On the continuous Zauner conjecture

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SICs and Zauner's conjecture

Denote by $H = \mathbb{C}^d$ a complex Hilbert space of dimension d.

Zauner's (weak) conjecture:
For any $d \geqslant 2$ there exist d^2 unit vectors $\{|x_i\rangle\}_{i=1}^{d^2} \in H$ such that

1. the frame is tight: $\sum_{i=1}^{d^2} |x_i\rangle\langle x_i| = dI$
2. it's equiangular: $|\langle x_i|x_j\rangle|^2 = \mathrm{const} = \frac{1}{d+1}$ for $i \neq j$

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Such a set $\{|x_i\rangle\}$ is a maximal ETF. It's also called a SIC, since $\{\frac{1}{d}|x_i\rangle\langle x_i|\}$ forms a symmetric, informationally complete, positive operator-valued measure.

Symmetric and asymmetric subspaces

The symmetric subspace of $H \otimes H$ is defined by

$$H_{\mathrm{sym}} = \mathrm{span}\{|\phi\rangle\,|\phi\rangle\}_{|\phi\rangle \in H}.$$

It's the $(+1)$ eigenspace of the swap (flip) operator U_{sw},

$$U_{\mathrm{sw}}\,|i\rangle\,|j\rangle = |j\rangle\,|i\rangle\,, \quad \forall i, j \in [d].$$

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It's complement, the asymmetric subspace H_{asym}, is the (-1) eigenspace of U_{sw},

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Their dimensions are

$$\mathrm{dim} H_{\mathrm{sym}} = \frac{d(d+1)}{2}, \quad \mathrm{dim} H_{\mathrm{asym}} = \frac{d(d-1)}{2}.$$

By Π_{sym}, Π_{asym} we denote the projectors onto H_{sym}, H_{asym} respectively.

Projective 2-designs

A set of unit vectors $\{|x_i\rangle\}_{i=1}^n \in \mathbb{C}^d$ is a weighted projective 2-design if

$$\sum_{i=1}^{n} w_i \left(|x_i\rangle\langle x_i| \right)^{\otimes 2} = \frac{2}{d(d+1)} \Pi_{\mathrm{sym}} = \mathop{\mathbb{E}}_{\phi} \left(|\phi\rangle\langle\phi| \right)^{\otimes 2},$$

where $w_i \geqslant 0$, $\sum_i w_i = 1$.
It's a projective 2-design if $w_i = \frac{1}{n}$, $i \in [n]$.
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A SIC is a minimal projective 2-design.
Conversely[1], a weighted projective 2-design with $n = d^2$ elements is a SIC (which means the weights must be equal).

[1]Scott, 2006

Quantum channels, Choi's matrix, Kraus decomposition

A linear map $\Phi : \mathbb{C}^{d\times d} \to \mathbb{C}^{r\times r}$ is a quantum channel if it is

1. trace preserving: $\mathrm{Tr}(\Phi(X)) = \mathrm{Tr}(X)$ for any X
2. completely positive: $\mathcal{I}_n \otimes \Phi$ is a positive map for any n, i.e. $(\mathcal{I}_n \otimes \Phi)(\rho) \geqslant 0$ for any $\rho \geqslant 0$ from $\mathbb{C}^{n\times n} \otimes \mathbb{C}^{d\times d}$.

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The Choi matrix of a linear map Φ is defined by

$$\mathcal{C}(\Phi) = (\mathcal{I}_d \otimes \Phi)(\frac{1}{d}\sum_{ij} E_{ij} \otimes E_{ij}) = \frac{1}{d}\sum_{ij} E_{ij} \otimes \Phi(E_{ij}).$$

Note that the map $\mathcal{C}$ is linear and invertible.

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Note that the map $\mathcal{C}$ is linear and invertible.

The following are equivalent[2]:

- a map Φ is completely positive
- $\mathcal{C}(\Phi) \geqslant 0$
- there exists a Kraus decomposition

$$\Phi(X) = \sum_k A_k X A_k^\dagger, \quad A_k \in \mathbb{C}^{r\times d}.$$

[2]Choi, 1972

Separability and its length

A state $\rho \in \mathbb{C}^{d\times d} \otimes \mathbb{C}^{r\times r}$ is separable if there exists a decomposition

$$\rho = \sum_{k=1}^{m} \lambda_k \rho_k^{(1)} \otimes \rho_k^{(2)},$$

where $\lambda_k \geqslant 0$, $\sum_k \lambda_k = 1$, and $\rho_k^{(1)}$, $\rho_k^{(2)}$ are states on the corresponding subsystems.
Otherwise ρ is called entangled.

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Clearly, any separable ρ also has a pure separable decomposition, where all $\rho_k^{(1)}$, $\rho_k^{(2)}$ are pure states.
The length of separability, denoted by $\mathrm{len}(\rho)$, is the minimum number of summands in a pure separable decomposition of ρ.
An immediate consequence is that $\mathrm{len}(\rho) \geqslant \mathrm{rank}(\rho)$. It follows from Caratheodory's theorem that $\mathrm{len}(\rho) \leqslant d^2 r^2$.

Entanglement breaking channels

A quantum channel Φ is entanglement breaking if $(\mathcal{I}_n \otimes \Phi)(\rho)$ is separable for any n and any state ρ on $\mathbb{C}^{n\times n} \otimes \mathbb{C}^{d\times d}$.

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The following are equivalent[3]:

- a channel Φ is entanglement breaking
- $\mathcal{C}(\Phi)$ is separable
- there exists a Kraus decomposition

  $$\Phi(X) = \sum_k A_k X A_k^\dagger, \quad A_k \in \mathbb{C}^{r\times d},$$

  where Kraus operators A_k are rank one matrices.

[3] Horodecki, Shor, Ruskai, 2003

Quantum depolarising channels

Consider the maps $\Phi_t : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$ defined by

$$\Phi_t(X) = t \cdot X + (1-t) \cdot \mathrm{Tr}(X)\frac{1}{d}I_d, \quad t \in \mathbb{R}.$$

In other words, Φ_t is a linear combination of

$$\Phi_0(X) = \mathrm{Tr}(X)\frac{1}{d}I_d, \quad \Phi_1(X) = X.$$

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Maps Φ_t are quantum channels for $t \in [-\frac{1}{d^2-1}, 1]$.
But entanglement breaking only for $t \in [-\frac{1}{d^2-1}, \frac{1}{d+1}]$.

Entanglement breaking rank and Zauner's conjecture

Entanglement breaking rank[4] of EB map Φ, denoted by $\mathrm{ebr}(\Phi)$, is the minimum number of summands in the Kraus decomposition of Φ, where A_k are rank one.

[4] Pandey, Paulsen, Prakash, Rahaman 2020; Paulsen CodEx Talk 21.07.2020

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Conjecture: $\mathrm{ebr}(\Phi_t) = d^2$ for any $d \geqslant 2$ and $t \in [-\frac{1}{d^2-1}, \frac{1}{d+1}]$.

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Proved for $d = 2$ and $d = 3$ via explicit construction of Kraus decompositions, where rank one A_k are continuous over t.

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Proved for $d = 2$ and $d = 3$ via explicit construction of Kraus decompositions, where rank one A_k are continuous over t.

Since ebr is lower semi-continuous it's enough to prove the conjecture only for $t \in [-\frac{1}{d^2-1}, \frac{1}{d+1})$.

[4] Pandey, Paulsen, Prakash, Rahaman 2020; Paulsen CodEx Talk 21.07.2020

EBR and length of separability

Lemma: $\Phi(X) = |a\rangle\langle b| X |b\rangle\langle a| \iff \mathcal{C}(\Phi) = (|b\rangle\langle b|)^T \otimes |a\rangle\langle a|$.
Corollary: If Φ is entanglement breaking then

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In total, we have

$$\mathrm{ebr}(\Phi) = \mathrm{len}(\mathcal{C}(\Phi)) = \mathrm{len}((I \otimes T)\mathcal{C}(\Phi)) = \mathrm{len}((T \otimes I)\mathcal{C}(\Phi)).$$

Werner and isotropic states

For $t \in [-\frac{1}{d^2-1}, \frac{1}{d+1}]$ the states

$$\mathcal{C}(\Phi_t) = t \cdot \frac{1}{d} \sum_{ij} E_{ij} \otimes E_{ij} + (1-t) \cdot \frac{1}{d^2} I_{d^2}$$

are known as separable isotropic states[5].

[5] Watrous, TQI, 2018

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Their partial transpose

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are known as separable Werner states – linear combinations of Π_{sym} and Π_{asym}. In particular,

$$(I \otimes T)\mathcal{C}(\Phi_{\frac{1}{d+1}}) = \frac{2}{d(d+1)} \Pi_{\mathrm{sym}} \implies \mathrm{ebr}(\Phi_{\frac{1}{d+1}}) = \mathrm{len}(\Pi_{\mathrm{sym}}).$$

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The size of minimal weighted projective 2-designs

Let $\{|x_i\rangle\}_{i=1}^n \in H$ be a weighted projective 2-design, i.e.

$$\sum_{i=1}^{n} w_i \big(|x_i\rangle\langle x_i| \big)^{\otimes 2} = \frac{2}{d(d+1)} \Pi_{\mathrm{sym}},$$

where $w_i \geqslant 0, \sum_i w_i = 1$. It means $\mathrm{len}(\Pi_{\mathrm{sym}}) \leqslant n$.

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Theorem[6]: The size of a minimal weighted projective 2-design equals $\mathrm{ebr}(\Phi_{\frac{1}{d+1}})$.

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Theorem[6]: The size of a minimal weighted projective 2-design equals $\mathrm{ebr}(\Phi_{\frac{1}{d+1}})$.

Proof sketch: 1. Use $\mathrm{ebr}(\Phi_{\frac{1}{d+1}}) = \mathrm{len}(\Pi_{\mathrm{sym}})$.

2. Let $\mathrm{len}(\Pi_{\mathrm{sym}}) = m$, that is $\exists\ \{|x_i\rangle\}, \{|y_i\rangle\}, w_i \geqslant 0, \sum_i w_i = 1$

$$\sum_{i=1}^{m} w_i \, |x_i\rangle\langle x_i| \otimes |y_i\rangle\langle y_i| = \frac{2}{d(d+1)} \Pi_{\mathrm{sym}}.$$

It follows $|x_i\rangle |y_i\rangle \in H_{\mathrm{sym}} \implies |x_i\rangle |y_i\rangle = |x_i'\rangle |x_i'\rangle$.

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Choi's map cheat sheet

Linear $\Phi : \mathbb{C}^{d\times d} \to \mathbb{C}^{r\times r}$	$\iff$	$\mathcal{C}(\Phi) \in \mathbb{C}^{d\times d} \otimes \mathbb{C}^{r\times r}$
Φ is completely positive	$\iff$	$\mathcal{C}(\Phi) \geqslant 0$
Φ is a quantum channel	$\implies$	$\mathcal{C}(\Phi)$ is a state

Assuming Φ is a quantum channel:

Kraus rank of Φ	$\iff$	rank of $\mathcal{C}(\Phi)$
Φ is entanglement breaking	$\iff$	$\mathcal{C}(\Phi)$ is separable
$\mathrm{ebr}(\Phi)$	$\iff$	$\mathrm{len}(\mathcal{C}(\Phi))$
$T \circ \Phi$, $\Phi \circ T$	$\iff$	$(I \otimes T)\mathcal{C}(\Phi)$, $(T \otimes I)\mathcal{C}(\Phi)$
EB depolarising channels Φ_t	$\iff$	separable isotropic states
EB channels $T \circ \Phi_t = \Phi_t \circ T$	$\iff$	separable Werner states
$T \circ \Phi_{\frac{1}{d+1}}$	$\iff$	$\Pi_{\mathrm{sym}} \cdot \frac{2}{d(d+1)}$

Mutually unbiased frames (kind of)

Let $\mathrm{ebr}(\Phi_t) = d^2$ for some $t \in [\frac{-1}{d^2-1}, \frac{1}{d+1}]$. Equivalently, $\mathrm{len}((I \otimes T)\mathcal{C}(\Phi_t)) = d^2$, which means there exist unit frames $\{|x_i\rangle\}$, $\{|y_i\rangle\}$ and $w_i \geqslant 0, \sum_i w_i = 1$ for $i \in [d^2]$ such that

$$\sum_{i=1}^{d^2} w_i \, |x_i\rangle\langle x_i| \otimes |y_i\rangle\langle y_i| = (I \otimes T)\mathcal{C}(\Phi_t) = t \cdot \frac{1}{d} U_{\mathrm{sw}} + (1-t) \cdot \frac{1}{d^2} I_{d^2}.$$

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Then[7] all w_i are equal $1/d^2$ and the frames $\{|x_i\rangle\}$, $\{|y_i\rangle\}$ are

1. tight,
2. informationally-complete if $t \neq 0$,
3. kind of mutually unbiased: $|\langle x_i|y_j\rangle|^2 = \begin{cases} \frac{t(d^2-1)+1}{d}, & i = j, \\ \frac{1-t}{d}, & i \neq j, \end{cases}$
4. reciprocal: $|\langle x_i|x_j\rangle\langle y_i|y_j\rangle| = |t|, \quad i \neq j.$

[7] D.Y., 2022

Mutually unbiased frames (kind of)

Conversely, let for $t \in [\frac{-1}{d^2-1}, \frac{1}{d+1}]$ unit frames $\{|x_i\rangle\}_{i=1}^{d^2}$, $\{|y_i\rangle\}_{i=1}^{d^2}$ are tight, informationally-complete, and kind of mutually unbiased:

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Then

$$\frac{1}{d^2}\sum_{i=1}^{d^2} |x_i\rangle\langle x_i| \otimes |y_i\rangle\langle y_i| = t \cdot \frac{1}{d} U_{\mathrm{sw}} + (1-t) \cdot \frac{1}{d^2} I_{d^2}, \qquad (*)$$

i.e. $\mathrm{len}((I \otimes T)\mathcal{C}(\Phi_t)) = \mathrm{ebr}(\Phi_t) = d^2$.

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i.e. $\mathrm{len}((I \otimes T)\mathcal{C}(\Phi_t)) = \mathrm{ebr}(\Phi_t) = d^2$.

Corollary: the PPPR conjecture is equivalent to the existence of such a pair of frames (except the case $t = 0$).

Weyl-Heisenberg group

Let $\omega = e^{2\pi i/d}$, $\tau = -e^{\pi i/d}$. Define clock and shift matrices by

$$Z = \sum_{i=0}^{d-1} w^i \,|i\rangle\langle i| \,, \quad X = \sum_{i=0}^{d-1} |i+1\rangle\langle i| \,, \quad |d\rangle := |0\rangle \,.$$

For $\mathbf{a} = (a_1, a_2) \in \mathbb{Z}_d^2$ unitaries $D_\mathbf{a} = \tau^{a_1 a_2} X^{a_1} Z^{a_2}$ form a projective representation of $\mathbb{Z}_d^2$. A frame $\{D_\mathbf{a} \,|v\rangle\}_{\mathbf{a}\in\mathbb{Z}_d^2}$ is called WH-covariant.

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For $d = 2$ and $d = 3$ there are WH-covariant solutions to Eq. $(*)$, that is,

$$|x_\mathbf{a}(t)\rangle = D_\mathbf{a} \, |x(t)\rangle \,, \; |y_\mathbf{a}(t)\rangle = D_\mathbf{a} \, |y(t)\rangle \,,$$

where $|x(t)\rangle$, $|y(t)\rangle$ are fiducial vectors of both frames. Moreover, $|x(t)\rangle$, $|y(t)\rangle$ are continuous and differentiable over t.

The case where $t = 0$

The equality for WH-covariant frames becomes

$$\sum_{\mathbf{a}\in\mathbb{Z}_d^2} D_{\mathbf{a}}^{\otimes 2}(|x(0)\rangle\langle x(0)| \otimes |y(0)\rangle\langle y(0)|)D_{\mathbf{a}}^{\dagger\otimes 2} = I_{d^2}$$

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There is a nice looking solution to this:

$$|x(0)\rangle = |0\rangle, \quad |y(0)\rangle = F\,|0\rangle = \frac{1}{\sqrt{d}}\sum_i |i\rangle,$$

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It turns out that this solution is not a good starting point for a series of WH-covariant solutions of Eq. $(*)$ for $t \in [0, \varepsilon]$ because it can't be differentiable at $t = 0^+$.

WH bases

For a WH SIC with fiducial $|x\rangle$ we have

$$\frac{1}{d^2}\sum_{\mathbf{a}\in\mathbb{Z}_d^2} D_{\mathbf{a}}^{\otimes 2}(|x\rangle\langle x|\otimes|x\rangle\langle x|)D_{\mathbf{a}}^{\dagger\otimes 2}=\frac{2}{d(d+1)}\Pi_{\mathrm{sym}}.$$

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By Naimark's theorem there exists an ONB $\{|b_{\mathbf{a}}\rangle\}$ on $H\otimes H$ such that

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But in fact[8], we can always find a WH-covariant type of such a basis, that is $|b_{\mathbf{a}}\rangle = D_{\mathbf{a}}^{\otimes 2}|b\rangle$. Equivalently,

$$|x\rangle|x\rangle = \sqrt{\frac{2d}{d+1}}\Pi_{\mathrm{sym}}|b\rangle,$$

where $|b\rangle \in H^{\otimes 2}$ is a fiducial basis vector.

[8] Ostrovsky, D.Y., "Geometric properties of SIC-POVM tensor square", 2022

This result can be extended. Let for some $t \in [\frac{-1}{d^2-1}, \frac{1}{d+1}]$:

$$\frac{1}{d^2} \sum_{\mathbf{a} \in \mathbb{Z}_d^2} D_{\mathbf{a}}^{\otimes 2} (|x(t)\rangle\langle x(t)| \otimes |y(t)\rangle\langle y(t)|) D_{\mathbf{a}}^{\dagger \otimes 2} = t \cdot \frac{1}{d} U_{\mathrm{sw}} + (1-t) \cdot \frac{1}{d^2} I_{d^2}.$$

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Let $M_t = d\cdot\sqrt{t\cdot\frac{1}{d}U_{\mathrm{sw}} + (1-t)\cdot\frac{1}{d^2}I_{d^2}} \geqslant 0.$
Then there exists a WH-covariant basis $|b_{\mathbf{a}}(t)\rangle = D_{\mathbf{a}}^{\otimes 2}|b(t)\rangle$ of $H^{\otimes 2}$ such that

$$|x(t)\rangle|y(t)\rangle = M_t|b(t)\rangle.$$

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- it is probably not true for $d \geqslant 4$ except $t = 0$, $t = \frac{1}{d+1}$
- it's still interesting to find ebr (equivalently, len) for maps Φ_t and EB maps in general

Thank you!