Mathematical Modeling, Proof of Concept

Mathematical Modeling

Gerhard Dangelmayr and Michael Kirby

Department of Mathematics, Colorado State University, Fort Collins, Colorado 80524

E-mail address: gerhard@math.colostate.edu, kirby@math.university.edu

This project is supported in part by NSF Grant DUE-#0126650. ©2002.

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Preface

This is the first draft of a set of course notes in mathematical modeling. These materials are the result of the National Science Foundation DUE-0126650 award entitled "Mathematical Modeling Program for Undergraduates in Science, Mathematics, Engineering & Technology".

These notes are incomplete and not intended for general dissemination at this time. If you would like to learn more about this project please contact one of the authors.

Gerhard Dangelmayr and Michael Kirby

CHAPTER 1

Mathematical Modeling

Mathematical modeling is becoming an increasingly important subject as computers expand our ability to translate mathematical equations and formulations into concrete conclusions concerning the world, both natural and artificial, that we live in.

1.1. Examples of Modeling

Here we do a quick tour of several examples of the mathematical process. We present the models as finished results as opposed to attempting to develop the models.

1.1.1. Modeling with Difference Equations. Consider the situation in which a variable changes in discrete time steps. If the current value of the variable is a_n then the predicted value of the variable will be a_{n+1}. A mathematical model for the evolution of the (still unspecified) quantity a_n could take the form

$$a_{n+1} = \alpha a_n + \beta$$

In words, the new value is a scalar multiple of the old value offset by some constant β. This model is common, e.g., it is used for modeling bank loans. One might amend the model to make the dependence depend on more terms and to include the possibility that every iteration the offset can change, thus,

$$a_{n+1} = \alpha_1 a_n + \alpha_2 a_n^2 + \beta_n$$

This could correpsond to, for example, a population model where the the migration levels change every time step. In some instances, it is clear that information required to predict a new value goes back further than the current value, e.g.,

$$a_{n+1} = a_n + a_{n-1}$$

Note now that two initial values are required to evolve this model. Finally, it may be that the form of the difference equations are unknown and the model must be written

$$a_{n+1} = f(a_n, a_{n-1}, a_{n-M-1})$$

Determining the nature of f and the step M is at the heart of model formulation with difference equations. Often observed data can be employed to assist in this effort.

1.1.2. Modeling with Ordinary Differential Equations. Although modeling with ordinary differential equations shares many of the ideas of modeling with the difference equations discussed above, there are many fundamental differences. At the center of these differences is the assumption that time is a continuous variable.

One of the simplest differential equations is also an extremely important model, i.e.,

$$\frac{dx}{dt} = \alpha x$$

In words, the rate of change of the quantity x depends on the amount of the quantity. If $\alpha > 0$ then we have exponential growth. If $\alpha < 0$ the situation is exponetial decay. Of course additional terms can be added that fundamentally alter the evolution of $x(t)$. For example

$$\frac{dx}{dt} = \alpha_1 x + \alpha_2 x^2$$

The model formulation again requires the development of the appropriate right-hand side.

In the above model the value x on the right hand side is implicitly assumed to be evaluated at the time t. It may be that there is evidence that the instantaneous rate of change at time t is actually a function of a previous time, i.e.,

$$\frac{dx}{dt} = f(x(t)) + g(x(t-\tau))$$

This is referred to as a delay differential equation.

1.1.3. Modeling with Partial Differential Equation. In the previous sections on modeling the behaviour of a variable as a function of time we assumed that there was only one independent variable. Many situations arise in practice where the number of independent variables is larger than two. For spatio-temporal models we might have time and space (hence the name!), e.g.,

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$$

or

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

1.1.4. Optimization. In many modeling problems the goal is to compute the "best" solution. This may correspond to maximizing profit in a company, or minimizing loss in a conflict. It is no surprise that optimization techniques take a central seat in the mathematical modeling literature.

Now one may allow $x \in \mathbb{R}^n$ and require that

$$x^* = \arg\min f(x)$$

The quantity $f(x)$ is referred to as the *objective function* while the vector x consists of decision variables. Because x sits in $\mathbb{R}^n$ the problem is referred to as unconstrained.

Alternatively, one might require that the solution x have all positive components. If we refer to this set as S then the optimization problem is constrained

$$x^* = \arg\min_{x \in S} f(x)$$

If the objective function as well as the equations that define the constraint set are linear, than the optimization problem is called a *linear programming problem.* Otherwise, the problem is referred to as a nonlinear programming problem. As we shall see, solution methods for linear and nonlinear programming problems are very different.

1.1.5. Modeling with Simulations. Many problems may afford a mathematical formulation yet be analytically intractable. In these situations a computer can implement the mathematics literally and repetitively often times to extreme advantage.

1.1.5.1. *Simulating Games.*

- What is the probability that you can win a game of solitaire?
- What is the best strategy for playing blackjack?
- Given a baseball team consisting of certain players, in what order should they hit?

On the other hand, computer simulations can be employed to model evolution equations. Applications in the realm of fluid dynamics and weather prediction are well established. A striking new example of such simulation modeling is attempting to model electrical activity in the brain.

1.1.6. Function Fitting: Data Modeling. Often data is available from a process to assist in the modeling. How can functions be computed that reflect the relationships between variables in the data. Produce a model

$$y = f(x; w)$$

and using the set of input output pairs compute the parameters w. In some cases the form of f may be guessed. In other cases a model free approach can be used.

1.2. The Modeling Process

The goal in all modeling problems is *added value.* Something novel must be learned from the modeling process or one has completed an exercise in futility, or mathematical wheel spinning, depending on your perspective. There are many obvious questions the answers to which have inherent added value. For example:

- Should a stock be bought or sold?
- Is the earth becoming warmer?
- Does creating a law have a positive or negative societal effect?
- What is the most valuable property in monopoly?

Clearly this is a very small start to an extremely long list.

1.2.1. An Algorithm for Modeling? The modeling process has a sequence of common steps that serve as an abstraction for the modeler:

- Identify the problem and questions.
- Identify the relevant variables in a problem.
- Simplify until tractable.
- Relate these variables mathematically.
- Solve.
- Does the solution provide added value?
- Tweak model and compare solutions.

1.3. Purpose of this Course

The primary goal of this course is to assist the student in developing an understanding of why mathematics is useful as a language for characterizing the interaction among variables.

The arena for mathematical modeling is so large that there can be a trade-off between learning the specifics of a particular problem and the mathematical tools for solving it. Here we emphasize not the special knowledge often required to make expert conclusions concerning models, but rather the efficacy of certain mathematics for constructing models.

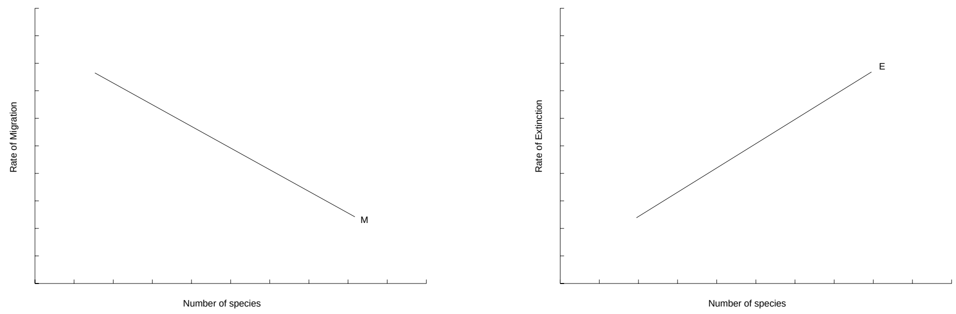
Problems

PROBLEM 1.1. Name three problems that might be modeled mathematically. Why do you think mathematics may provide a key to each solution. What is the added value in each case?

PROBLEM 1.2. Consider the differential equation

$$\frac{dx}{dt} = x$$

Translate this model to a difference equation. Compare the solutions and discuss.



(a) Migration curve.

(b) Extinction curve

FIGURE 1. Qualitative form of the migration and extinction curves.

CHAPTER 2

Qualitative Modeling with Functions

It is often surprising that very simple mathematical modeling ideas can produce results with added value. Indeed, the solutions may be elegant and provide quality of understanding that obviates further exploration by more technical or complex means. In this chapter we explore a few simple approaches to qualitatively modeling phenomena with well-behaved functions.

2.1. Modeling Species Propagation

This problem concerns the factors that influence the number of species existing on an island. The discussion is adapted from [**?**].

One might speculate that factors affecting the number of species could include

- Distance of the island from the mainland
- Size of the island

Of course limiting ourselves to these influences has the dual effect of making a tractable model that needs to be recognized as omitting many possible factors.

The number of species may increase due to new species discovering the island as a suitable habitat. We will refer to this as the *migration rate*. Alternatively, species may become extinct due to competition. We will refer to this as the *extinction rate*. This discussion will be simplified by employing an aggregate total for the number of species and not attempting to distinguish the nature of each species, i.e., birds versus plants.

Now we propose some basic modeling assumptions that appear reasonable.

> *The migration rate of new species decreases as the number of species on the island increases.*

The argument for this is straight forward. The more species on an island the smaller the number of new species there is to migrate. See Figure 1 (a) for a qualitative picture.

> *The extinction rate of species increases as the number of species on the island increases.*

Clearly the more species there are the more possibilities there are for species to die out. See Figure 1 (b) for a qualitative picture.

If we plot the extinction rate and the migration rate on a single plot we identify the point of intersection as an equilibrium, i.e., the migration is exactly offset by the extinction and the number of species on the island is a constant. We will assume in this discussion that we are considering islands for which the number of species is roughly constant over time, i.e., they are in a state of equilibrium.

Now we consider whether this simple model provides any added value. In particular, can it be used to address our questions posed at the outset.

First, what is the effect of the distance of the island from the mainland on the number of bird species? One can characterize this effect by a shift in the migration curve. The further the island is away from the mainland, the less likely a species is to successfully migrate. Thus the migration curve is shifted down for *far* islands and shifted up for *near* islands. Presumably, this distance of the island from the mainland has no impact on the extinction curve. Thus, by examining the shift in the equilibrium, we may conclude that the number of species on an island decreases as the island's distance from the mainland increases. See Figure 2.

Note in this model we assume that the time-scales are small enough that new species are not developed via evolution. While this may seem reasonable there is evidence that in some extreme climates, such as those found in the Galapagos Islands, variation may occur over shorter periods. There have been 140 different species of birds

2.2. Supply and Demand

In this section we sketch a well-known concept in economics, i.e., supply and demand. We shall see that relatively simple laws, when taken together, afford interesting insight into the relationship between producers and consumers. Furthermore, we may use this framework to make predictions such as

- What is the impact of a tax on the sale price?
- What is the impact an increase in employees wages on sales price? Can the owner of the business pass this increase on to the consumer?

> *Law of Supply:* An increase in the price of a commodity will result in an increase of the amount supplied.

> *Law of Demand:* If the price of a commodity increases, then the quantity demanded will decrease.

Thus, we may model the supply curve qualitatively by a monotonically increasing function. For simplcity we may assume a straight line with positive slope. Analogously, we may model the demand curve qualitatively by a monotonically decreasing function, which again we will take as a straight line.

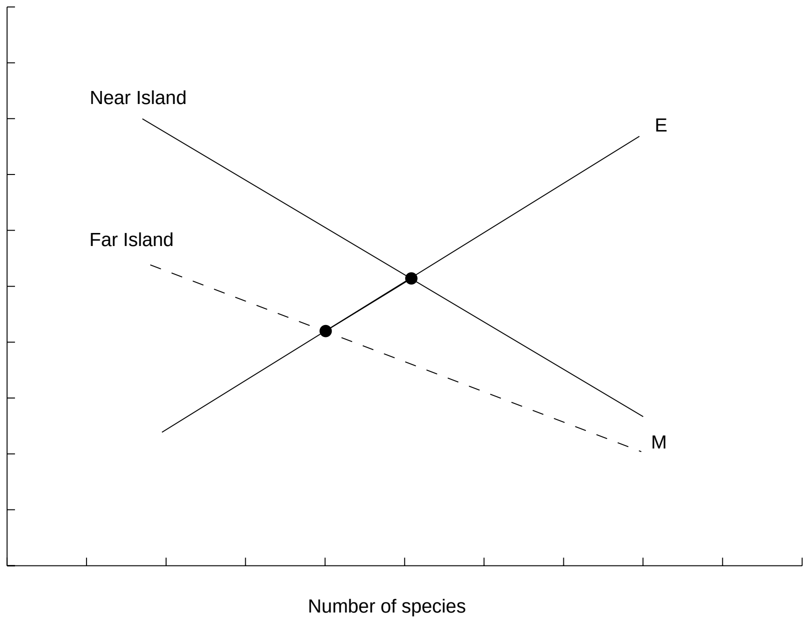


FIGURE 2. The effect of distance of the island from the mainland is to shift the migration curve. Consequently the equilibrium solution dictates a smaller number of species will be supported for islands that are farther away from the mainland.

A flat demand curve may be interpreted as consumers being very sensitive to the price of a commodity. If the price goes up just a little, then the quantity in demand goes down significantly. Steep and flat supply and demand curves all have similar qualitative interpretations (see the problems).

2.2.1. Market Equilibrium. Given a supply curve and a demand curve we may plot them on the same axis and note their point of intersection (q_*, p_*). This point is special for the following reason:

- The seller is willing to supply q_* at the price p_*
- The demand is at the price p_* is q_*

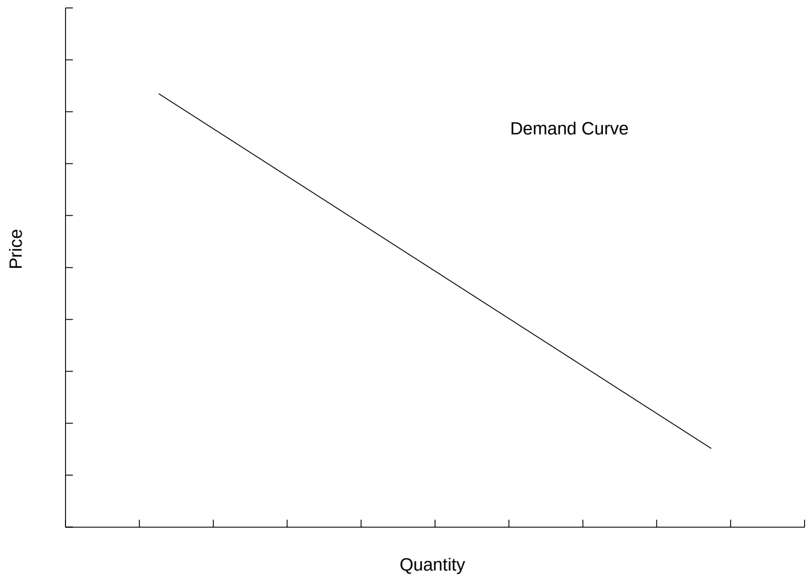
So both the supplier(s) and the purchaser(s) are happy economically speaking.

2.2.2. Market Adjustment. Of course, in general markets do not exist in the perfect economic utopia described above. We may model the market adjustment as a sequence of points on the demand and supply curves.

Based on market research it is estimated that consumers will demand a quantity q_1 at a price p_1. The supply and demand curves will permit a prediction of how the market will evolve. For simplicity, we will assume that the initial point (q_1, p_1) is on the demand curve to the right of the equilibrium point.

At the price p_1 the supplier looks to his supply curve and proposes to sell a reduced quantity q_2. Thus we move from right to left horizontally. Note that moving vertically to the supply curve would not make sense as this would correspond to offering the quantity q_1 at an increased price. These goods will not sell at this price.

(a) Supply curve



(b) Demand curve

FIGURE 3. (a) Qualitative form of supply and demand curves.

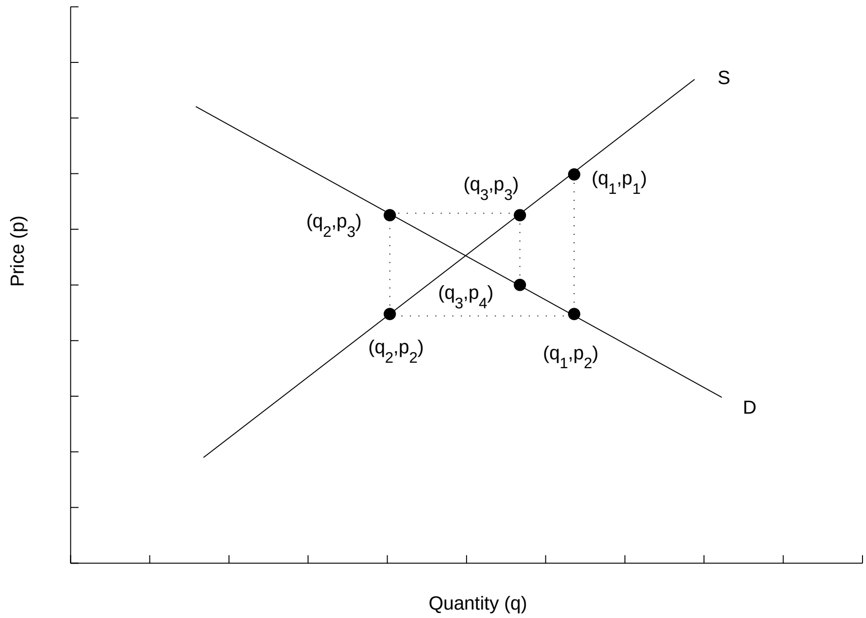


FIGURE 4. The cobweb model illustrating a sequence of market adjustments.

From the point (q_1, p_2) the consumer will respond to the new reduced quantity q_2 by being willing to pay more. This corresponds to moving vertically upward to the new point (q_2, p_2) on the supply curve.

Now the supplier adjusts to the higher price being paid in the market place by increasing the quantity produced to q_3. This process then continues, in theory, until an equilbrium is reached. It is possible that this will never happen, at least not without a basic adjustment to the shape of either the supply of demand curves, for example through cost cutting methods such as improved efficiency, or layoffs.

2.2.3. Taxation. The effect of a new tax on a product is to shift the demand curve down because consumers will not be willing to pay as much for the product (before the tax). Note that this leads to a new equilibrium point which reduces the price paid to the seller per item and reduces the quantity supplied by the producer. Thus one may conclude from this picture that the effect of a tax on alcohol is to reduce consumption as well as profit for the supplier. See Figure 5.

2.3. Modeling with Proportion and Scale

In the previous sections we have considered how simple functions may be employed to qualitatively model various situations and produce added value. Now we turn to considerations that assist in determining the nature of these functional dependencies in more complex terms.

2.3.1. Proportion. If a quantity y is *proportional* to a quantity x then we write

$$y \propto x$$

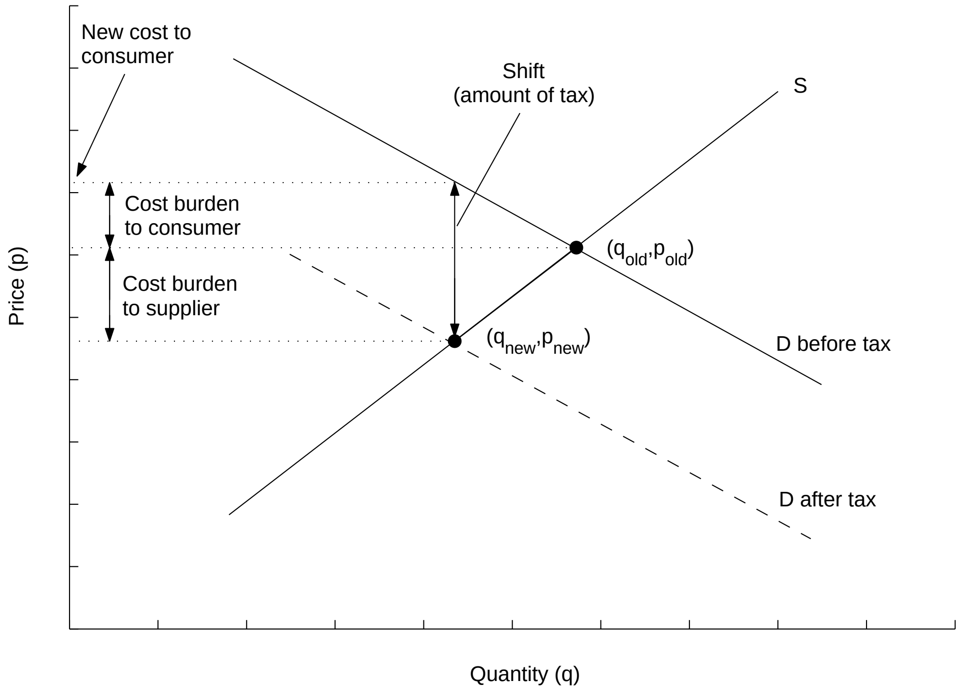


FIGURE 5. A tax corresponds to a downwards shift in the demand curve.

by which is meant

$$y = kx$$

for some constant of proportionality k.

EXAMPLE 2.1. In 1678 Robert Hooke proposed that the restoring force F of a spring is proportional to its elongation e, i.e.,

$$F \propto e$$

or,

$$F = ke$$

where k is the *stiffness* of the spring.

Note that the property of proportionality is symmetric, i.e.,

$$y \propto x \to x \propto y \tag{2.1}$$

and transitive, i.e.,

$$y \propto x \quad \text{and} \quad z \propto y \to z \propto x \tag{2.2}$$

EXAMPLE 2.2. If $y = kx + b$ where k, b are constants, then

$$y \not\propto x$$

but

$$y - b \propto x$$

Inverse proportion. If $y \propto 1/x$ then y is said to be *inversely* proportional to x.

EXAMPLE 2.3. If y varies inversely as the square-root of x then

$$y = \frac{k}{\sqrt{x}}$$

Joint Variation. The volume of a cylinder is given by

$$V = \pi r^2 h$$

where r is the radius and h is the height. The volume is said to vary *jointly* with r^2 and h, i.e.,

$$V \propto r^2 \quad \text{and} \quad V \propto h$$

EXAMPLE 2.4. The volume of a given mass of gas is proportional to the temperature and inversely proportional to the pressure, i.e., $V \propto T$ and $V \propto 1/P$, or,

$$V = k\frac{T}{P}$$

EXAMPLE 2.5. Frictional drag due to the atmosphere is jointly proportional to the surface area S and the velocity v of the object.

Superposition of Proportions. Often a quantity will vary as the sum of proportions.

EXAMPLE 2.6. The stopping distance of a car when an emergency situation is encountered is the sum of the reaction time of the driver and the amount of time it takes for the breaks to dissipate the energy of the vehicle. The reaction distance is proportional to the velocity. The distance traveled once the breaks have been hit is proportional to the velocity squared. Thus,

$$\text{stopping distance} = k_1 v + k_2 v^2$$

EXAMPLE 2.7. Numerical error in the computer estimation of the center difference formula for the derivative is given by

$$e(h) = \frac{c_1}{h} + c_2 h^2$$

where the first term is due to roundoff error (finite precision) and the second term is due to truncation error. The value h is the distance δx in the definition of the derivative.

2.3.1.1. *Direct Proportion.* If

$$y \propto x$$

we say y varies in *direct* proportion to x. This is not true, for example, if $y \propto r^2$. On the other hand, we may construct a direct proportion via the obvious change of variable $x = r^2$. This simple trick always permits the investigation of the relationship between two variables such as this to be recast as a direct proportion.

2.3.2. Scale. Now we explore how the size of an object can be represented by an appropriate length scale if we restrict our attention to replicas that are *geometrically similar*. For example, a rectangle with sides l_1 and w_1 is geometrically similar to a rectangle with sides l_2 and w_2 if

$$\frac{l_1}{l_2} = \frac{w_1}{w_2} = k \tag{2.3}$$

(a) Plot of y against r for $y = r^2$

(b) Plot of y against r^2

(c) Plot of y against r for $y = kr(r+1)$

(d) Plot of y against $r(r+1)$

FIGURE 6. Simple examples of how a proportion may be converted to a direct proportion.

As the ratio $\kappa = l_1/w_1$ characterizes the geometry of the rectangle it is referred to as the *shape factor*. If two objects are geometrically similar, then it can be shown that they have the same shape factor. This follows directly from multiplying Equation (2.3) by the factor l_2/w_1, i.e.,

$$\frac{l_1}{w_1} = \frac{l_2}{w_2} = k\frac{l_2}{w_1}$$

2.3.2.1. *Characteristic Length.* Characteristic length is useful concept for characterizing a family of geometrically similar objects. We demonstrate this with an example.

(a) (b) (c)

FIGURE 7. Examples of characteristic lengths for the rectangle.

Consider the area of a rectangle of side l and width w where l and w may vary under the restriction that the resulting rectangle be geometrically similar to the rectangle with length l_1 and width w_1. An expression for the area of the varying triangle can be simplified as a consequence of the constraint imposed by geometric similarity. To see this

$$\begin{aligned} A &= lw \\ &= l(\frac{w_1 l}{l_1}) \\ &= \kappa l^2 \end{aligned}$$

where $\kappa = w_1/l_1$, i.e., the shape factor.

EXAMPLE 2.8. Watering a farmer's rectangular field requires an amount of area proportional to the area of the field. If the characteristic length of the field is doubled, how much additional water q will be needed, assuming the new field is geometrically similar to the old field? Solution: $q \propto l^2$, i.e., $q = kl^2$. Hence

$$\begin{aligned} q_1 &= kl_1^2 \\ q_2 &= kl_2^2 \end{aligned}$$

Taking the ratio produces

$$\frac{q_1}{q_2} = \frac{l_1^2}{l_2^2}$$

Now if $q_2 = 100$ acre feet of water are sufficient for a field of length $l_2 = 100$, how much water will be required for a field of length $l_1 = 200$? Sol.

$$q_1 = q_2 \frac{l_1^2}{l_2^2} = 100 \frac{200^2}{100^2} = 400 \, \text{acre feet} \quad \square$$

EXAMPLE 2.9. Why are gymnasts typically short? It seems plausible that the ability A, or natural talent, of gymnast would be proportional to strength and inversely proportional to weight, i.e.,

$$A \propto \text{strength}$$

and

$$A \propto \frac{1}{\text{weight}}$$

and taken jointly

$$A \propto \frac{\text{strength}}{\text{weight}}$$

One model for strength is that the strength of a limb is proportional to the cross-sectional area of the muscle. The weight is proportional to the volume (assuming constant density of the gynmast). Now, assuming all gymnasts are geometrically similar with characteristic length l

$$\text{strength} \propto \text{muscle area} \propto l^2$$

and

$$\text{weight} \propto \text{volume} \propto l^3$$

so the ability A follows

$$A \propto \frac{l^2}{l^3} \propto \frac{1}{l}$$

So shortness equates to a talent for gymnastics.

□

EXAMPLE 2.10. Proportions and terminal velocity. Consider a uniform density spherical object falling under the influence of gravity. The object will travel will constant (terminal) velocity if the accelerating force due to gravity $F_g = mg$ is balance exactly by the decelerating force due to atmospheric friction $F_d = kSv^2$; S is the cross-sectional surface area and v is the velocity of the falling object. Our equilibrium condition is then

$$F_g = F_d$$

Since surface area satisfies $S \propto l^2$ it follows $l \propto S^{1/2}$. Given uniform density $m \propto w \propto l^3$ so it follows $l \propto m^{1/3}$. Combining proportionalities

$$m^{1/3} \propto S^{1/2}$$

from which it follows by substitution into the force equation that

$$m \propto m^{2/3} v^2$$

or, after simplifying,

$$v \propto m^{1/6}$$

□

EXAMPLE 2.11. In this example we will attempt to model observed data displayed in Table 1 that relates the heart rate of mammals to there body weight. From the table we see that we would like to relate the heart rate as a function of body weight. Smaller animals have a faster heart rate than larger ones. But how do we estimate this proportionality?

We begin by assuming that all the energy E produced by the body is used to maintain heat loss to the environment. This heat loss is in turn proportional to the surface area s of the body. Thus,

$$E \propto s$$

The energy available to the body is produced by the process of respiration and is assumed to be proportional to the oxygen available which is in turn proportional to the blood flow B through the lungs. Hence, $B \propto s$ If we denote the pulse rate as r we may assume

$$B \propto rV$$

mammal	body weight (g)	pulse rate
shrew[2]	3.5	782
pipistrelle bat[1]	4	660
bat[2]	6	588
mouse[1]	25	670
hamster[2]	103	347
kitten[2]	117	300
rat[1]	200	420
rat[2]	252	352
guinea pig[1]	300	300
guinea pig[1]	437	269
rabbit[2]	1,340	251
rabbit[1]	2,000	205
oppossum[2]	2,700	187
little dog[1]	5,000	120
seal[2]	22,500	100
big dog[1]	30,000	85
goat[2]	33,000	81
sheep[1]	50,000	70
human[1]	70,000	72
swine[2]	100,000	70
horse[2]	415,000	45
horse[1]	450,000	38
ox[1]	500,000	40
elephant[1]	3,000,000	48

TABLE 1. Superscript 1 data source A.J. Clark; superscript 2 data source Altman and Dittmer.

where V is the volume of the heart.

We still need to incorporate the body weight w into this model. If we take W to be the weight of the heart assuming constant density of the heart it follows

$$W \propto V$$

Also, if the bodies are assumed to be geometrically similar then $w \propto W$ so by transitivity $w \propto V$ and hence

$$B \propto rw$$

Using the geometric similarity again we can relate the body surface area s to its weight w. From characteristic length scale arguments

$$v^{1/3} \propto s^{1/2}$$

so

$$s \propto w^{2/3}$$

from which we have $rw \propto w^{2/3}$ or

$$r = kw^{-1/3}$$

To validate this model we plot $w^{-1/3}$ versus r for the data Table 8. We see that for the larger animals with slower heart rates that this data appears linear and suggests this rather crude model actually is supported by the data. For much

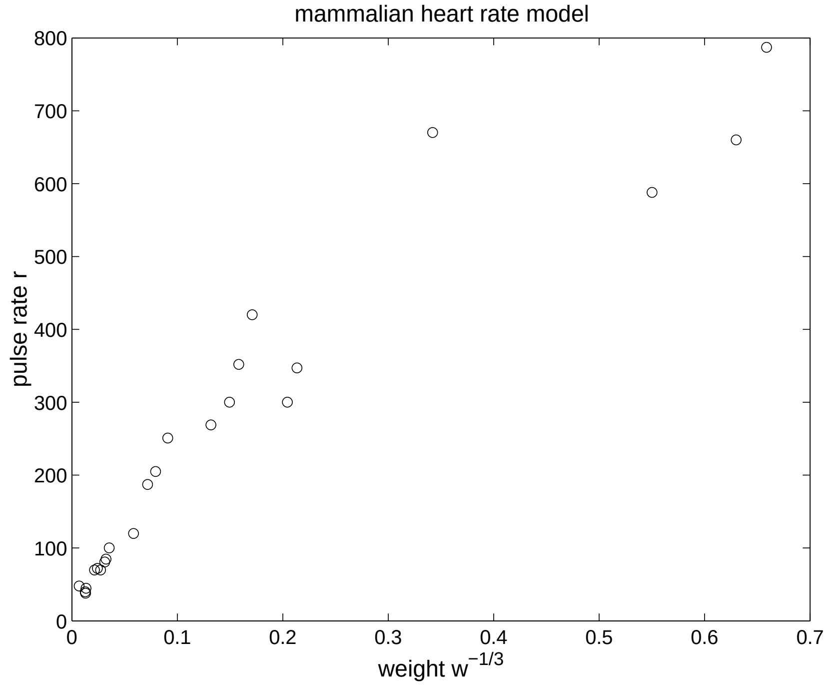


FIGURE 8. Testing the model produced by proportionality. For the model to fit, the data should sit on a straight line eminating from the origin.

smaller animals there appear to be factors that this model is not capturing and the data falls off the line.

2.4. Dimensional Analysis

In this chapter we have explored modeling with functions and proportion. In some instances, such as the mammalian heart rate, it is possible to cobble enough information together to actually extract a model; in particular, to identify the functional form for the relationship between the dependent and independent variables. Now we turn to a surprisingly powerful and simple tool known as *dimensional analysis*[1].

Dimensional analysis operates on the premise that equations contain terms that have units of measurement and that the validity of these equations, or laws, are not dependent on the system of measurement. Rather these equations relate variables that have inherent physical dimensions that are derived from the fundamental dimensions of *mass, length* and *time.* We label these dimensions generically as M, L and T, respectively.

As we shall see, dimensional analysis provides an effective tool for mathematical modeling in many situations. In particular, some benefits include

- determination of the form of a joint proportion
- reduce number of variables in a model

[1]This dimension should not be confused with the usual notion of geometric dimension.

- enforcement of dimensional consistency
- ability to study scaled versions of models

2.4.1. Dimensional homogeneity. An equation is said to be *dimensionally homogeneous* if all the terms in the equation have the same physical dimension.

EXAMPLE 2.12. All the laws of physics are dimensionally homogeneous. Consider Newton's law

$$F = ma$$

The units on the right side are

$$M \cdot \frac{L}{T^2}$$

so we conclude that the physical dimension of a force must be MLT^{-2}. □

EXAMPLE 2.13. The equation of motion of a linear spring with no damping is

$$m\frac{d^2x}{dt^2} + kx = 0$$

What are the units of the spring constant? Dimensionally we can recast this equation as

$$MLT^{-2} + M^aL^bT^cL = 0$$

Matching exponents for each dimension permits the calculation of a, b and c.

$$\begin{aligned} M: &\quad a = 1 \\ L: &\quad 1 = b + 1 \\ T: &\quad -2 = c \end{aligned}$$

Thus we conclude that the spring constant has the dimensions MT^{-2}. □

EXAMPLE 2.14. Let v be velocity, t be time and x be distance. The model equation

$$v^2 = t^2 + \frac{x}{t}$$

is dimensionally inconsistent.

EXAMPLE 2.15. An angle may be defined by the formula

$$\theta = \frac{s}{r}$$

where the arclength s subtends the angle θ and r is the radius of the circle. Clearly this angle is dimensionless.

2.4.2. Discovering Joint Proportions. If in the formulation of a problem we are able to identify a dependent and one or more independent variables, it is often possible to identify the form of a joint proportion. The form of the proportion is actually constrained by the fact that the equations must be dimensionally consistent.

EXAMPLE 2.16. In this problem we consider the drag force F_D on an airplane. As our model we propose that this drag force (dependent variable) is proportional to the independent variables

- cross-sectional area A of airplane
- velocity v of airplane
- density ρ of the air

As a joint proportion we have

$$F_D = kA^a v^b \rho^c$$

where a, b and c are unknown exponents. As a consequence of dimensional consistency we have

$$\begin{aligned} MLT^{-2} &= (M^0 L^0 T^0)(L^2)^a (\frac{L}{T})^b (\frac{M}{L^3})^c \\ &= M^c L^{2a-3c+b} T^{-b} \end{aligned}$$

From the M exponent we conclude $c = 1$. From the T exponent $b = 2$ and from the L exponent it follows that $1 = 2a - 3c + b$, whence $a = 1$. Thus the only possibility for the form of this joint proportion is

$$F_D = kAv^2\rho$$

Note that if the density of were a constant it would be appropriate to simplify this dependency as

$$F = \tilde{k}Av^2$$

but now the constant $\tilde{k}$ actually has dimensions. □

2.4.3. Procedure for Nondimensionalization. Consider the nonlinear model for a pendulum

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

Based on the terms in this model we may express the solution very generally as a relationship between these included terms, i.e.,

$$\phi(\theta, g, l, t) = 0$$

Note that the angle in this model is dimensionless but the other variables all have dimensions. We can convert this equation into a new equation where none of the terms have dimensions. This will be referred to, for obvious reasons, as a dimensional form of the model.

To accomplish this, let

$$\tau = \frac{t}{\sqrt{l/g}}.$$

The substitution of variables may be accomplished by noting that

$$\frac{d^2\theta}{dt^2} = \frac{d^2\theta}{\frac{l}{g}d\tau^2}$$

Thus, after cancelation, the dimensionless form for the nonlinear pendulum model is

$$\frac{d^2\theta}{d\tau^2} = -\sin\theta$$

Now the solution has the general form

$$f(\theta, \tau) = 0,$$

or equivalently,

$$f(\theta, \sqrt{\frac{l}{g}}t) = 0$$

This is a special case of a more general theory.

The Buckingham π-theorem. Any dimensionally homogeneous equation with physical variables $x_1, \ldots, x_m$ expressed

$$\phi(x_1, \ldots, x_m)$$

may be rewritten in terms of its associated dimensionless variables $\pi_1, \ldots, \pi_n$ as

$$f(\pi_1, \ldots, \pi_n) = 0$$

where

$$\pi_k = x_1^{a_{k1}} \ldots x_m^{a_{km}}$$

2.4.4. Modeling with Dimensional Analysis. Now we consider two examples of the application of the ideas described above concerning dimensional analysis. In each of these examples there is more than one dimensionless parameter and it is appropriate to apply the Buckingham π-theorem.

2.4.4.1. *The Pendulum.* In this example the goal is to understand how the period of a pendulum depends on the other parameters that describe the nature of the pendulum. The first task is to identify this set of parameters that act as the independent variables on which the period P depends.

Obvious candidates include From this list we are motivated to write

variable	symbol	dimensions
mass	m	M
length	l	L
gravity	g	LT^{-2}
angle	θ_0	$M^0L^0T^0$
period	P	T

TABLE 2. Parameters influencing the motion of a simple pendulum.

$$P = \phi(m, l, g, \theta_0)$$

As we shall see, attempting to establish the form of ϕ directly is unnecessarily complicated. Instead, we pursue the idea of dimensional analysis.

To begin this modeling procedure, we compute the values of a, b, c, d and e that make the quantity

$$\pi = m^a l^b g^c \theta_0^d P^e$$

a dimensionless parameter. Again, this is done by equating exponents on the fundamental dimensions

$$M^0L^0T^0 = M^aL^b(LT^{-2})^c(M^0L^0T^0)^dT^e$$

From M^0: $0 = a$.
From L^0: $0 = b + c$.
From T^0: $0 = -2c + e$.
From this we may conclude that

$$\pi = m^0 l^{-c} g^c \theta_0^d P^{2c}$$

or, after collecting terms,

$$\pi = \theta_0^d (\frac{gP^2}{l})^c$$

where π is dimensionless for any values of d and c. Thus we have found a complete set of dimensionless parameters

$$\pi_1 = \theta_0$$

and

$$\pi_2 = \sqrt{\frac{g}{l}}P$$

Since the period P of the pendulum is based on dimensionally consistent physical laws we may apply the Buckingham π-theorem. In general,

$$f(\pi_1, \pi_2) = 0$$

which we rewrite as

$$\pi_2 = h(\pi_1)$$

which now becomes

$$P = \sqrt{\frac{l}{g}}h(\theta_0)$$

We may draw two immediate conclusions from this model.

- The period depends on the square root of the length of the pendulum.
- The period is independent of the mass

Of course we have not really shown these conclusions to be "true". But now we have something to look for that can be tested. We could test these assertions and if they contradict our model then we would conclude that we are missing an important factor that governs the period of the pendulum. Indeed, as we have neglected drag forces due to friction it seems our model will have limited validity.

The functional form of h may now be reasonable calculated as there is only one independent variable θ_0. If we select several different initial displacements $\theta_0(i)$ and measure the period for each one we have a set of domain–range values

$$h(\theta_0(i)) = P_i\sqrt{\frac{g}{l}}$$

to which a data fitting procedure may now be applied.

2.4.4.2. *The damped pendulum.* We assumed that there was no damping of this pendulum above due to air resistance. We can include a drag force F_D by augmenting the list of relevant parameters to

$$m, l, g, \theta_0, P, F_D$$

Now our dimensionless parameter takes the form

$$\pi = m^a l^b g^c \theta_0^d P^e F_D^f$$

Converting to dimensions

$$M^0L^0T^0 = M^aL^b(LT^{-2})^c(M^0L^0T^0)^dT^e(MLT^{-2})^f$$

As

$$0 = a + f$$

it is no longer possible to immediately conclude that $a = 0$. In fact, it is not. (See problems).

variable	symbol	dimensions
velocity	v	LT^{-1}
density	ρ	ML^{-3}
gravity	g	LT^{-2}
radius	l	L
viscosity	μ	$ML^{-1}T^{-1}$

TABLE 3. Parameters influencing the motion of a fluid around a submerged body.

2.4.4.3. *Fluid Flow.* Consider the parameters governing the motion of an oil past a spherical ball bearing. Let's assume they include:

The dimensionless combination has the form

$$\pi = v^a \rho^b l^c g^d \mu^e$$

Using the explicit form of the physical dimensions for each term we have

$$M^0 L^0 T^0 = (LT^{-1})^a (ML^{-3})^b (L)^c (LT^{-2})^d T^e (ML^{-1}T^{-1})^f$$

Again, matching exponents

$$M: \quad 0 = b + e$$
$$L: \quad 0 = a - 3b + c + d - e$$
$$T: \quad 0 = -a - 2d - e$$

Sinc there are three equations and five unknowns the system is said to be undetermined. Given these numbers, we anticipate that there we can solve for three variable in terms of the other two. Of course, we can solve in terms of *any* of the two variables. For example,

$$a = -2d - e$$
$$b = -e$$
$$c = d - e$$

Plugging these constraints into our expression for π gives

$$\pi = \Big(\frac{v^2}{lg}\Big)^d \Big(\frac{\rho l v}{\mu}\Big)^{-e}$$

Thus, our two dimensionless parameters are

$$\pi_1 = \frac{v^2}{lg}$$

and

$$\pi_2 = \frac{v\rho l}{\mu}$$

For further discussion see Giordano, Wells and Wilde, UMAP module 526.

Problems

PROBLEM 2.1. By drawing a new graph, show the effect of the size of the island on the

- extinction curve
- migration curve

Now predict how island size impacts the number of species on the island. Does this seem reasonable?

PROBLEM 2.2. Give an example of a commodity that does not obey the

- law of supply
- law of demand

and justify your claim.

PROBLEM 2.3. Translate into words the qualitative interpretation of the slope of the supply and demand curves. In particular, what is the meaning of a

- flat supply curve?
- steep supply curve?
- steep demand curve?

PROBLEM 2.4. Consider the table of market adjustments below. Assuming the first point is on the demand curve, compute the equations of both the demand and supply curve. Using these equations, find the missing values A, B, C, D. What is the equilibrium point? Do you think the market will adjust to it?

quantity	price
3	0.7
0.14	0.7
0.14	0.986
0.1972	0.986
$A =?$	$B =?$
$C =?$	$D =?$

PROBLEM 2.5. Using the cobweb plot show an example of a market adjustment that oscillates wildly out of control. Can you describe a qualitative feature of the supply and demand curves that will ensure convergence to an equilibrium?

PROBLEM 2.6. Consider the effect of a price increase on airplane fuel (kerosene) on the airline industry. What effect does this have on the supply curve? Will the airline industry be able to pass this cost onto the flying public? How does your answer differ if the demand curve is flat versus steep?

PROBLEM 2.7. Prove properties 2.1 and 2.2.

PROBLEM 2.8. Is the temperature measured in degrees Fahrenheit proportional to the temperature measured in degrees centigrade?

PROBLEM 2.9. Consider the Example 2.6 again. Demonstrate the proportionalities stated. For the case of the breaking distance equate the work done by the breaks to the dissipated kinetic energy of the car.

PROBLEM 2.10. Items at the grocery store typically come in various sizes and the cost per unit is generally smaller for larger items. Model the cost per unit weight by considering the superposition of proportions due to the costs of

- production
- packaging
- shipping

the product. What predictions can you make from this model. This problem was adapted from Bender [?].

PROBLEM 2.11. Go to your nearest supermarket and collect data on the cost of items as a function of size. Do these data behave in a fashion predicted by your model in the previous problem?

PROBLEM 2.12. In this problem take the diagonal of a rectangle as it's length scale l. Show by direct calculation that this can be used to measure the area, i.e.,

$$A = \alpha l^2$$

Determine the constant of proportionality α in terms of the shape factor of the rectangle.

PROBLEM 2.13. Consider a radiator designed as a spherical shell. If the characteristic length of the shell doubles (assume the larger radiator is geometrically similar to the smaller radiator) what is the effect on the amount of heat loss? What if the design of the radiator is a parallelpiped instead?

PROBLEM 2.14. How does the argument in Example 2.10 change if the falling object is not spherical but some other irregular shape?

PROBLEM 2.15. Extend the definition of geometric similarity for

- parallelpipeds
- irregularly shaped objects

Can you propose a computer algorithm for testing whether two objects are geomtrically similar?

PROBLEM 2.16. Consider the force on a pendulum due to air friction modeled by

$$F_D = \kappa v^2$$

Determine the units of κ.

PROBLEM 2.17. Newton's law of gravitation states that

$$F = \frac{G m_1 m_2}{r^2}$$

where F is the force between two objects of masses m_1, m_2 and r is the distance between them.

a) What is the physical dimension of G?
b) Compute two dimensionless products π_1 and π_2 and show explicitly that they satisfy the Buckingham π-theorem.

PROBLEM 2.18. This problem concerns the pendulum example described in subsection 2.4.4.1. Repeat the analysis to determine the dimensionless parameter(s) but now omit the gravity term g. Discuss.

PROBLEM 2.19. This problem concerns the pendulum example described in subsection 2.4.4.1. Repeat the analysis for determining all the dimensionless parameters but now include a parameter κ associated with the drag force of the form $F_D = \kappa v$. Hint: first compute the dimensions of κ.

PROBLEM 2.20. Convert the equation governing the distance traveled by a projectile

$$\frac{d^2x}{dt^2} = \frac{-gR^2}{(x+R)^2}$$

with the initial conditions

$$x(0) = 0 \qquad \frac{dx}{dt}(0) = V$$

to non-dimensional form using the transformation

$$y = \frac{x}{R} \qquad \tau = \frac{Vt}{R}$$

Verify the resulting equation is dimensionless. See [?] for further details.

PROBLEM 2.21. Reconsider the example in subsection 2.4.4.3. Instead of solving for a, b, c in terms of d, e solve for c, d, e in terms of a and b. Show that now

$$\pi'_1 = \frac{v}{\sqrt{lg}}$$

and

$$\pi'_2 = \frac{\rho l^{3/2} g^{1/2}}{\mu}$$

Show also that both π'_1 and π'_2 can be written in terms of π_1 and π_2.

PROBLEM 2.22. Consider an object with surface area A traveling with a velocity v through a medium with kinematic viscosity μ and density ρ.

a) Assuming the effect of μ is small compute the drag force due to the density F_ρ.
b) Assuming the effect of ρ is small compute the drag force due to the kinematic viscosity F_μ.
c) Compute the dimensionless ratio of these drag forces and discuss what predictions you can make.

CHAPTER 3

Data Fitting

In this chapter the model building is *empirical* in nature, i.e., the relationship between the dependent and independent variables is found by direct examination of data related to the process.

Data fitting problems have several common elements. The *model* has the general form

$$y = f(x_1, \ldots, x_N; w_1, \ldots, w_M)$$

and the *parameters* w_i are determined empirically from the *observations*

$$\{(x^{(i)}, y^{(i)})\}_{i=1}^{P}$$

by requiring f to be such that

$$y^{(i)} = f(x_1^{(i)}, \ldots, x_N^{(i)}; w)$$

or at least that the *error function* $E(w)$ defined by

$$S(w) = \sum_i (y^{(i)} - f(x_1^{(i)}, \ldots, x_N^{(i)}; w_1, \ldots, w_M))^2$$

be *small*. This error function seeks to minimize the sum of squares of the *residuals*. As we shall see, other error functions are possible but least squares is certainly the most widely used and we will focus on this approach at the outset.

EXAMPLE 3.1. A *Radial basis function* model has the form

$$f(x; w, c) = w_0 + \sum_{k=1}^{N_c} w_k \phi(\|x - c_k\|)$$

where the w_k are the weights and the c_k are the centers of the basis functions. An example of a radial basis function is

$$\phi(r) = \exp(-r^2)$$

The norm $\|\cdot\|$ is generally taken to be the Euclidean distance.

3.1. Linear Least Squares

In this section we begin by revisiting an example from the previous chapter followed by a general formulation of linear least squares and some simple extentions to exponential fits.

3.1.1. The Mammalian Heart Revisited. Recall from Example 2.11 in Section 2.3.2.1 that a sequence of proportionalities produced the model

$$r = kw^{-1/3}$$

where w is the body weight of a mammal and r is its heart rate. The data on Figure 8 corresponds to observations

$$\{(w_i^{-1/3}, r_i)\}$$

collected for various measured rates and weights. The residual error for the ith measurement is

$$\epsilon_i = r_i - kw_i^{-1/3}$$

and the total squared error is

$$E = \sum_{i=1}^{P} \epsilon_i^2$$

We rewrite this error as a function of the unknown slope parameter k as

$$E(k) = \sum_{i=1}^{P} (r_i - kw_i^{-1/3})^2$$

To minimize E as a function of k we compute the derivative of E w.r.t. k, i.e.,

$$\frac{dE}{dk} = \sum_{i=1}^{P} 2(r_i - kw_i^{-1/3}) \cdot (-w_i^{-1/3}) = 0$$

From which it follows that

$$k = \frac{\sum_{i=1}^{P} r_i w_i^{-1/3}}{\sum_{i=1}^{P} w_i^{-2/3}}$$

Thus we can obtain an estimate for the slope of the line empirically from the data.

3.1.2. General Formulation. In this section we focus our attention to one of the most widely used models

$$f(x; m, b) = mx + b$$

To clean-up the notation we now use subsripts to label points for domain data that is one dimensional; we used superscripts in the previous section when the dimension of the domain could exceed one. For a set of observations $\{(x_i, y_i)\}$, $i = 1, \ldots, P$, the total squared error is given by

$$E(m, b) = \sum_{i=1}^{P} (y_i - mx_i - b)^2$$

Now because there are two parameters that determine the error function the necessary condition for a minimum is now

$$\frac{\partial T}{\partial m} = 0$$
$$\frac{\partial T}{\partial b} = 0$$

Solving the above equations gives the slope of the line as

$$m = \frac{(\sum y_i)(\sum x_i) - P \sum y_i x_i}{(\sum x_i)^2 - P \sum x_i^2}$$

FIGURE 1. Linear least squares. The line $y = mx + b$ is determined such that the residuals ϵ_i^2 are minimized.

and its intercept to be

$$b = \frac{-(\sum y_i)(\sum x_i^2) + (\sum x_i)(\sum y_i x_i)}{(\sum x_i)^2 - P \sum x_i^2}$$

3.1.2.1. *Inpterpolation Condition.* In this section we present another route to the equations for m and b produced in the previous section. Again, the input data is taken as $\{x_i\}$, the output data is $\{y_i\}$ and the model equation is $y = mx + b$. Applying the *interpolation condition* for each observation we have

$$\begin{aligned} y_1 &= mx_1 + b \\ y_2 &= mx_2 + b \\ y_3 &= mx_3 + b \\ &\vdots \\ y_P &= mx_P + b \end{aligned}$$

In terms of matrices

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_P \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_P \end{pmatrix}$$

In terms of matrices we can summarize the above as

$$Xb = y$$

We can reveal the relationship between the previous approach using calculus and this approach with the interpolation condition by hitting both sides of the above matrix equation with the transpose X^T

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_P \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_P \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_P \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_P \end{pmatrix}$$

In terms of the matrices,

$$X^T X b = X^T y$$

Multiplying out produces the equations that are seen to be the same as those in the above section, i.e.,

$$\begin{pmatrix} P & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

In linear algebra these equations are referred to as the *normal* equations.

There are many algorithms in the field of numerical linear algebra developed precisely for solving the problem

$$Xb = y$$

We will consider these more in the problems.

3.1.3. Exponential Fits. We have already seen models of the form

$$y = kx^n$$

where n was given. How about if n is unknown? Can it be determined empirically from the data? Note now that the computation of the derivative of the error function w.r.t. n is now quite complicated. This problem is resolved by converting it to a linear least squares problem now in terms of logarithms. Specifically,

$$\begin{aligned} \ln y &= \ln(kx^n) \\ &= \ln k + \ln x^n \\ &= \ln k + n \ln x \end{aligned}$$

This is now seen to be a linear least squares problem

$$y' = nx' + k'$$

where we have made the substitutions $y' = \ln y, k' = \ln k$ and $x' = \ln x$. Now one can apply the standard least squares solution to determine n and k'. The value of k can be found as well by

$$k = \exp(k')$$

APPENDIX A

Matlab Code for Data Fitting

A.1. Mammalian Heart Rate Problem

```
File: ls_mammals.m
---------------------Start of actual file contents------------------------
%LEAST SQUARES ANALYSIS OF MAMMALIAN HEART RATE

%w body weights
%r corresponding heart rates
%data is a row vector
w = [3.5 4 6 25 103 117 200 252 300 437 1340 2000 2700 5000
     22500 30000 33000 50000  70000 100000 415000 450000 500000 3000000];
r = [787 660 588 670 347 300 420 352 300 269 251 205 187 120 100
     85 81 70 72 70 45 38 40 48];

x1 = w.^(-1/3) %raise each component of vector w to -1/3 power.
x2 = w.^(-2/3) %the result of this operation is a vector the same size as w
```

Now calculate the slope given by the formula:

$$k = \frac{\sum_{i=1}^{P} r_i w_i^{-1/3}}{\sum_{i=1}^{P} w_i^{-2/3}}$$

We will use the variables numerator and denominator to split up the calcution in the obvious fashion. The numerator is expressed as the vector dot product of r, the row vector of heart rates, and x1 as found above.

```
numerator = r*x1';%apply transpose operator ' to x1 to compute dot product.
denominator = sum(x2);%compute the sum of each component
k1 = numerator/denominator;
```

Now we reproduce the above calculation reproducing all the steps but by using a different data set to compute the slope. It would be more effiecient to pass the data to a subroutine rather than repeat all the code. We examine this in the next section.

```
%%Now build the model on the first 2/3 of the data (16 points)
ws = w(8:24)%the notation 8:24 is equivalent to [8 9 10 11 12 .... 24]
rs = r(8:24)

x1 = ws.^(-1/3)%raise each component of vector w to -1/3 power.
```

```
x2 = ws.^(-2/3)

numerator = rs*x1';%apply transpose operator ' to x1 to compute dot product.
denominator = sum(x2);%compute the sum of each component
k2 = numerator/denominator;%see formula in section 3.1.1

hold on
plot(w.^(-1/3),r,'o')%plot raw data
plot(w.^(-1/3),k1*w.^(-1/3),'--x')%plot first model
plot(w.^(-1/3),k2*w.^(-1/3),'--v')%plot second model
title('mammalian heart rate model')
xlabel('weight w^{(-1/3)}')
ylabel('pulse rate')
legend('raw data','least squares fit (all data)', 'least squares fit 2/3 data')
-----------------------End of actual file contents---------------------------------
```

A.2. Least Squares with Normal Equations

This program consists of two parts: a subroutine called ls_normal.m and a driver called run_ls.m.

```
File:  ls_normal.m
---------------------Start of actual file contents------------------------
%Input:
%   x is a column vector of domain (input) variables
%   y is a column vector of range (output) variables
%
%Output:
%   m is the slope of the line
%   b is the intercept of the line

function [m,b] = ls_normal (x,y)

P = size(x,1)%how many points are there in this column vector?
```

Now we compute the terms required in the evaluation of m and b in the normal equations. Recall

$$m = \frac{(\sum y_i)(\sum x_i) - P\sum y_i x_i}{(\sum x_i)^2 - P\sum x_i^2}$$

$$b = \frac{-(\sum y_i)(\sum x_i^2) + (\sum x_i)(\sum y_i x_i)}{(\sum x_i)^2 - P\sum x_i^2}$$

We set sy $= \sum y_i$, dp_xy $= \sum y_i x_i$ and x_sq $= \sum x_i^2$.

```
sy = sum(y);%sum y_i (scalar)
sx = sum(x);%sum x_i (scalar)
dp_xy = x'*y; %y dot product x (scalar)
x_sq = sum(x.*x);%sum x_i^2 term (scalar)
denom = sx^2 - P*x_sq; %(scalar)

m = (sy*sx-P*dp_xy)/denom
b = (-sy*x_sq+sx*dp_xy)/denom

%plot results
plot(x,y,'o')
hold
plot(x,m*x+b,'--v')
legend('data','model')
-----------------------End of actual file contents--------------------------------
```

The above subroutine is called using the following driver:

```
File: run_ls.m
---------------------Start of actual file contents------------------------
w = [3.5 4 6 25 103 117 200 252 300 437 1340 2000 2700 5000
     22500 30000 33000 50000 70000 100000 415000 450000 500000 3000000];
r = [787 660 588 670 347 300 420 352 300 269 251 205 187 120 100
     85 81 70 72 70 45 38 40 48];

x = w.^(-1/3)'%note that the transpose operator ' turns the row vec into col vec.
y = r'% the subroutine expects column vectors by design.oo

[m,b] = ls_normal (x,y)
-----------------------End of actual file contents--------------------------------
```

A.3. Least Squares with Overdetermined System

```
File: ls_interp.m
---------------------Start of actual file contents------------------------
%Input:
%   x is a column vector of domain (input) variables
%   y is a column vector of range (output) variables
%
%Output:
%   m is the slope of the line
%   b is the intercept of the line

function [m,b] = ls_interp(x,y)

%Compute the matrix X
```

Recall the equation we are solving in this problem:

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_P \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_P \end{pmatrix}$$

In terms of matrices we can summarize the above as

$$\mathrm{Xvec} = \mathrm{y}$$

This equation will be solved via matlab's backslash routine.

```
P = size(x,1)%how big is the data set?

c1 = ones(P,1)%create a column vector of ones of length P

X = [c1 x]% construct the "interpolation matrix"

vec = X\y% solve the least squares problem

b = vec(1)%obtain the first component (intercept)
m = vec(2)%obtain the slope

%plot results
plot(x,y,'o')
hold
plot(x,m*x+b,'--v')
legend('data','model')
-----------------------End of actual file contents--------------------------------
```

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