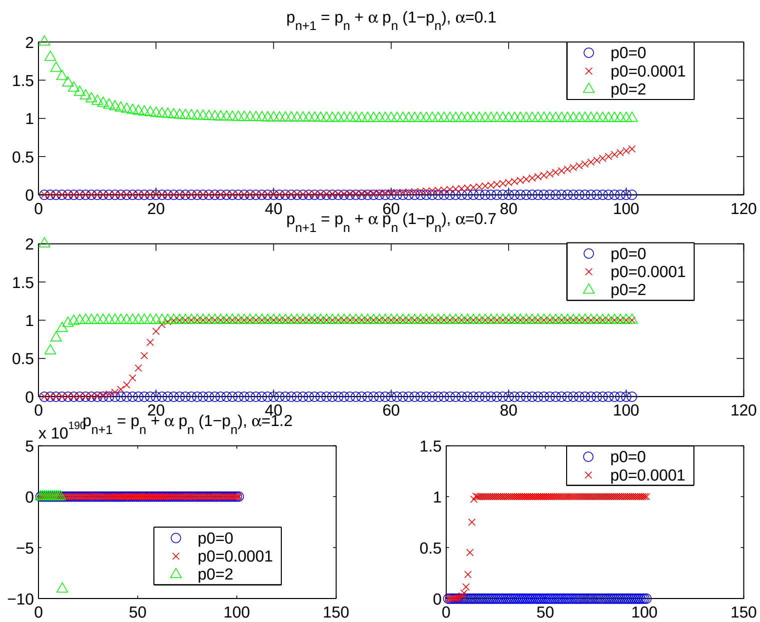


FIGURE 1. problem 4.5

```
   p(i+1) = p(i) + alpha*p(i)*(1-p(i))*(2-p(i));
end
a1=p;
p(1) = 0.9999
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i))*(2-p(i));
end
a2=p;
p(1) = 1
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i))*(2-p(i));
end
a3=p;
p(1) = 1.0001
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i))*(2-p(i));
end
a4=p;
p(1) = 1.9999
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i))*(2-p(i));
end
```

```
a5=p;
p(1) = 2
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i))*(2-p(i));
end
a6=p;
p(1) = 2.0001
for i = 1:100
   p(i+1) = p(i) + alpha*p(i)*(1-p(i))*(2-p(i));
end
a7=p;

x=1:size(a2,2);
subplot(2,1,1);
plot(x,a0,’o’,x,a1,’*’,x,a2,’^’,x,a3,’o’,x,a4,’*’,x,a5,’^’,x,a6,’*’)
legend(’p0=0’,’p0=0.0001’,’p0=.9999’,’p0=1’,’p0=1.0001’,’p0=1.9999’,’p0=2’,0);
title(’p_{n+1} = p_n + \alpha p_n (1-p_n) (2-p_n), \alpha=0.1’);
subplot(2,1,2);
plot(x,a7)
legend(’p0=2.0001’,0);
```

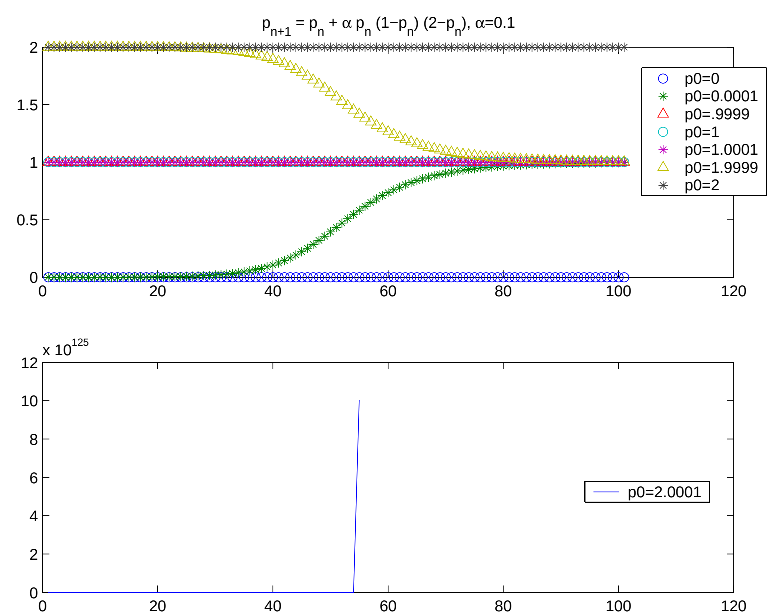


FIGURE 2. problem 4.6

Problem 4.7. (1 pt) Here we can find that the values tend to oscillate around an equilibrium value instead of actually converging.

Matlab code:

```
p(1) = 6
p(2) = 1
p(3) = 2.5
p(4) = -3

for i = 1:100
   p(i+4) = sin( p(i+3) + p(i+2) + p(i+1) - p(i) ) + 3;
end
a0=p;
for i = 1:100
   p(i+4) = sin( p(i+3) - p(i) ) + 1;
end
a1=p;

x=1:size(a1,2);
subplot(2,1,1);
plot(x,a0)
title('p_{n+4} = sin(p_{n+3}^2+p_{n+2}+p_{n+1}-p_n) + 3');
subplot(2,1,2);
plot(x,a1)
title('p_{n+4} = sin(p_{n+3}-p_n) + 1');
```

Problem 4.8. (1 pt). In this system the equilibrium values depend upon the initial conditions chosen. But it turns out that $\bar{x} + \bar{y}$ will always be constant if the coefficients of $x_n \text{and} y_n$ sum to 1 in the system of equations (initial problem and part (b)). If however the coefficients sum to not be equal to one, then the solutions become unstable and experience continuous growth (or decay).

Verify the points you have are equilibrium points by plugging the values into the right-hand side of the equation and you should get the same values out of the left-hand side of the equation.

Matlab code:

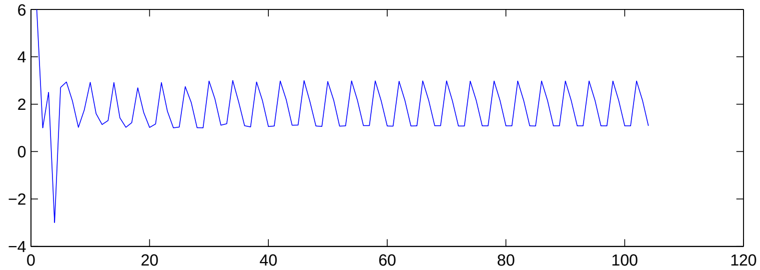
```
% coefficients for xn+1=a*xn+b*yn and yn+1=c*xn+d*yn
x(1) = 5
y(1) = 2

a = 0.3; b = 0.8; c = 0.7; d = 0.2
for i = 1:100;
```

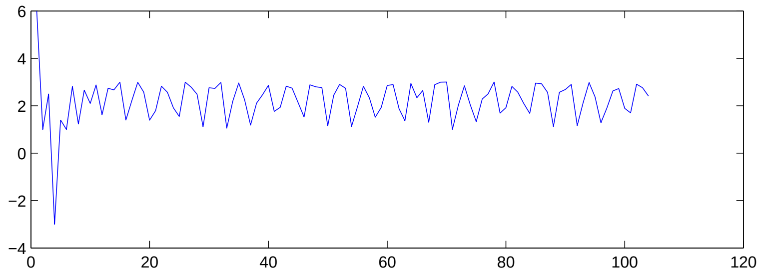
```
   x(i+1) = a*x(i) + b*y(i);
   y(i+1) = c*x(i) + d*y(i);
end
x1=x; y1=y;
a = 0.31; b = 0.8; c = 0.7; d = 0.2
for i = 1:100;
   x(i+1) = a*x(i) + b*y(i);
   y(i+1) = c*x(i) + d*y(i);
end
x2=x; y2=y;
a = 0.31; b = 0.8; c = 0.69; d = 0.2
for i = 1:100;
   x(i+1) = a*x(i) + b*y(i);
   y(i+1) = c*x(i) + d*y(i);
end
x3=x; y3=y;

p=1:101;
subplot(3,1,1)
plot(p,x1,'o',p,y1,'^')
legend('x','y',0);
title('original problem')
subplot(3,1,2)
plot(p,x2,'o',p,y2,'^')
legend('x','y',0);
title('part a')
subplot(3,1,3)
plot(p,x3,'o',p,y3,'^')
legend('x','y',0);
title('part b')
```

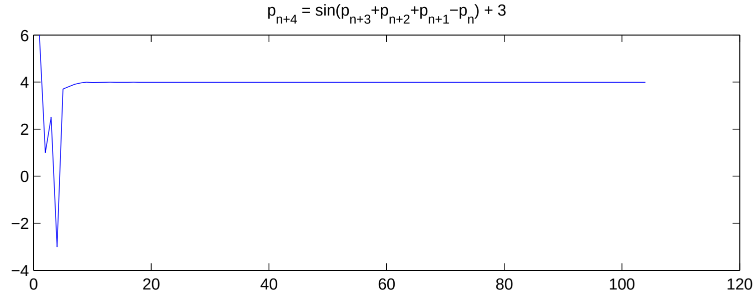
$p_{n+4} = \sin(p_{n+3}+p_{n+2}+p_{n+1}-p_n) + 2$



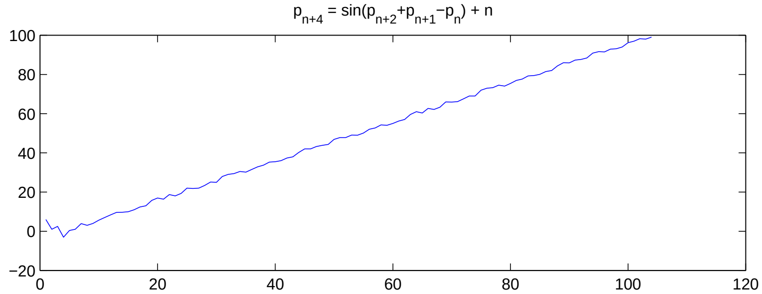
$p_{n+4} = \sin(p_{n+2}+p_{n+1}-p_n) + 2$



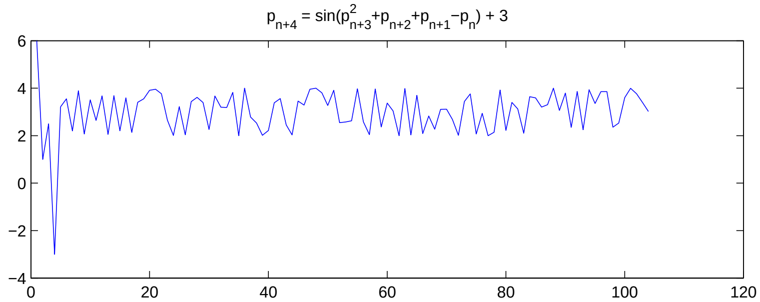
$p_{n+4} = \sin(p_{n+3}+p_{n+2}+p_{n+1}-p_n) + 3$



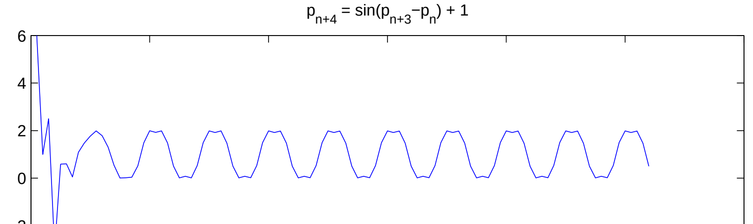
$p_{n+4} = \sin(p_{n+2}+p_{n+1}-p_n) + n$



$p_{n+4} = \sin(p^2_{n+3}+p_{n+2}+p_{n+1}-p_n) + 3$



$p_{n+4} = \sin(p_{n+3}-p_n) + 1$



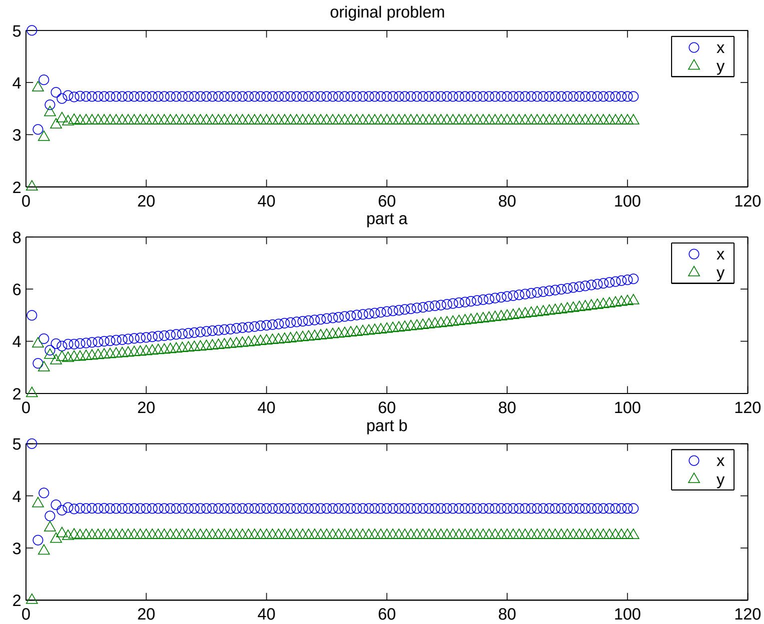


FIGURE 4. problem 4.8