M331 Exam II

Instructions: this is a closed book quiz; calculators are permitted. Do all work on the sheets provided. Required physical dimensions provided on back page.
October 18, 2002.

Problem 1. Assume the temperature of a roast in the oven increase at a rate proportional to the difference between the oven (set to 400 degrees F) and the roast. If the roast enters the oven at 50 degrees F and is measured one hour later to be at 90 when should the table be set if the eating temperature is 166 degrees F? Hint: write down the difference equation and solve analytically.

16 pts – $\Delta T_n \propto (400 - T_n)$

$T_{n+1} = (1-k)T_n + 400k$

h.s. : $h_n = c\cdot(1-k)^n$

p.s. : $p_n = A$

$A = (1-k)A + 400k$

$A = 400$

$T_n = c(1-k)^n + 400$

$T_0 = 50 = c + 400 \Rightarrow c = -350$

$T_1 = 90 = 400 - 350(1-k) \Rightarrow k = 4/35$

$T_n = 400 - 350(31/35)^n$

done : $166 = 400 - 350(31/35)^n$

$n = 3.317 \sim 3$ hrs, 19 min.

Problem 2. Consider the first order

$$x_{n+1} = \frac{x_n}{3} - n + \frac{1}{3}$$

a) Determine the general solution to the nonhomogeneous problem.

b) Verify your solution is in fact a solution to the original problem.

c) Does this equation possess any equilibrium points?

d) Find the unique solution associated with the initial condition $x_0 = -2$.

$h_n = c\left(\frac13\right)^n$ $\quad p_n = An + B$

$An + A + B = \frac13 An + \frac13 B - n + \frac13$

n: $A = \frac13 A - 1$ $\quad$ c: $A + B = \frac13 B + 1/3$

$2/3\, A = -1$ $\quad$ $2/3\, B = 3/2 + 1/3$

$A = -3/2$ $\quad$ $B = 9/4 + 1/2 = 11/4$

$p_n = -3/2\, n + 11/4$

15 pts a) $x_n = c\left(\frac13\right)^n - \frac32 n + \frac{11}{4}$

2 pts b) LHS: $c\frac13\left(\frac13\right)^n - \frac32 n - \frac32 + \frac{11}{4} = \frac13 c\left(\frac13\right)^n - \frac32 n + \frac54$ ✓

RHS: $\frac13 c\left(\frac13\right)^n - \frac12 n + \frac13\cdot\frac{11}{4} - n + \frac13 = \frac13 c\left(\frac13\right)^n - \frac32 n + \frac13\left(\frac{11}{4} + \frac44\right)$ ✓ $\quad \frac{15}{4}$ $\quad \frac54$ ✓

3 pts c) $\bar{x} = x_{n+1} = x_n$

$\bar{x} = \frac13 \bar{x} - n + \frac13$

$\frac23 \bar{x} = -n + \frac13 \Rightarrow \bar{x} = \frac12 - \frac32 n$ $\quad$ no eq. pts

5 pts d) $x_0 = -2 = c + \frac{11}{4}$

$c = \frac{-19}{4} \quad \Rightarrow x_n = -\frac{19}{4}\left(\frac13\right)^n - \frac32 n + \frac{11}{4}$

Problem 3. Consider the model

$$f(x; c_1, c_2) = c_1 x^{\sqrt{2}} + c_2 x^{\sqrt{3}}$$

Using this model, apply the interpolation condition to each observation in the data set of P points

$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_P, y_P)\}$$

to deduce P equations in 2 unknowns. Next, write the resulting system in matrix form

$$Xc = y$$

where c is a column vector with components (c_1, c_2) and y is a column vector with components $(y_1, \ldots, y_P)$. (Confirm that your matrix X has P rows and 2 columns.) Lastly, show that the equation $Xc = y$ can be reduced to a 2×2 system for c_1 and c_2. (Note this system is what we referred to as the *normal form* of the least squares equations.)

$$\left.\begin{aligned} y_1 &= c_1 x_1^{\sqrt2} + c_2 x_1^{\sqrt3} \\ y_2 &= c_1 x_2^{\sqrt2} + c_2 x_2^{\sqrt3} \\ &\vdots \\ y_P &= c_1 x_P^{\sqrt2} + c_2 x_P^{\sqrt3} \end{aligned}\right\} \text{P eq}^{\text{ns}} \text{ 2 unk.}$$

17 pts

$$\begin{bmatrix} x_1^{\sqrt2} & x_1^{\sqrt3} \\ x_2^{\sqrt2} & x_2^{\sqrt3} \\ \vdots & \vdots \\ x_P^{\sqrt2} & x_P^{\sqrt3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_P \end{bmatrix} \quad : Xc = y$$

2×2 system by $X^T X c = X^T y$

5 pts
$$\begin{bmatrix} x_1^{\sqrt2} & x_2^{\sqrt2} & \cdots & x_P^{\sqrt2} \\ x_1^{\sqrt3} & x_2^{\sqrt3} & \cdots & x_P^{\sqrt3} \end{bmatrix} \begin{bmatrix} x_1^{\sqrt2} & x_1^{\sqrt3} \\ x_2^{\sqrt2} & x_2^{\sqrt3} \\ \vdots & \vdots \\ x_P^{\sqrt2} & x_P^{\sqrt3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_1^{\sqrt2} & \cdots & x_P^{\sqrt2} \\ x_1^{\sqrt3} & \cdots & x_P^{\sqrt3} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_P \end{bmatrix}$$

3 pts
$$\begin{bmatrix} \sum x_i^{2\sqrt2} & \sum x_i^{\sqrt2+\sqrt3} \\ \sum x_i^{\sqrt2+\sqrt3} & \sum x_i^{2\sqrt3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sum x_i^{\sqrt2} y_i \\ \sum x_i^{\sqrt3} y_i \end{bmatrix}$$

Problem 4. Consider the linear system of difference equations

$$a_{n+1} = c_1 a_n + d_1 b_n^4$$

$$b_{n+1} = c_2 a_n^3$$

Given a set of observations $\{(a_0, b_0), (a_1, b_1), \ldots, (a_P, b_P)\}$ estimate via least squares the coeffients c_1, d_1, c_2. Your solution should consist of 3 equations for the three unknowns c_1, d_1, c_2. Solve explicity only for c_2. [Write dowm but do not solve the 2×2 system for c_1, d_1.]

15 pts $E = \sum (a_{n+1} - c_1 a_n - d_1 b_n^4)^2 + \sum (b_{n+1} - c_2 a_n^3)^2$

$\frac{\partial E}{\partial c_1} = \sum 2(a_{n+1} - c_1 a_n - d_1 b_n^4)(-a_n) = 0$

$\Rightarrow \sum a_{n+1} a_n - c_1 \sum a_n^2 - d_1 \sum a_n b_n^4 = 0$

5 pts $\frac{\partial E}{\partial d_1} = \sum 2(a_{n+1} - c_1 a_n - d_1 b_n^4)(-b_n^4)$

$\Rightarrow \sum a_{n+1} b_n^4 - c_1 \sum a_n b_n^4 - d_1 \sum b_n^8 = 0$

3 pts $\Rightarrow \begin{bmatrix} \sum a_n^2 & \sum a_n b_n^4 \\ \sum a_n b_n^4 & \sum b_n^8 \end{bmatrix} \begin{bmatrix} c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} \sum a_{n+1} a_n \\ \sum a_{n+1} b_n^4 \end{bmatrix}$

$\frac{\partial E}{\partial c_2} = \sum 2(b_{n+1} - c_2 a_n^3)(-a_n^3)$

$\Rightarrow \sum b_{n+1} a_n^3 - c_2 \sum a_n^6 = 0$

2 pts. $\quad c_2 = \dfrac{\sum b_{n+1} a_n^3}{\sum a_n^6}$