DECOMPOSING (EQUIANGULAR) TIGHT FRAMES INTO (EQUIANGULAR) TIGHT FRAMES FOR THEIR SPANS

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Given a frame, it is of interest to ask if a subset of frame vectors also have certain frame properties. The answer to this question has implications for the robustness of the frame representation to erasures [GKK01, CK03] and appropriateness of the frame in compressed sensing applications [Ela10, BDE09] and is sometimes characterized by combinatorial structures [FJKM18, Cah09, Kin15, BLR15]. In order to state our main research question, we need a definition.

Definition 0.1. (see, e.g., [FMJ16]) Let $\Phi = \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \end{pmatrix} \in \mathbb{C}^{d\times n}$. We call Φ (or, more precisely, its columns) a *tight frame for its span* if there exists an $A > 0$ such that $(\Phi\Phi^*)^2 = A\Phi\Phi^*$; that is, $\Phi\Phi^*$ is a positive multiple of an orthogonal projection. If $\Phi\Phi^* = AI$, then Φ is a *tight frame*. If further each of the columns of Φ has unit norm and there exists an $\lambda \geq 0$ such that $|\langle \varphi_j, \varphi_k\rangle| = \lambda$ for all $j \neq k$, then we call Φ an *equiangular tight frame for its span* (respectively, *equiangular tight frame*).

Question 0.2. If Φ is an (equiangular) tight frame, can it non-trivially be decomposed into the disjoint union of (equiangular) tight frames for their spans?

This generalizes questions worked on in [BLR15, FJKM18, CFM+11, CFH+12, ABDF17, Hug07, DBBA13, LMO14, DS06] and more. Since Φ itself is assumed to be an (equiangular) tight frame, it is also an (equiangular) tight frame for its span, yielding one trivial decomposition. On the other extreme, a non-zero (unit norm) vector is trivially an (equiangular) tight frame for its span, and Φ is the disjoint union of its vectors.

A selection of results proven during the mini-workshop includes the following.

Proposition 0.3. *Let Φ be a tight frame. Then if $\Phi = \Psi_1 \sqcup \Psi_2$, with Ψ_1, Ψ_2 tight frames for their spans, one of the following must hold:*

(1) $\operatorname{span}\Psi_1 \perp \operatorname{span}\Psi_2$ *or*
(2) $\operatorname{span}\Psi_1 = \operatorname{span}\Psi_2 = \operatorname{span}\Phi$.

If further Φ as an equiangular tight frame with Ψ_1, Ψ_2 equiangular tight frames for their spans, then

(1') *Φ is an orthonormal basis of at least 2 vectors.*

Proof. It follows from the hypotheses that there exist $A, \alpha, \beta > 0$ and orthogonal projections P_1, P_2 such that

$$AI = \Phi\Phi^* = \Psi_1\Psi_1^* + \Psi_2\Psi_2^*, \quad \Psi_1\Psi_1^* = \alpha P_1 \quad \text{and} \quad \Psi_2\Psi_2^* = \beta P_2 \quad \Rightarrow \quad I - \frac{\alpha}{A}P_1 = \frac{\beta}{A}P_2.$$

By comparing the spectra of $I - \frac{\alpha}{A}P_1$ and $\frac{\beta}{A}P_2$, one obtains that either $P_1 = P_2 = I$ or $1 - \frac{\alpha}{A} = 0$ and thus $P_2 = P_1^\perp$. The second claim follows by comparing the inner products of the vectors using the so-called Welch bound (see, e.g. [Wal18, SH03]). $\square$

Theorem 0.4. *Let $\Phi \in \mathbb{C}^{d\times n}$ be a tight frame with frame bound A. Assume that $\Phi = \bigsqcup_{j=1}^3 \Psi_j$, with Ψ_j, $j = 1, 2, 3$ tight frames for their spans. Then for each $j = 1, 2, 3$ there exists $r_j, n_j \in \mathbb{N}$, $\alpha_j > 0$ and a rank-r_j orthogonal projection P_j such that $\Psi_j \in \mathbb{C}^{d\times n_j}$ and $\Psi_j\Psi_j^* = \alpha_j P_j$. Further, if Φ does not decompose in the sense of (2) of Proposition 0.3, then exactly one of the following must hold:*

(1) $r_1 + r_2 + r_3 = d$ *and* $\alpha_1 = \alpha_2 = \alpha_3 = A$;
(2) $r_1 = r_2$, $r_3 = d - r_1$, $\alpha_1 + \alpha_2 = A$, *and* $\alpha_3 = A$;
(3) $(d - r_1) + (d - r_2) + (d - r_3) = d$ *and* $\alpha_1 = \alpha_2 = \alpha_3 = \frac{A}{2}$; *or*
(4) $r_1 = r_2 = r_3 = \frac{d}{2}$, $\alpha_1 + \alpha_2 + \alpha_3 = 2A$, *and* $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq A$.

If further Φ is an equiangular tight frame with Ψ_j, $j = 1, 2, 3$ equiangular tight frames for their spans, then one of the following must hold:

(1') Φ *is an orthonormal basis of at least* 3 *vectors.*
(3') $n \equiv 9 \mod 12$, $d = \frac{n+3}{4}$ *and for* $j = 1, 2, 3$, $n_j = \frac{n}{3}$, *and* $r_j = \frac{n+3}{6}$; *or*
(4') $n \equiv 3 \mod 12$, $d = \frac{n+1}{2}$ *and for* $j = 1, 2, 3$, $n_j = \frac{n}{3}$, *and* $r_j = \frac{n+1}{4}$.

Proof. The proof uses Knutson-Tao honeycombs [KT99, KT01, KTW04] and the Welch bound. □

There are examples of each of the configurations in Theorem 0.4.

(1') Trivial.

(3') Let ζ be a primitive 3rd root of unity. Set

$$\Phi = \left(\begin{array}{c|c|c} \Psi_1 & \Psi_2 & \Psi_3 \end{array} \right) = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & -1 & -\zeta & -\zeta^2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -\zeta & -\zeta^2 \\ -1 & -\zeta & -\zeta^2 & 1 & 1 & 1 & 0 & 0 & 0 \end{array} \right).$$

Then Φ is an equiangular tight frame of 9 vectors in $\mathbb{C}^3$, and for each $j = 1, 2, 3$, $(\Psi_j \Psi_j^*)^2 = \frac{3}{2}\Psi_j \Psi_j^*$. This Φ is an example of a symmetric, informationally complete, positive operator-valued measure [Hug07, DBBA13] and a Gabor-Steiner equiangular tight frame [BK18, FMT12].

(4') Let ζ be a primitive 15th root of unity. Set

$$\Phi = \frac{1}{\sqrt{8}} \left(\zeta^{k\ell}\right)_{k \in D, \ell \in \{0,1,\ldots,14\}}, \quad D = \{0, 1, 2, 3, 5, 7, 8, 11\}$$

and for $j = 0, 1, 2$

$$\Psi_j = \frac{1}{\sqrt{8}} \left(\zeta^{k(j+3\ell)}\right)_{k \in D, \ell \in \{0,1,\ldots,4\}}.$$

Then Φ is an equiangular tight frame of 15 vectors in $\mathbb{C}^8$, and for each $j = 0, 1, 2$, $(\Psi_j \Psi_j^*)^2 = \frac{5}{4}\Psi_j \Psi_j^*$. This Φ is an example of a Naimark complement (see, e.g. [Wal18]) of an equiangular tight frame generated by $(3, 2)$-Singer difference set (see, e.g. [XZG05]) and the decomposition appears as Example 7.2 in [FJKM18].

In on-going work started at Oberwolfach, we are concerned with higher order decompositions, nested decompositions, the relationship of the decompositions with various dualities in frame theory, and constructions of infinite classes of (equiangular) tight frames which yield each possible type of configuration.

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