Homework # 7 solutions – M331 – F2002
Avg: 13/15, Median: 13.5/15
Note problem 4.23 was added in as extra credit.

Problem 4.14

$$\Delta a_n = \underbrace{k_1 a_n b_n}_{\text{A eats B}} + \underbrace{k_2 a_n c_n}_{\text{A eats C}} - \underbrace{k_3 a_n c_n}_{\text{C eats A}} + \underbrace{k_8 a_n}_{\text{A increases in absence of B \& C}}$$

$$\Delta b_n = -\underbrace{k_4 a_n b_n}_{\text{A eats B}} + \underbrace{k_5 b_n}_{\text{B eats B's waste}} + \underbrace{k_6 a_n}_{\text{B eats A's waste}} + \underbrace{k_7 b_n}_{\text{B increases in absence of A \& C}}$$

$$\Delta c_n = -\underbrace{k_9 a_n c_n}_{\text{A eats C}} + \underbrace{k_{10} a_n c_n}_{\text{C eats A}} - \underbrace{k_{11} c_n}_{\text{C decreases in absence of A \& B}} - \underbrace{k_{12} c_n^2}_{\text{C competes with C for food.}}$$

Problem 4.20

Part a:

a) $$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \sin x = 0$$

using given approximations for derivatives

$$\frac{x_{n+1} + x_{n-1} - 2x_n}{(\Delta t)^2} + \alpha\,\frac{x_n - x_{n-1}}{\Delta t} + \sin x_n = 0$$

$$x_{n+1} + x_{n-1} - 2x_n + \alpha\Delta t\,(x_n - x_{n-1}) + (\Delta t)^2 \sin x_n = 0$$

$$x_{n+1} + (\alpha\Delta t - 2)x_n + (1 - \alpha\Delta t)x_{n-1} + (\Delta t)^2 \sin x_n = 0$$

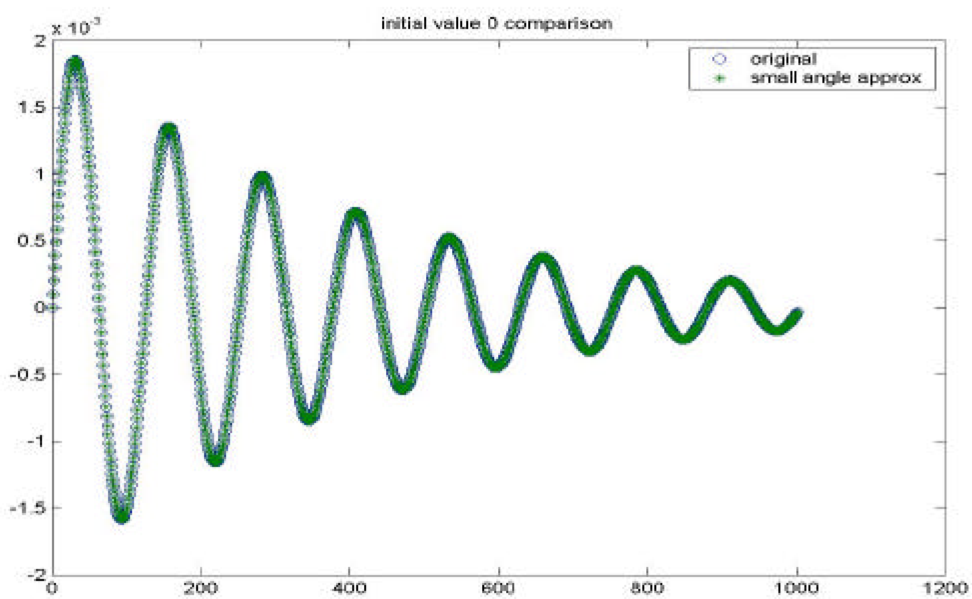
$$x_{n+1} = \underline{(2 - \alpha\Delta t)x_n + (\alpha\Delta t - 1)x_{n-1} - (\Delta t)^2 \sin x_n}$$

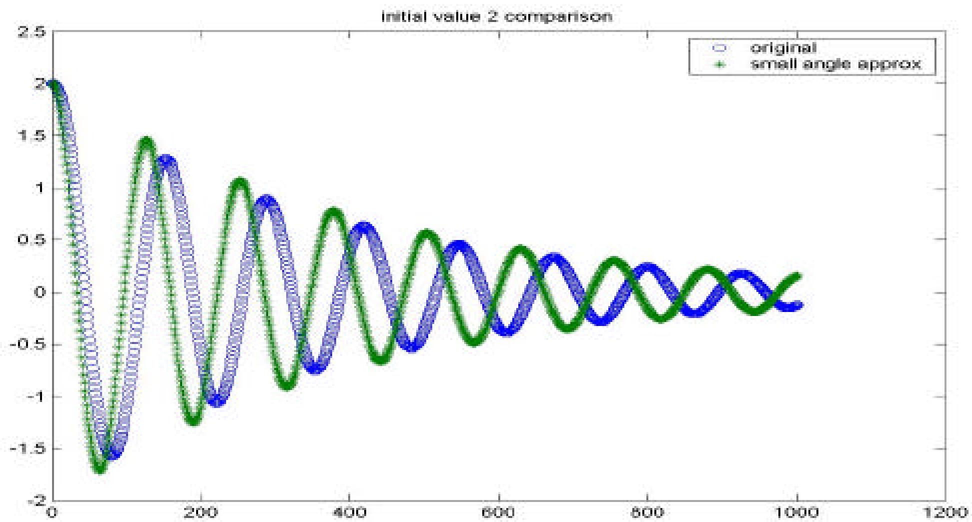
Parts b & c:

MATLAB CODE:

```
        close all; clear all;
alpha = 0.1; dt = 0.05;
x(1) = 0; x(2) = 0.0001;
for i = 1:1000;
  x(i+2) = (2-alpha*dt)*x(i+1) + (alpha*dt-1)*x(i)-dt^2*sin(x(i+1));
end;
y1=x;
```

```
x(1) = 2; x(2) = 2.0001;
for i = 1:1000;
   x(i+2) = (2-alpha*dt)*x(i+1) + (alpha*dt-1)*x(i)-dt^2*sin(x(i+1));
end;
x1=x;
z=1:1002;
            plot(z,y1,'o',z,x1,'*')
            legend('close to 0', 'close to 2');
            title('equation 4.16')
%small angle
alpha = 0.1; dt = 0.05;
x(1) = 0; x(2) = 0.0001;
for i = 1:1000;
   x(i+2) = (2-alpha*dt)*x(i+1) + (alpha*dt-1)*x(i)-dt^2*x(i+1);
end;
y2=x;
x(1) = 2; x(2) = 2.0001;
for i = 1:1000;
   x(i+2) = (2-alpha*dt)*x(i+1) + (alpha*dt-1)*x(i)-dt^2*x(i+1);
end;
x2=x;
figure;
            plot(z,y2,'o',z,x2,'*')
            legend('close to 0', 'close to 2');
            title('equation 4.17')
figure;
            plot(z,x1,'o',z,x2,'*')
            legend('original', 'small angle approx');
            title('initial value 2 comparison')
figure
            plot(z,y1,'o',z,y2,'*')
            legend('original', 'small angle approx');
            title('initial value 0 comparison')
```





The small angle approximation is not a good approximation for the initial values near two. Makes sense since then we are not in the small angle domain.

Part d:

d) 1st eqn: $y_{n+1} = x_n \Rightarrow y_n = x_{n-1}$

so sub. into (4.16) get 2nd eqn: $x_{n+1} = (2 - a\Delta t)x_n + (a\Delta t - 1)y_n - (\Delta t)^2 \sin x_n$

equilibria: $\bar{y} = y_{n+1} = y_n$, $\bar{x} = x_{n+1} = x_n$

(1): $\bar{y} = \bar{x}$

(2): $\bar{x} = (2 - a\Delta t)\bar{x} + (a\Delta t - 1)\bar{y} - (\Delta t)^2 \sin \bar{x}$

(1) into (2):

$$\bar{x} = 2\bar{x} - a\Delta t\,\bar{x} + a\Delta t\,\bar{x} - \bar{x} - (\Delta t)^2 \sin \bar{x}$$

$$\bar{x} = \bar{x} - (\Delta t)^2 \sin \bar{x}$$

$$0 = \sin \bar{x}$$

$$\bar{y} = \bar{x} = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Part e:

e) Jacobian: $Dh(x,y) = \begin{vmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{vmatrix}$

$h_1(x,y) = (2-\alpha\Delta t)x + (\alpha\Delta t - 1)y - (\Delta t)^2 \sin x$
$h_2(x,y) = x$
} from system of eq^ns found in (d).

$$Dh(x,y) = \begin{bmatrix} (2-\alpha\Delta t) - (\Delta t)^2 \cos x & (\alpha\Delta t - 1) \\ 1 & 0 \end{bmatrix}$$

at even π multiples $(0, 2\pi, 4\pi \ldots)$ $\cos(x) = 1$
at odd " " $(\pi, 3\pi, 5\pi, \ldots)$ $\cos(x) = -1$

even π mult:

$$Dh(2n\pi, 2n\pi) = \begin{bmatrix} 2-\alpha\Delta t - (\Delta t)^2 & \alpha\Delta t - 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1.9925 & -.995 \\ 1 & 0 \end{bmatrix}$$

eig in matlab $\Rightarrow$ $\lambda_{\pm} = .9963 \pm .0499i$

$|\lambda_{\pm}| = \sqrt{(.9963)^2 + (.0499)^2} = .9975 < 1$

$\therefore$ stable

odd π mult:

$$Dh((2n+1)\pi, (2n+1)\pi) = \begin{bmatrix} 2-\alpha\Delta t + (\Delta t)^2 & \alpha\Delta t - 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1.9975 & -.995 \\ 1 & 0 \end{bmatrix}$$

eig in matlab $\Rightarrow$ $\lambda_1 = 1.0488 > 1$
$\lambda_2 = 0.9487$

$\therefore$ unstable

Problem 4.22 & 4.23

Solution to Problem 4.22: Analytically solve the linear difference equation from the previous problem

$$x_{n+1} = (2 - \alpha\Delta t)x_n + (\alpha\Delta t - 1)x_{n-1} - (\Delta t)^2 x_n$$

and compare with your numerical simulation above. For simplicity you may take $\Delta t = 0.05, \alpha = 0.1, x_1 = 2, x_2 = 2.0001$.

Let's write the problem as

$$x_{n+1} + ax_n + bx_{n-1} = 0$$

where

$$a = \alpha\Delta t + (\Delta)^2 - 2 = -1.9925$$

and

$$b = 1 - \alpha\Delta t = 0.995$$

The associated characteristic equation is

$$\lambda^2 + a\lambda + b = 0$$

which has solutions (after factoring out i)

$$\lambda_{\pm}^n = (-\frac{a}{2} \pm i\frac{\sqrt{4b - a^2}}{2})^n$$

Observe that since $a^2 - 4b = -0.00994375 < 0$ the roots are imaginary. Thus, our solutions can be written in the imaginary form

$$x_n = c_1\lambda_+^n + c_2\lambda_-^n$$

To reformulate the solution as a real solution we compute λ_+ in polar form. Let

$$\lambda_+ = x + iy = -a/2 + i\frac{\sqrt{4b - a^2}}{2}$$

so $x = -a/2$ and $y = \sqrt{4b - a^2}/2$. We showed in Section 4.1.2 that given

$$x = r\cos\theta, y = r\sin\theta$$

the

$$r = \sqrt{b} = 0.99749686716300, \quad \theta = \tan^{-1}(\frac{\sqrt{4b - a^2}}{-\alpha}) = 0.05000513115822$$

Using

$$\lambda_+^n = (r\exp(i\theta))^n = r^n \exp(in\theta) = r^n\cos(n\theta) + ir^n\sin(n\theta)$$

and recalling that both the imaginary and real parts are solutions we obtain

$$x_n = c_1 r^n\cos(n\theta) + c_2 r^n\sin(n\theta)$$

To simplify the calculation (try it both ways to see why) we will shift the indices of the initial conditions to $0, 1$ so

$$x_0 = c_1$$

$$x_1 = c_1 r \cos(\theta) + c_2 r \sin\theta$$

which gives

$$c_2 = (x_1 {\scriptstyle\frac{a}{2}} x_0)\frac{2}{\sqrt{4b - a^2}} = 0.15242931201143$$

So the analytical solution to the homogeneous problem is

$$x_n = 2(0.995)^{n/2}\cos(0.05n) + 0.1524(0.995)^{n/2}\sin(0.05n)$$

□

Solution to Problem 4.23: Analytically solve the linear nonhomogeneous difference equation

$$x_{n+1} = (2 - \alpha\Delta t)x_n + (\alpha\Delta t - 1)x_{n-1} - (\Delta t)^2 x_n + 0.01\sin(n/50)$$

Simulate this problem numerically and compare with your analytical solution for 2000 iterations. Can you identify a transient (i.e., a term that goes to zero) and steady state (persisent) components of your solution? Again, for simplicity you may take $\Delta t = 0.05, \alpha = 0.1, x_1 = 2, x_2 = 2.0001$. Hint: combine your solution to the homogeneous problem found above with a particular solution of the form

$$p_n = A\cos(n/50) + B\sin(n/50)$$

Solve for the undetermined coefficients A and B.

Plugging $p_n = A\cos(n/50) + B\sin(n/50)$ into the nonhomogeneous problem we have

$$A\cos\frac{n+1}{50} + B\sin\frac{n+1}{50} + a(A\cos\frac{n}{50} + B\sin\frac{n}{50}) + b(A\cos\frac{n-1}{50} + B\sin\frac{n-1}{50}) = .01\sin\frac{1}{50}$$

Letting $n = 1$ gives

$$A(\cos\frac{2}{50} + a\cos\frac{1}{50} + b) + B\sin(\frac{2}{50} + a\sin\frac{1}{50}) = .01\sin\frac{1}{50}$$

and letting $n = 2$ gives

$$A(\cos\frac{3}{50} + a\cos\frac{2}{50} + b\cos\frac{1}{50}) + B(\sin\frac{3}{50} + a\sin\frac{2}{50} + b\sin\frac{1}{50}) = .01\sin\frac{1}{50}$$

Recall the values of $a = -1.9925$ and $b = 0.995$ now we can solve to find

$$A = -0.22601165839997, \quad B = 4.74885158590701$$

So the solution is the sum of the homogenous part from above (before the constants are evaluated!!) and this particular solution

$$x_n = c_1(0.995)^{n/2}\cos(.05n) + c_2(0.995)^{n/2}\sin(.05n) - 0.2260\cos(n/50) + 4.7489\sin(n/50)$$

Setting $x_0 = 2$ and $x_1 = 2.0002$ one gets

$$c_1 = 2.22601165839997, \quad c_2 = -1.73625725291089$$

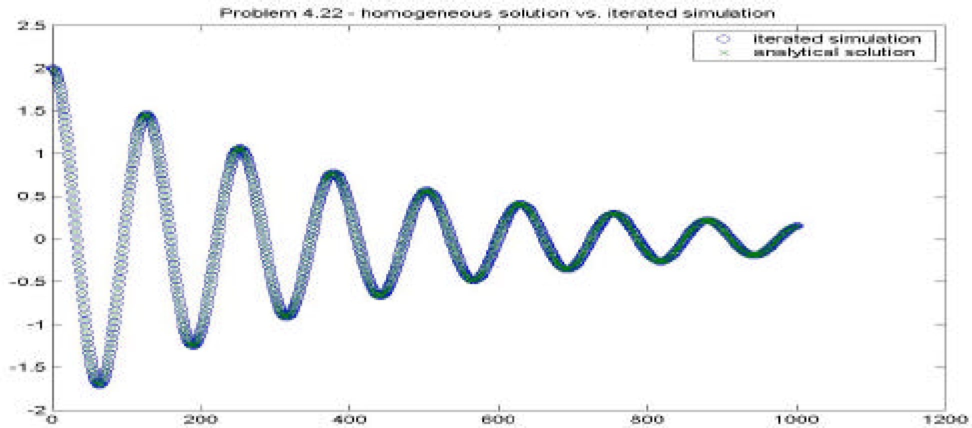
Note that if you use $x_1 = 2$ and $x_2 = 2.0002$ you should obtain

$$c_1 = 2.22108702188432, c_2 = -1.63986237742405$$

Note that the following codes verify that all is right since the numerical and analytical solutions are the same.

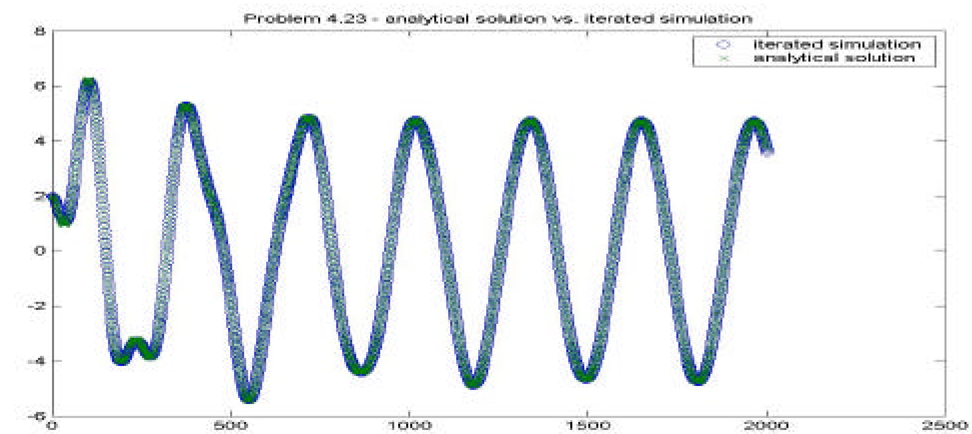
MATLAB code (4.22):

```
close all; clear all;
alpha = 0.1; dt = 0.05;
x(1) = 2; x(2) = 2.0001;
for i = 1:1000;
   y(i) = 2*(.9975)^i*cos(.05*i) + .1525*(.9975)^i*sin(.05*i);
   x(i+2) = (2-alpha*dt)*x(i+1) + (alpha*dt-1)*x(i) - dt^2*x(i+1);
end;
z=1:1002;
plot(z,x,'o',z(1:1000),y,'x');  legend('iterated simulation','analytical solution')
title('Problem 4.22 - homogeneous solution vs. iterated simulation')
```



MATLAB CODE (4.23):

```
close all; clear all;
alpha = 0.1; dt = 0.05;
x(1) = 2; x(2) = 2.0001;
for i = 1:2000;
   y(i) = 2.226*(.9975)^i*cos(.05*i)  - 1.7363*(.9975)^i*sin(.05*i) - .226*cos(i/50) + 4.7489*sin(i/50);
   x(i+2) = (2-alpha*dt)*x(i+1) + (alpha*dt-1)*x(i) - dt^2*x(i+1) + .01*sin((i+1)/50);
end;
z=1:2002;
plot(z,x,'o',z(1:2000),y,'x'); legend('iterated simulation','analytical solution')
title('Problem 4.23 - analytical solution vs. iterated simulation')
```



The homogeneous term is the transient part of the solution (it decays) and the particular solution is the persistent part of the solution (it doesn't decay).