Conjugate Phase Retrieval in the Paley-Wiener Space

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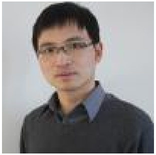
Acknowledgements



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Phase Retrieval

Problem (Phase Retrieval)

Can a signal f be reconstructed from the magnitudes of linear measurements of f, up to the ambiguity of uniform phase factor α?

Formally, we define an equivalence class on the signal space $\mathcal{H}$ by:

$$f \sim g \text{ if } f = \alpha g \text{ for some } |\alpha| = 1.$$

then ask whether the mapping

$$\mathcal{A} : \mathcal{H}/\sim \to \ell^2(\mathcal{I}) : f \mapsto (|\phi_n(f)|)_n$$

is injective, where ϕ_n are linear functionals on $\mathcal{H}$.
If so, the next question is: how to invert $\mathcal{A}$?

Conjugate Phase Retrieval

Suppose the signal space $\mathcal{H}$ has the property that if $f \in \mathcal{H}$, then $\overline{f} \in \mathcal{H}$ (e.g. $\mathbb{C}^d$ or the Paley-Wiener space). We formulate a weaker variation:

Problem (Conjugate Phase Retrieval–Evans & Lai [EL17])

Can a signal f be reconstructed from the magnitudes of linear measurements of f, up to the ambiguity of uniform phase factor α and the ambiguity of conjugation?

Formally, we define an equivalence class on the signal space $\mathcal{H}$ by:

$$f \sim g \text{ if } f = \alpha g, \text{ or } f = \alpha \overline{g} \text{ for some } |\alpha| = 1.$$

then ask whether the mapping

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Paley-Wiener Space

Definition

For $\beta > 0$, we denote

$$PW_\beta = \{f \in L^2(\mathbb{R}) | \hat{f}(\xi) = 0 \ a.e. |\xi| > \beta\}.$$

Definition

If $f \in PW_\beta$, then

$$f^\sharp(z) = \overline{f(\overline{z})} \in PW_\beta$$

Note that for real z, $f^\sharp = \overline{f}$. From here on, our equivalence relation is on PW_β:

$$f \sim g \text{ if } f = \alpha g, \text{ or } f = \alpha g^\sharp \text{ for some } |\alpha| = 1.$$

The Problem with Phase Retrieval in the PW_β

Goal

Design a sampling regime $\{\phi_n\}$ on PW_β that:

1. *does conjugate phase retrieval;*
2. *admits a numerical reconstruction method.*

Preferably, the sampling regime occurs on the real axis.

Problem

If $f \in PW_\beta$, $|f(x)|$ does not determine $f(x)$ up to unimodular scalar, or conjugation either (here, $x \in \mathbb{R}$).

If $\hat{f}$ is supported in an interval smaller than 2β, $\hat{f}$ can be shifted to remain in $(-\beta, \beta)$, which modulates $f(x)$.

Phase Retrieval: A Short History

1. Optics: Gerchberg-Saxton [GS72], Fienup [Fie78], Rosenblatt [Ros84], Levi-Stark [LS84]
2. Inverse Spectral Theory [KS92, KST95]
3. Frames: Balan et. al. [BCE06, BBCE09] Bandeira et. al. [BCMN14]
4. Reconstructions: Alternating projections; Wirtinger Flow; PhaseLift; PhaseMax; AltMinPhase; Kaczmarz; etc

Phase Retrieval: A Short History (cont'd)

1. Paley-Wiener space: Thakur [Tha11], Pohl-Yang-Boche [PYB14].
   1. [Tha11] considers the case of real phase retrieval in PW_π. The reconstruction occurs off of the real axis, where there are no zeros of the function.
   2. [PYB14] considers the case of (complex) phase retrieval in PW_π by designing a sampling scheme that occurs off of the real axis. In particular, the sampling scheme as presented in [PYB14] takes the form

$$\phi_n(f) = \sum_j c_{j,n} f(z_n - b_{j,n}) \tag{1}$$

      for complex scalars $c_{j,n}, z_n, b_{j,n}$. Sampling schemes such as this are referred to as *structured modulations* in [PYB14] because the authors there consider the reconstruction in the Fourier domain, where the shifts become modulations.

Conjugate Phase Retrieval: A Short History

Proposition (McDonald [McD04])

Suppose $f, g \in PW_\beta$.

1. *If* $b < \beta/\pi$, *and for all* $x \in \mathbb{R}$, $|f(x)| = |g(x)|$ *and* $|f(x+b) - f(x)| = |g(x+b) - g(x)|$, *then* $f \sim g$.
2. *If for all* $x \in \mathbb{R}$, $|f(x)| = |g(x)|$ *and* $|f'(x)| = |g'(x)|$, *then* $f \sim g$.

Proposition (Evans-Lai [EL17])

If $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^2$ *is written as* $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ *then* $\vec{v}_1, \vec{v}_2, \vec{v}_3$ *does conjugate phase retrieval in* $\mathbb{C}^2$ *if and only if*

$$\det \begin{bmatrix} a_1^2 & 2a_1 a_2 & a_2^2 \\ b_1^2 & 2b_1 b_2 & b_2^2 \\ c_1^2 & 2c_1 c_2 & c_2^2 \end{bmatrix} \neq 0. \quad (2)$$

Conjugate Phase Retrieval in $\mathbb{C}^d$

1. CPR is weaker than PR: there exist vectors in $\mathbb{C}^d$ that do CPR but not PR.
2. Q: Can CPR be done with $3d$ vectors?
3. No proven reconstruction method exists for Conjugate Phase Retrieval in $\mathbb{C}^d$.
4. We will use Gerchberg-Saxton for reconstruction. We show experimentally that it works well.
5. Q: Can CPR be made robust to noise?

Gerchberg-Saxton for Reconstruction

Given A that does conjugate phase retrieval, $|A^T\vec{v}|$, reconstruct $\vec{w} \in [\vec{v}]$.

1. Choose phases $\lambda_1, \dots, \lambda_n$;
2. Apply $(A^T)^\dagger$ to
$$\left(\lambda_1|\langle \vec{v}, \vec{v}_1\rangle|, \dots, \lambda_n|\langle \vec{v}, \vec{v}_n\rangle|\right)^T$$
to obtain estimate $\vec{w}$;
3. Replace $\vec{w}$ with $(A^T)^\dagger$ applied to
$$\left(\frac{\langle \vec{w}, \vec{v}_1\rangle}{|\langle \vec{w}, \vec{v}_1\rangle|}|\langle \vec{v}, \vec{v}_1\rangle|, \dots, \frac{\langle \vec{w}, \vec{v}_n\rangle}{|\langle \vec{w}, \vec{v}_n\rangle|}|\langle \vec{v}, \vec{v}_n\rangle|\right)^T;$$
4. Repeat step 3.

Gerchberg-Saxton Method Results

Observed behavior:

1. the alternating projections method converges (Levi-Stark)
2. the method may not converge to a solution (i.e. Traps)
3. Traps/Tunnels were observed in numerical simulations (reconstruction failed; however, reseeding succeeds)
4. for the 3×6 matrix A used in the PW example, reconstruction error was $< 10^{-4}$ after 400 iterations for approximately 80% of instances (performed 1000 random examples)

A Corollary of McDonald's Theorem

Lemma

If $f \in PW_\beta$*, then:* $f' \in PW_\beta$*;* $ff^\sharp \in PW_{2\beta}$*;* $f'(f')^\sharp \in PW_{2\beta}$*.*

Theorem

Suppose $\{t_n\} \subset \mathbb{R}$ *is a set of sampling for* $PW_{2\beta}$*. Then the mapping*

$$\mathcal{A} : PW_\beta / \sim \to \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) : f \mapsto (|f(t_n)|, |f(t_n + b) - f(t_n)|)_n$$

is one-to-one whenever $b < \frac{\beta}{\pi}$*.*
Similarly, the mapping

$$\widetilde{\mathcal{A}} : PW_\beta / \sim \to \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) : f \mapsto (|f(t_n)|, |f'(t_n)|)_n$$

is one-to-one.

Inversion

Fundamental Question: How to invert these mappings?

$$\mathcal{A} : PW_\beta / \sim \to \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) : f \mapsto (|f(t_n)|, |f(t_n + b) - f(t_n)|)_n$$
$$\widetilde{\mathcal{A}} : PW_\beta / \sim \to \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) : f \mapsto (|f(t_n)|, |f'(t_n)|)_n$$

1. We can reconstruct $|f(x)|^2$ and $|f(x+b) - f(x)|^2$ (or $|f'(x)|^2$) easily.
2. McDonald's Theorem guarantees injectivity of the mappings using Weierstrass factorizations of functions of finite order.
3. Reconstruction from McDonald's Theorem requires knowledge of the zeros of f.
4. Inversion is unstable: observed by Mallat-Waldspurger [MW15], proven by Cahill-Casazza-Daubechies [CCD16].

We propose a sampling and reconstruction method following Pohl-Yang-Boche [PYB14] using "structured convolutions".

Injectivity via Structured Convolutions

Theorem (Lai, Littmann & W.)

The following sampling scheme does conjugate phase retrieval on PW_π:

$$\{|\alpha_m * f(t_n)| : m = 0, 1, \dots, M-1;\ n \in \mathbb{Z}\}$$

where

$$\alpha_m * f = \sum_{k=0}^{K-1} \overline{a_{km}} f(\cdot - b_k) \qquad (3)$$

provided:

1. $A = (a_{km})$ *be a* $K \times M$ *matrix which does conjugate phase retrieval on* $\mathbb{C}^K$
2. $\{t_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ *is a set of sampling for the space* $PW_{2\pi}$
3. $\{b_j\}_{j=0}^{K-1} \subset \mathbb{R}$ *be such that the group* $\mathbb{Z}(\{b_0, b_1, \dots, b_{K-1}\})$ *has finite upper Beurling density and lower Beurling density greater than one.*

Reconstruction via Structured Convolutions

For $f \in PW_\pi$ we sample

$$\{|\alpha_m * f(t_n)| : m = 0, 1, \ldots, M-1;\ n \in \mathbb{Z}\} \tag{4}$$

where $\{t_n\}$ and α_m satisfy the hypotheses of Theorem 3. Choose $b_j = j/B$ for some integer B.

Reconstruction Algorithm

1. *From the samples in Equation (4), reconstruct the functions*

$$|\alpha_m * f(x)|^2, \quad m = 0, 1, \ldots, M-1,$$

   using the Shannon sampling theorem.
2. *Choose β at random*†*.*

By Lemma 4, with probability 1,

$$f(\frac{n}{B} - b_j - \beta) \neq 0 \text{ for all } j = 0, \ldots, K-1,\ n \in \mathbb{Z}.$$

Reconstruction via Structured Convolutions

Reconstruction Algorithm (Continued)

3. *Calculate the following samples using Step 1:*

$$|\alpha_m * f(\frac{n}{B} - \beta)|^2, \quad m = 0, 1, \ldots, M-1, \ n \in \mathbb{Z}.$$

4. *Use the fact that the matrix A does conjugate phase retrieval to calculate for each $n \in \mathbb{Z}$ the vector*

$$\vec{F}_n := \lambda(\frac{n}{B} - \beta) \begin{pmatrix} f(\frac{n}{B} - b_0 - \beta) \\ \vdots \\ f(\frac{n}{B} - b_{K-1} - \beta) \end{pmatrix} \tag{5}$$

up to the unknown phase $\lambda(\frac{n}{B} - \beta)$ and unknown conjugation.

Reconstruction via Structured Convolutions

Reconstruction Algorithm (Continued)

5. *For adjacent vectors $\vec{F}_n$ and $\vec{F}_{n+1}$, choose the conjugations and phase factors so that the entries that appear in both vectors agree. This can be done, since by Lemma 5, the choice of β makes these choices unique (with probability 1).*
6. *We now obtain the samples*

$$\{\lambda f(\frac{n}{B} - \beta) : n \in \mathbb{Z}\} \text{ or } \{\overline{\lambda f(\frac{n}{B} - \beta)} : n \in \mathbb{Z}\}$$

up to unknown unimodular scalar λ, depending on whether our choice for conjugation was correct. Using these samples, we reconstruct $\lambda f(x - \beta)$ (if our choice of conjugation was correct) or $\lambda f^\sharp(x - \beta)$ (if our choice of conjugation was incorrect).

An Example

1. $f \in PW_\pi$, generated with 20 random complex valued coefficients $(f(n))$, and $f(0) = 0$
2. we choose $b_0 = 0$, $b_1 = 1/2$, $b_2 = 1$, and the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix}$$

3. we sample $\alpha_m * f$ at $\dfrac{\mathbb{Z}}{2}$
4. we select $\beta \in (0,1)$ at random, then reconstruct for $n \in [-20, 20]$:

$$|\alpha_m * f(\frac{n}{2} - \beta)|^2$$

An Example (Continued)

5. Apply alternating projections (the Gerchberg-Saxton method) to

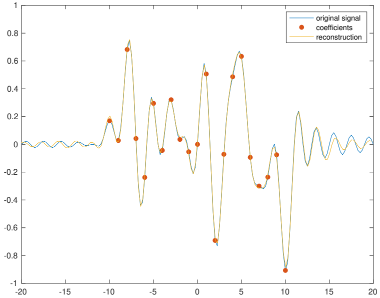
$$\left| \left\langle \begin{pmatrix} f(\frac{n+1}{2} - \beta) \\ f(\frac{n}{2} - \beta) \\ f(\frac{n-1}{2} - \beta) \end{pmatrix}, \vec{a}_m \right\rangle \right| \quad m = 1, \ldots, 6$$

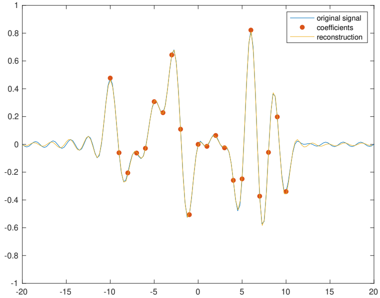
6. for n we obtain, up to unknown conjugation and phase λ_n:

$$\vec{F}_n = \lambda_n \begin{pmatrix} f(\frac{n+1}{2} - \beta) \\ f(\frac{n}{2} - \beta) \\ f(\frac{n-1}{2} - \beta) \end{pmatrix}$$

7. choose conjugation and phase so that $\vec{F}_{n+1}$ is consistent with $\vec{F}_n$.
8. Reconstruct f from the consistent samples $f(\frac{n}{2} - \beta)$.

Graphs of the Reconstruction





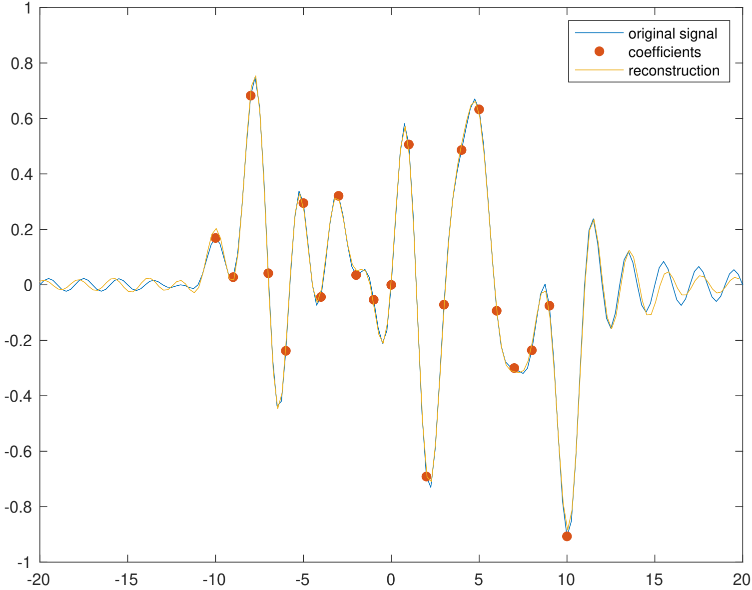
Real part Imaginary part

Reconstruction error: (f-original signal; r-reconstructed signal; Fröbenius norm)

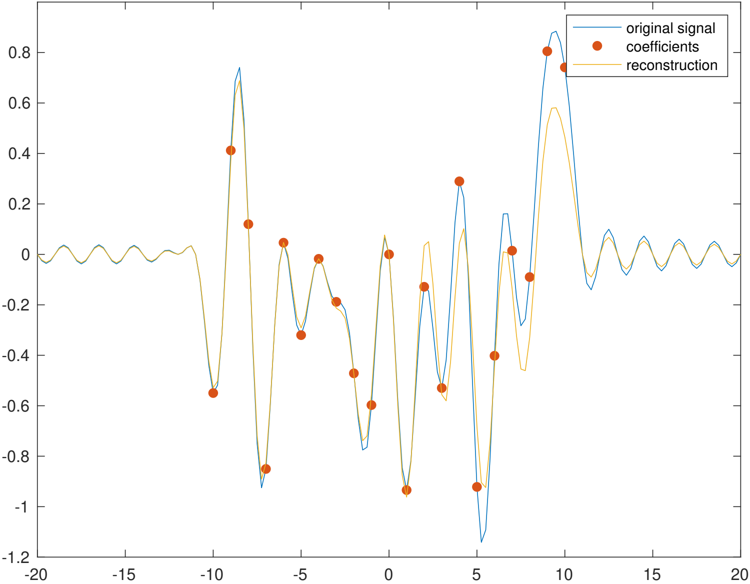
$$\|f \otimes f - r \otimes r\| / \|f \otimes f\| = 0.026$$

$\beta = 0.2119$ (chosen using the MATLAB command `rand`).
Choice of β affected the reconstruction error.

Reconstruction (Magnified)



Failed Reconstruction



Sample Complexity

The "McDonald Corollary" samples 2 convoluted copies of f at twice the Nyquist rate. Thus, the sample complexity is 4 times the Nyquist rate.
Our example with a numerical reconstruction samples 3 convoluted copies of the signal f at twice the Nyquist rate. Thus, the sample complexity is 6 times the Nyquist rate.
However, for *generic* signals, we can reduce the sample complexity to 3 times the Nyquist rate:

Reconstruction Algorithm

1. *Sample* $|f(n)|$, $|f(n+1) - f(n-1)|$, *and* $|f(n+1) - f(n)|$ *at* $\mathbb{Z}$*;*
2. *reconstruct* $\{f(n+1), f(n), f(n-1)\}$ *up to phase and conjugation from*
$$\{|f(n+1)|, |f(n)|, |f(n-1)|, |f(n+1) - f(n-1)|, \\ |f(n+1) - f(n)|, |f(n) - f(n-1)|\}; \quad (6)$$
3. *If* $\mathcal{Z}(f) \cap \mathbb{Z} = \emptyset$*, the samples can be made consistent.*

Several Lemmas

Lemma

The set of all $\beta \in \mathbb{R}$ *such that*

$$f(\frac{n}{B} - b_j - \beta) = 0,$$

for some $j = 0, \ldots, K-1$ *and for some* $n \in \mathbb{Z}$ *is countable.*

Lemma

Suppose g *is an entire function. For fixed* $\{b_0, \ldots, b_{K-1}\} \subset \mathbb{R}$, *the set of* $x \in \mathbb{R}$ *for which the vectors*

$$\begin{pmatrix} g(x-b_0) \\ g(x-b_1) \\ \vdots \\ g(x-b_{K-1}) \end{pmatrix} \text{ and } \begin{pmatrix} \overline{g(x-b_0)} \\ \overline{g(x-b_1)} \\ \vdots \\ \overline{g(x-b_{K-1})} \end{pmatrix}$$

are colinear is either all of $\mathbb{R}$ *or at most countable.*

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