Linear PDE with Constant Coefficients

Bernd Sturmfels
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A polynomial is exactly the same thing as a homogeneous linear partial differential equation (PDE) with constant coefficients.

Polynomials in one variable are ODE:

Exercise: Find all functions $\phi(z)$ that satisfy the equation

$$\phi'''''(z) - 6\phi'''(z) + 10\phi''(z) - 6\phi'(z) + 9\phi(z) = 0.$$
Prologue

A polynomial is exactly the same thing as a homogeneous linear partial differential equation (PDE) with constant coefficients.

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Exercise: Find all functions \( \phi(z) \) that satisfy the equation

\[
\phi''''(z) - 6\phi'''(z) + 10\phi''(z) - 6\phi'(z) + 9\phi(z) = 0.
\]

Solution: ODE in operator form is the characteristic polynomial:

\[
(\partial^4 - 6\partial^3 + 10\partial^2 - 6\partial + 9) \cdot \phi(z) = 0
\]

\[
x^4 - 6x^3 + 10x^2 - 6x + 9 = (x - 3)^2 \cdot (x^2 + 1)
\]

Basis of solutions: \( \{ e^{3z}, z \cdot e^{3z}, e^{iz}, e^{-iz} \} \)

Basis of solutions: \( \{ e^{3z}, z \cdot e^{3z}, \sin(z), \cos(z) \} \)
Undergraduates study the one-dimensional wave equation

$$\phi_{tt}(z, t) = c^2 \phi_{zz}(z, t), \quad \text{where } c \in \mathbb{R}\{0\}.$$ 

for functions $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$. The corresponding polynomial is

$$x_1^2 - c^2 x_2^2 = (x_1 - cx_2)(x_1 + cx_2).$$
18th Century

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In 1747, Jean Le Rond D’Alembert found that the general solution is the superposition of traveling waves:

$$\phi(z, t) = f(z + ct) + g(z - ct),$$

where $f$ and $g$ are twice differentiable functions, or distributions.

Q: How to deal with the special parameter value $c = 0$?
A: Replace $g(z - ct)$ with $\frac{1}{2c}(h(z+ct) - h(z-ct))$ and take limit:

$$\phi(z, t) = f(z) + t \cdot h'(z).$$
20th Century

In his 1938 article on foundations of algebraic geometry, Gröbner introduced differential operators to characterize membership in a polynomial ideal. He derived this for zero-dimensional ideals (Macaulay’s inverse systems), and he envisioned it for all ideals.

Gröbner wanted algorithmic solutions. We provide them.

Wolfgang Gröbner: Über die algebraischen Eigenschaften der Integrale von linearen Differentialgleichungen mit konstanten Koeffizienten, Monatshefte für Mathematik und Physik, 1939

In the 1960s, Ehrenpreis and Palamodov studied linear PDE with constant coefficients. A main step was the characterization of membership in a primary ideal by Noetherian operators.

Their celebrated Fundamental Principle appears in the books Leon Ehrenpreis: Fourier Analysis in Several Complex Variables, 1970
Victor Palamodov: Linear Differential Operators w Constant Coeffs, 1970
In his 1938 article on foundations of algebraic geometry, Gröbner introduced **differential operators** to characterize **membership in a polynomial ideal**. He derived this for zero-dimensional ideals (Macaulay’s inverse systems), and he envisioned it for all ideals. Gröbner wanted **algorithmic solutions**. We provide them.


In the 1960s, Ehrenpreis and Palamodov studied linear PDE with constant coefficients. A main step was the characterization of membership in a primary ideal by **Noetherian operators**.

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Leon Ehrenpreis: *Fourier Analysis in Several Complex Variables*, 1970
Section 3.3

Theorem 3.27. Let $I$ be a zero-dimensional ideal in $\mathbb{C}[x_1, \ldots, x_n]$, here interpreted as a system of linear PDEs. The space of holomorphic solutions has dimension equal to the degree of $I$. There exist nonzero polynomial solutions if and only if the maximal ideal $M = \langle x_1, \ldots, x_n \rangle$ is an associated prime of $I$. In that case, the polynomial solutions are precisely the solutions to the system of PDEs given by the $M$-primary component $(I : (I : M^\infty))$. 
Prime Ideals

Let $P$ be a prime ideal in $\mathbb{C}[x_1, \ldots, x_n]$ and $V(P)$ its variety in $\mathbb{C}^n$. A polynomial $f$ is in the ideal $P$ if and only if $f$ vanishes on $V(P)$. Setting $x_i = \partial_{z_i}$, view $P$ as PDE for an unknown function $\phi(z_1, \ldots, z_n)$.

Remark

For $u \in \mathbb{C}^n$, the exponential function

$$z \mapsto \exp(u \cdot z) = \exp(u_1 z_1 + \cdots + u_n z_n)$$

satisfies the PDE given by $P$ if and only if $u \in V(P)$.
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Proposition
Each solution to $P$ admits an integral representation

$$\phi(z) = \int_{V(P)} \exp(x \cdot z) \, d\mu(x),$$

where $\mu$ is a measure on the irreducible variety $V(P)$. 
Primary Ideals

Set $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$. An ideal $Q$ is primary if it has only one associated prime $P$. The variety $V(Q) = V(P)$ in $\mathbb{C}^n$ is irreducible.

Theorem (Ehrenpreis-Palamodov)

Fix a prime ideal $P$ in $\mathbb{C}[x]$. For any $P$-primary ideal $Q$, there exist polynomials $B_1, \ldots, B_m$ in $2n$ variables such that the function

$$\phi(z) = \sum_{i=1}^m \int_{V(P)} B_i(x, z) \exp(x \cdot z) \, d\mu_i(x)$$

is a solution to the PDE, for any measures $\mu_1, \ldots, \mu_m$ on $V(P)$.

Conversely, every solution $\phi(z)$ of the PDE given by $Q$ admits such an integral representation. The minimal number is

$$m = \text{length}(R_P / QR_P) = \frac{\text{degree}(Q)}{\text{degree}(P)}.$$

The Noetherian multipliers $B_i(x, z)$ depend only on the ideal $Q$. 
Palamodov’s Example

Let \( n = 3, P = \langle x_1, x_2 \rangle, m = 2 \) and \( Q = \langle x_1^2, x_2^2, x_1 - x_2x_3 \rangle \):

\[
PDE: \quad \frac{\partial^2 \phi}{\partial z_1^2} = \frac{\partial^2 \phi}{\partial z_2^2} = \frac{\partial \phi}{\partial z_1} - \frac{\partial^2 \phi}{\partial z_2 \partial z_3} = 0.
\]

Writing \( \xi, \psi \) for functions in one variable, the general solution is

\[
\phi(z) = \xi(z_3) + z_2 \psi(z_3) + z_1 \psi'(z_3),
\]

The Noetherian multipliers of \( Q \) are \( B_1 = 1 \) and \( B_2 = z_2 + x_3z_1 \).

Their integrals in the Ehrenpreis-Palamodov Theorem are

\[
\phi_1(z) = \int 1 \cdot \exp(0z_1 + 0z_2 + x_3z_3) \, d\mu_1(x) = \xi(z_3)
\]

and

\[
\phi_2(z) = \int (z_2 + z_1x_3) \cdot \exp(0z_1 + 0z_2 + x_3z_3) \, d\mu_2(x) = z_2 \int \exp(0z_1 + 0z_2 + x_3z_3) \, d\mu_2(x) + z_1 \int x_3 \exp(0z_1 + 0z_2 + x_3z_3) \, d\mu_2(x)
\]

\[
= z_2 \psi(z_3) + z_1 \psi'(z_3).
\]
Noetherian Operators

The Noetherian multipliers $B_i(x, z)$ of a primary ideal $Q$ furnish a finite representation of the (infinite-dimensional) vector space of all solutions to the PDE. We now recycle them for ideal membership.

Switching the roles of $x$ and $z$, we set $z_1 = \partial x_1$, $\ldots$, $z_n = \partial x_n$ in $B_i(x, z)$, with $z$-variables to the right of the $x$-variables in each monomial. This gives the Noetherian operators $B_i(x, \partial_x)$. These are elements in the Weyl algebra. They act on polynomials in $\mathbb{C}[x]$. 
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Switching the roles of $x$ and $z$, we set $z_1 = \partial x_1$, $\ldots$, $z_n = \partial x_n$ in $B_i(x, z)$, with $z$-variables to the right of the $x$-variables in each monomial. This gives the Noetherian operators $B_i(x, \partial x)$. These are elements in the Weyl algebra. They act on polynomials in $\mathbb{C}[x]$.

Proposition

Noetherian operators characterize ideal membership. Namely, a polynomial $f(x)$ lies in the primary ideal $Q$ if and only if

$$B_i(x, \partial x) \bullet f(x) \text{ lies in } P \text{ for } i = 1, \ldots, m.$$ 

Example

A polynomial $f$ lies in the primary ideal $Q = \langle x_1^2, x_2^2, x_1 - x_2 x_3 \rangle$ if and only if both $f$ and $(x_3 \partial x_1 + \partial x_2) \bullet f$ vanish on the $x_3$-axis $V(P)$.
Towards an Algorithm

**Input:** Generators of a (primary) ideal $Q$ in the polynomial ring $\mathbb{C}[x]$.

**Output:** Noetherian multipliers (resp. Noetherial operators) for $Q$.


B. Sturmfels: Solving Systems of Polynomial Equations, AMS, 2002


J. Chen, M. Härkönen, R. Krone and A. Leykin: Noetherian operators and primary decomposition, 2006.13881

J. Chen, Y. Cid-Ruiz, M. Härkönen, R. Krone and A. Leykin: Noetherian operators in Macaulay2, 2101.01002

R. Ait El Manssour, M. Härkönen and BSt.: Linear PDE with constant coefficients, Glasgow Math J., 2022
Current Perspective

Fix a prime $P$ of codimension $c$ in $R = \mathbb{C}[x_1, \ldots, x_n]$, in Noether position. Write $\mathbb{F} = \mathbb{C}(u_1, \ldots, u_n)$ for the field of fractions of $R/P$.

Theorem

The following sets are in bijective correspondences:

(a) $P$-primary ideals $Q$ in $R$ of multiplicity $m$,
(b) points in the punctual Hilbert scheme $\text{Hilb}^m(\mathbb{F}[[y_1, \ldots, y_c]])$,
(c) $m$-dimensional $\mathbb{F}$-subspaces of $\mathbb{F}[z_1, \ldots, z_c]$ that are closed under differentiation, Inverse systems
(d) $m$-dimensional $\mathbb{F}$-subspaces of the Weyl-Noether module $\mathbb{F} \otimes_R D_{n,c}$ that are $R$-bi-modules, where $D_{n,c} = R \langle \partial x_1, \ldots, \partial x_c \rangle$.

(c) $\rightarrow$ Noetherian multipliers (d) $\rightarrow$ Noetherian operators

Yairon Cid-Ruiz, Roser Homs Pons and BSt: Primary ideals and their differential equations, *Foundat. Computational Math*, 2021
Solving Gröbner’s Problem

If \( I = Q_1 \cap \cdots \cap Q_k \) is a primary decomposition then we may simply

- aggregate Noetherian operators to get a membership test for \( I \)
- aggregate Noetherian multipliers to solve the PDE given by \( I \)

Works fine if \( I \) has no embedded primes. Can do better in general.

Example (Fat point on a double line)

\[
I = \langle x_1^2, x_2^2, x_1 x_3 - x_2 x_3^2 \rangle = \langle x_1^2, x_2^2, x_1 - x_2 x_3 \rangle \cap \langle x_1^2, x_2^2, x_3 \rangle
\]

The naive method gives six Noetherian multipliers, namely two for the line and four for the point. But we need only four of them:

- **Prime multipliers**
  - \( \langle x_1, x_2 \rangle \)
  - \( \langle x_1, x_2, x_3 \rangle \)

- **Multipliers**
  - \( 1, z_2 + x_3 z_1 \)
  - \( z_1, z_1 z_2 \)

- **Operators**
  - \( 1, \partial_{x_2} + x_3 \partial_{x_1} \)
  - \( \partial_{x_1}, \partial_{x_1} \partial_{x_2} \)
Commutative Algebra

Fix $R = \mathbb{C}[x]$. Consider any ideal $I \subset R$. Associated primes $P_1, \ldots, P_k$. A differential primary decomposition of $I$ is a list $(P_1, A_1), \ldots, (P_k, A_k)$ where $A_i$ is a finite subset of $D_{n,n}$ with

$$I = \{ f \in R \mid \delta \cdot f \in P_i \text{ for all } \delta \in A_i \text{ and } i = 1, \ldots, k \}.$$ 

Its arithmetic multiplicity is $\text{amult}(I) = \sum_{j=1}^{k} \text{mult}_I(P_j)$, where

$$\text{mult}_I(P) = \frac{\text{degree}(\text{saturate}(I,P)/I)}{\text{degree}(P)}$$

is the length of the largest ideal of finite length in $R_P/IR_P$.

**Theorem**

There exists a differential primary decomposition of size $\text{amult}(I)$. This is a lower bound on the size of any such decomposition.

Yairon Cid-Ruiz and BSt: Primary decomposition with differential operators, 2101.03643
**Macaulay 2**

Computing a **minimal differential primary decomposition**:

```plaintext
i1 : needsPackage "NoetherianOperators"

i2 : R = QQ[x,y,z];

i3 : I = ideal(x^2,y^2,x*z-y*z^2);

o3 : Ideal of R

i4 : amult(I)

o4 = 4

i5 : solvePDE(I)

o5 = {{ideal (y, x),     {| 1 |, | zdx+dy |}},
      {ideal (z, y, x),  {| dx |, | dxdy | }}
```

**This is double line with a fat point:**

\[ P_1 = \langle x, y \rangle, \quad A_1 = \{1, z \partial_x + \partial_y\} \]

\[ P_2 = \langle x, y, z \rangle, \quad A_2 = \{\partial_x, \partial_x \partial_y\} \]
The treatment of Ehrenpreis-Palamodov in books on analysis emphasizes PDE for vector-valued functions $\psi : \mathbb{R}^n \to \mathbb{C}^k$.

[J.-E. Björk: Rings of Differential Operators], [L. Hörmander: An Introduction to Complex Analysis in Several Variables]

In calculus we learn how to rewrite one higher-order ODE as a system of first order ODE, and in algebraic geometry we learn how to appreciate matrix representations of geometric objects:

Ideals $\longrightarrow$ Schemes

Modules $\longrightarrow$ Coherent Sheaves

A system of $\ell$ linear PDE for $\psi$ is represented by an $\ell \times k$ matrix $A$ with entries in $R = \mathbb{C}[x_1, \ldots, x_n]$. We consider the submodule $M$ of $R^k$ spanned by the rows of $A$, and its primary decomposition

$$M = M_1 \cap \cdots \cap M_s.$$

Macaulay2 computes the **differential primary decomposition**
Coherent Sheaves

needsPackage "NoetherianOperators";
R = QQ[x1,x2,x3,x4];
M = image matrix {
{x1*x3, x1*x2, x1^2*x2},
{ x1^2,  x2^2, x1^2*x4}};
amult(M)
solvePDE(M)

We consider a PDE for functions $\psi : \mathbb{R}^4 \rightarrow \mathbb{C}^2$, or distributions:

$$\frac{\partial^2 \psi_1}{\partial z_1 \partial z_3} + \frac{\partial^2 \psi_2}{\partial z_1^2} = \frac{\partial^2 \psi_1}{\partial z_1 \partial z_2} + \frac{\partial^2 \psi_2}{\partial z_2^2} = \frac{\partial^3 \psi_1}{\partial z_1^2 \partial z_2} + \frac{\partial^3 \psi_2}{\partial z_1^2 \partial z_4} = 0.$$ 

We represent this by the matrix

$$A = \begin{bmatrix} \partial_1 \partial_3 & \partial_1^2 \\
\partial_1 \partial_2 & \partial_2^2 \\
\partial_1^2 \partial_2 & \partial_1^2 \partial_4 \end{bmatrix}.$$ 

The module $M \subset R^2$ has six associated primes, namely

$P_1 = \langle \partial_1 \rangle$, $P_2 = \langle \partial_2, \partial_4 \rangle$, $P_3 = \langle \partial_2, \partial_3 \rangle$, $P_4 = \langle \partial_1, \partial_3 \rangle$, $P_5 = \langle \partial_1, \partial_2 \rangle$, $P_6 = \langle \partial_1^2 - \partial_2 \partial_3, \partial_1 \partial_2 - \partial_3 \partial_4, \partial_2^2 - \partial_1 \partial_4 \rangle$.

$P_4, P_5$ are embedded. **Multiplicity**: $1+1+1+1+4+1 = 9 = \text{amult}(M)$. 
Syzygies .... in Control Theory

Theorem
Suppose the sequence \( R^{k'} \xrightarrow{B} R^k \xrightarrow{A} R^\ell \) is exact. Then TFAE:

1. The sequence \( R^{k'} \xleftarrow{B^T} R^k \xleftarrow{A^T} R^\ell \) is exact.
2. The module \( M = \text{im}_R(A^T) \) is \( \{0\} \)-primary.
3. The quotient of \( R^k \) modulo \( M \) is torsion-free.
4. The PDE \( A \) admits a vector potential, i.e. every solution to this PDE is obtained by applying the differential operator \( B \).

Example \((n = 3, k = 4, \ell = 2, m = 2)\)

\[
A = \begin{bmatrix}
\partial_1 & \partial_2 & \partial_3 & 0 \\
0 & \partial_1 & \partial_2 & \partial_3
\end{bmatrix}
\]

\[
\frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} + \frac{\partial \psi_3}{\partial x_3} = \frac{\partial \psi_2}{\partial x_1} + \frac{\partial \psi_3}{\partial x_2} + \frac{\partial \psi_4}{\partial x_3} = 0
\]

\[
B = \begin{bmatrix}
0 & -\partial_2^2 & \partial_2 \partial_3 & \partial_1 \partial_3 - \partial_2^2 \\
\partial_2^2 & -\partial_3 & \partial_2 \partial_3 & \partial_1 \partial_3 \\
-\partial_2 \partial_3 & \partial_1 \partial_3 & 0 & \partial_1 \partial_2 \\
\partial_2^2 - \partial_1 \partial_3 & -\partial_1 \partial_2 & \partial_1^2 & 0
\end{bmatrix}
\]

admits vector potential
Compact Support and Waves

If the PDE $A$ admits a vector potential then we can build solutions $\psi : \mathbb{R}^n \to \mathbb{C}^k$ with compact support. The following converse holds:

**Theorem**

*The PDE $A$ has compactly supported solutions if and only if the zero ideal $\{0\}$ is an associated prime of $M = \text{im}(A^T)$.*

[S. Shankar: *Controllability and Vector Potential*, Steklov Lectures, 2019]

Analysts are interested in wave solutions, and in particular in distributional solutions with low-dimensional support:


We develop algebraic tools for such constructing solutions in

[M. Härkönen, J. Hirsch and B. Sturmfels: Making waves, 2111.14045]
A *simple wave* is a trigonometric function

\[ \phi_{\xi,u} : \mathbb{R}^n \to \mathbb{C}^k, \ x \mapsto \exp(i\xi \cdot x) \cdot u, \quad \text{where } i = \sqrt{-1}. \]

Here \( \xi \in \mathbb{R}^n \) is the *frequency* and \( u \in \mathbb{C}^k \) is the *amplitude*.

**Lemma**

A *variety* characterizes simple wave solutions to the PDE \( A \):

\[ A \odot \phi_{\xi,u} = 0 \quad \text{if and only if} \quad A(\xi) \cdot u = 0. \]

Call \((u, \pi)\) a *wave pair* for \( A \) if \( u \in \mathbb{C}^k \) and \( \pi \) is a linear subspace of \( \mathbb{R}^n \) such that \( A(\xi)u = 0 \) for all \( \xi \in \pi \). A *classical wave solution* is any superposition of such simple waves, with frequencies \( \xi_1, \ldots, \xi_p \) in \( \pi \):

\[ \phi(x) = \sum_{j=1}^{p} \lambda_j \phi_{\xi_j,u}(x) \]

A *wave solution* is any distribution that arises as a limit.
Varieties

Consider first order PDEs, i.e. the entries of $A$ are linear forms.

We study the variety of all wave pairs $(u, \pi)$ in $\mathbb{P}^{k-1} \times \text{Gr}(r, n)$. The projection into $\mathbb{P}^{k-1}$ corresponds to wave cones in analysis.

→ Fano schemes and determinantal varieties in algebraic geometry

Example ($n = 3, k = 4, r = 2$, a surface in $\mathbb{P}^3 \times \mathbb{P}^2$)

\[
\begin{bmatrix}
\xi_1 & \xi_2 & \xi_3 & 0 \\
0 & \xi_1 & \xi_2 & \xi_3 \\
u_1 & u_2 & u_3 & u_4 \\
u_3 & u_2 & u_3 & u_4 \\
\end{bmatrix} =
\begin{bmatrix}
u_1 & u_2 & u_3 \\
u_2 & u_3 & u_4 \\
\xi_1 \\
\xi_2 \\
\end{bmatrix} =
\begin{bmatrix}0 \\
0 \\
\end{bmatrix}.
\]

Rank one constraint on $u$-matrix gives twisted cubic curve in $\mathbb{P}^3$.

For every amplitude $u$ in that wave cone, we obtain distributions $\psi$ that are supported on a line in $\mathbb{R}^3$. These waves satisfy our PDE

\[
\frac{\partial\psi_1}{\partial x_1} + \frac{\partial\psi_2}{\partial x_2} + \frac{\partial\psi_3}{\partial x_3} = \frac{\partial\psi_2}{\partial x_1} + \frac{\partial\psi_3}{\partial x_2} + \frac{\partial\psi_4}{\partial x_3} = 0
\]
Cayley’s Cubic Surface

This picture is the logo of the Nonlinear Algebra group at MPI Leipzig:

Let’s think of this surface as a PDE constraint for $\psi : \mathbb{R}^4 \to \mathbb{C}^3$:

$$A = \begin{bmatrix} \partial_1 & \partial_2 & \partial_3 \\ \partial_2 & \partial_1 & \partial_4 \\ \partial_3 & \partial_4 & \partial_1 \end{bmatrix}$$

**Quiz:** What does the command `solvePDE` in Macaulay2 tell us? What is the wave cone? What can we say about wave solutions?

Many Thanks for Listening