CodEx seminar

# Finite quantum dilogarithms

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Based on joint works with Campbell Wheeler and Stavros Garoufalidis

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# Metric groups

Let $G$ be a finite abelian group.

**Definition**

A Gaussian on $G$ is a function $\langle\cdot\rangle\colon G \to \mathbb{C}^\times$ such that $\langle x\rangle = \langle -x\rangle$ and

$$\langle x; y\rangle := \frac{\langle x+y\rangle}{\langle x\rangle\langle y\rangle}$$

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is a non-degenerate symmetric bicharacter on $G$.

We call a pair $(G, \langle\cdot\rangle)$ a **metric group**. We also have a Fourier transform on $L^2(G)$:

$$\widehat{f}(y) := \frac{1}{\sqrt{|G|}} \sum_{x\in G} f(x)\langle x; y\rangle^{-1}.$$

# Metric groups parameterized by $\mathrm{SL}_2(\mathbb{Z})$ matrices

**Example**

$G = \mathbb{Z}/N\mathbb{Z}$ with

$$\langle k \rangle = (-1)^k e^{\pi i k^2/N}, \qquad \langle k; \ell \rangle = e^{2\pi i k \ell / N}.$$

If $(G_1, \langle\cdot\rangle_1)$ and $(G_2, \langle\cdot\rangle_2)$ are metric groups, then so is $(G_1 \times G_2, \langle\cdot\rangle_1 \times \langle\cdot\rangle_2)$.

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**Example (Key example)**

Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be a hyperbolic matrix with $\operatorname{tr}\gamma = N + 2$, $N > 0$. Define $G_\gamma$ to be the set of fixed points of $\gamma$ on $(\mathbb{Q}/\mathbb{Z})^2$:

$$G_\gamma := \mathbb{Z}^2(\gamma - 1)^{-1}/\mathbb{Z}^2.$$

$$\langle u \rangle = s(u(\gamma-1))\,\mathbf{e}\big(\tfrac{1}{2}\,\omega(u, u\gamma)\big), \qquad \langle u; v \rangle = \mathbf{e}\big(\omega(u, v\gamma - v)\big).$$

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Here $\omega(v, w) = v_1 w_2 - v_2 w_1$, $\mathbf{e}(t) = e^{2\pi i t}$, and $s(n) = (-1)^{n_1 + n_2 + n_1 n_2}$ for $n \in \mathbb{Z}^2$.

# Specific examples

**Example (cyclic)**

$\gamma = \begin{pmatrix} N+2 & -1 \\ 1 & 0 \end{pmatrix}$. Then $G_\gamma$ is cyclic of order $N$, and for $u_k = (\frac{k}{N}, -\frac{k}{N})$

$$\langle u_k \rangle = (-1)^k e^{\pi i k^2/N}, \qquad \langle u_k; u_\ell \rangle = e^{2\pi i k\ell/N},$$

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## Example (product of two cyclic groups)

$\gamma = \begin{pmatrix} m^2+1 & m \\ m & 1 \end{pmatrix}$, $N = m^2$. Then $\gamma \equiv 1 \pmod{m}$ and $G_\gamma = \frac{1}{m}\mathbb{Z}^2/\mathbb{Z}^2 \cong (\mathbb{Z}/m\mathbb{Z})^2$.

$$\langle \tfrac{u}{m} \rangle = (-1)^{u_1} e^{\pi i u_1^2/m} \cdot (-1)^{u_2} e^{-\pi i u_2^2/m}, \qquad \langle \tfrac{u}{m}; \tfrac{v}{m} \rangle = e^{2\pi i (u_1 v_1 - u_2 v_2)/m},$$

so $G_\gamma \cong (\mathbb{Z}/m\mathbb{Z}, \langle\cdot\rangle) \times (\mathbb{Z}/m\mathbb{Z}, \overline{\langle\cdot\rangle})$.

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**Definition (Finite quantum dilogarithm)**

A **finite quantum dilogarithm** on $G$ is a function $E\colon G \to \mathbb{C}$ satisfying

$$E(x)E(y)\langle x; y\rangle = C + \frac{1}{\sqrt{|G|}} \sum_{t\in G} E(x-t)E(t)E(y-t)\langle t\rangle\,, \qquad x, y \in G$$

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The $N$ values $E(x)$, $x \in G$ satisfy a system of $N^2$ cubic equations.

# Comments on the definition

- **Andersen–Kashaev (2014):** a quantum dilogarithm on a LCA metric group is a function $f$ satisfying a certain reflection identity together with

$$\widehat{f}(x)\widehat{f}(y)\langle x;y\rangle = \int_G \widehat{f}(x-t)\widehat{f}(t)\widehat{f}(y-t)\langle t\rangle dt\,, \qquad x,y\in G\,.$$

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- **Izumi (1993, 2001):** essentially the same system of equations is the input data for near-group fusion categories, with simple objects $g \in G$ and $\rho$,

$$[g][h] = [g+h]\,, \qquad [g][\rho] = [\rho][g] = [\rho]\,, \qquad [\rho]^2 = N[\rho] + \sum_{g\in G}[g]\,.$$

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- Non-commutative pentagon equation with a defect: for a Heisenberg pair

$$(\mathbf{p}^x f)(t) = f(x+t)\,, \qquad (\mathbf{q}^y f)(t) = \langle t; y\rangle f(t)\,, \qquad \mathbf{p}^x\mathbf{q}^y = \langle x; y\rangle\, \mathbf{q}^y\mathbf{p}^x\,,$$

the equation for $E$ is an operator identity

$$\widehat{E}(\mathbf{p})\widehat{E}(\mathbf{q}) - \widehat{E}(\mathbf{q})\widehat{E}(\mathbf{p}+\mathbf{q})\widehat{E}(\mathbf{p}) = C\,\delta(\mathbf{q})\delta(\mathbf{p})\,, \qquad C \neq 0\,.$$

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$$g(\mathbf{r}) = \frac{1}{\sqrt{N}}\sum_{x\in G}\widehat{g}(x)\mathbf{r}^x, \qquad (\mathbf{p}+\mathbf{q})^x = \langle x\rangle\, \mathbf{q}^x\mathbf{p}^x\,.$$

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The operator notation is chosen so that $g(\mathbf{q})$ is the operator of multiplication by $g(x)$.

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**Theorem (R.–Wheeler, 2026)**

*Let $E\colon G\to\mathbb{C}$ be a finite quantum dilogarithm. Then*

1. $E(0)^2 = \sqrt{N}\,E(0)+1$ *and* $C=-\widehat{E}(0)$;
2. $E(u)E(-u)=\langle u\rangle^{-1}$ *for all* $u\neq 0$;
3. $\widehat{E}(u)=\lambda\,\langle u\rangle E(u)$ *for all* $u\in G$, *where* $\lambda=\widehat{E}(0)/E(0)$ *satisfies*

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$$\lambda^3 = \frac{1}{\sqrt{N}}\sum_{x\in G}\langle x\rangle^{-1}.$$

In particular, $E(0)=\frac{\sqrt{N}\pm\sqrt{N+4}}{2}=\pm\varepsilon^{\pm 1/2}$, where $\varepsilon>1$ satisfies $\varepsilon^2-(N+2)\varepsilon+1=0$.

# Alternative forms of the pentagon relation

1. **Pentagon.**
$$\langle x;y\rangle E(x)E(y) = -\widehat{E}(0) + \frac{1}{\sqrt{N}}\sum_{t\in G} E(y-t)E(t)\langle t\rangle E(x-t).$$

2. **Dual pentagon.** For $\widetilde{E}(x) = \langle x\rangle E(x)$.
$$\langle x;y\rangle^{-1}\widetilde{E}(x)\widetilde{E}(y) = -\lambda^{-1}E(0) + \frac{1}{\sqrt{N}}\sum_{t\in G}\widetilde{E}(y-t)\widetilde{E}(t)\langle t\rangle^{-1}\widetilde{E}(x-t).$$

3. **2-3 product relation.**
$$\frac{1}{\sqrt{N}}\sum_{t\in G} E(t)\langle t\rangle E(x-t)\langle t;y\rangle = \langle x+y\rangle E(-x-y)E(x)E(y) - NE(0)\delta(x)\delta(y).$$

4. **Ratios.**
$$\frac{1}{\sqrt{N}}\sum_{t\in G}\frac{E(v+t)}{E(t)}\langle t;-u\rangle = E(v)\Big(\frac{E(u)}{E(u+v)} - 1\Big) + \sqrt{N}\frac{\delta(u+v)}{\langle v\rangle}$$

5. **Fivefold symmetry.** The operator $(\mathbf{K}f)(x) = E(x)\widehat{f}(-x)$ satisfies
$$\operatorname{im}(\mathbf{K}^5 - \lambda^2\,\mathrm{Id}) \subseteq \mathbb{C}1 + \mathbb{C}\delta + \mathbb{C}E.$$

# Some other interesting identities

Let $E\colon G \to \mathbb{C}$ be a finite quantum dilogarithm, $\lambda = \widehat{E}(0)/E(0)$.

- **Fourier transform of $1/E$:**

$$\widehat{1/E}(u) = \lambda^{-1} E(u) - 1\,, \qquad u \in G\,.$$

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- **Convolutions of $|E|^2$:** for all $x \neq 0$,

$$\sum_{u+v=x} |E(u)|^2|E(v)|^2 = |E(x)|^2 \sum_{u+v=-x} |E(u)|^2|E(v)|^2\,.$$

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No examples are known for $G$ that are products of three or more cyclic groups.

# Faddeev's modular quantum dilogarithm

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$$(z;q)_\infty = \prod_{n\geq 0}(1-zq^n), \qquad \mathbf{e}(z) = e^{2\pi i z}.$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

$$\Phi_{\gamma,m,n}(z;\tau) = \frac{(\mathbf{e}(z+m\tau);\mathbf{e}(\tau))_\infty}{(\mathbf{e}(\frac{z}{c\tau+d}+n\gamma\tau);\mathbf{e}(\gamma\tau))_\infty}.$$

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**Key property**: meromorphic continuation to $c\tau + d \notin \mathbb{R}_{\leq 0}$.

Real FQDs $E\colon G_\gamma \to \mathbb{C}$ are given by special values of $\Phi_{\gamma,m,n}$ at $(z,\tau)$ satisfying

$$\tau = \gamma\tau\,, \qquad \mathbf{e}(z+m\tau) = \mathbf{e}(\tfrac{z}{c\tau+d}+n\gamma\tau).$$

# Application ?: Heisenberg orbit of a unitary finite quantum dilogarithm

Let $E\colon G \to \mathbb{C}$ be a unitary finite quantum dilogarithm, so $E(0) = -\varepsilon^{-1/2}$. Let

$$A(x,y) := \frac{1}{\sqrt{N}} \sum_{t \in G} E(x+t)\langle t; y\rangle \overline{E(t)} = \langle \mathbf{q}^y \mathbf{p}^x E,\, E\rangle_{L^2(G)}\,.$$

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In particular

$$|A(x,y)| = \begin{cases} 1\,, & x \neq 0,\ y \neq 0,\ x \neq y\,, \\ \varepsilon^{-1/2}\,, & \text{exactly one of } x=0,\ y=0,\ x=y \text{ holds}\,, \\ \sqrt{N} - \varepsilon^{-1/2}\,, & x = y = 0\,. \end{cases}$$

# Cohn's problem

**Problem** (Harvey Cohn). Let $f : \mathbb{F}_p \to \mathbb{C}$ satisfy $f(0) = 0$, $|f(x)| = 1$ for $x \neq 0$, and

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Equivalently: $f(0) = \widehat{f}(0) = 0$ and $|f| = |\widehat{f}| = 1$ away from 0.

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Equivalently: $f(0) = \widehat{f}(0) = 0$ and $|f| = |\widehat{f}| = 1$ away from 0.

Characters are examples, since $\widehat{\chi}(x) = \frac{G(\chi)}{\sqrt{p}}\,\overline{\chi(-x)}$.

# Cohn's problem

**Problem** (Harvey Cohn). Let $f\colon \mathbb{F}_p \to \mathbb{C}$ satisfy $f(0) = 0$, $|f(x)| = 1$ for $x \neq 0$, and

$$\sum_t f(t+x)\overline{f(t)} = -1 \qquad (x \neq 0)\,.$$

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- Biró (1999): finitely many solutions for each $p$; a $(\pm 1)$-valued solution is the Legendre symbol.
- Choi–Siu (2000): the answer is no over $\mathbb{F}_{p^k}$, $k > 1$.
- Kurlberg (2002): yes if the non-zero values are roots of unity.
- Klurman–Mangerel (2018): an approximate version for multiplicative functions.
- Benoist (2024): the answer is **no** for every prime $p \geq 11$.

# Application I ($\gamma$): Cohn-like functions

**Unitary finite quantum dilogarithms** have the same shape, with a small value at 0:

$$E(0) = -\varepsilon^{-1/2}, \qquad |E(x)| = |\widehat{E}(x)| = 1 \quad (x \neq 0),$$

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- $\widehat{E}(x) = \lambda\, \overline{E(-x)}$ for $x \neq 0$, exactly as for characters.
- if $|f(0)| = a$ and $|f| = |\widehat{f}| = 1$ away from 0, then
  $a = 1$: biunimodular; $a = \varepsilon^{-1/2}$: unitary dilogarithms; $a = 0$: Cohn.

# Towers of finite quantum dilogarithms

A fixed point of $\gamma^k$ is a fixed point of $\gamma^\ell$ when $k \mid \ell$:

$$G_{\gamma^k} \subseteq G_{\gamma^\ell}\,, \qquad k \mid \ell\,.$$

The real finite quantum dilogarithms $E_{\gamma^k}$ on $G_{\gamma^k}$ satisfy connecting relations.

- $\gamma$**-invariance.**

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Applying one Galois automorphism to the whole tower gives a tower of unitary finite quantum dilogarithms, for the conjugated Gaussians.

# Application II ($\gamma^2$): biunimodular functions

$f : G \to \mathbb{C}$ is **biunimodular** if $|f| \equiv 1$ and $|\widehat{f}| \equiv 1$. Classical example: Gaussians.

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Inside $G_{\gamma^2}$, of order $N(N+4)$, $N$ odd, let

$$G_+ = \{x\colon x\gamma = x\} = G_\gamma\,, \qquad G_- = \{x\colon x\gamma = -x\}\,, \qquad |G_-| = N+4\,.$$

For $x \in G_{\gamma^2}$ define $g_x\colon G_- \to \mathbb{C}$ by $g_x(z) = E_{\gamma^2}(x+z)\,\langle z\rangle_-$, where $\langle z\rangle_-^2 = \langle z\rangle$.

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**Theorem (Garoufalidis–R.–Wheeler, 2026)**

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For a unitary tower and $x \notin G_-$ this gives $|g_x| \equiv |\widehat{g_x}| \equiv 1$:
$(N-1)$ biunimodular functions on a group of order $N+4$, with $\widehat{g_x} = c\,\overline{g_x}$.

# Application III ($\gamma^3$): complex equiangular lines

On $L^2(\mathbb{Z}/M\mathbb{Z})$ with $\langle f, g\rangle = \sum_k \overline{f(k)}g(k)$ and $\zeta_M = e^{2\pi i/M}$, let

$$(\mathbf{p}f)(k) = f(k+1)\,, \qquad (\mathbf{q}f)(k) = \zeta_M^{-k} f(k)\,, \qquad \mathbf{pq} = \zeta_M^{-1}\mathbf{qp}\,.$$

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The $M^2$ vectors $\mathbf{p}^k\mathbf{q}^\ell f$ span equiangular lines precisely when

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**Conjecture (Zauner)**

*For all $M \geq 2$ there exists $f \in L^2(\mathbb{Z}/M\mathbb{Z})$ satisfying (SIC).*

# Connection to class field theory

- Appleby, Yadsan-Appleby, Zauner: known SICs in dimensions $M > 3$ are defined over abelian extensions of $K = \mathbb{Q}\big(\sqrt{(M-3)(M+1)}\big)$.
- Appleby, Flammia, McConnell, Yard: precise conjectures relating SICs to ray class fields of $K$.
- Donnelly, McConnell: connection with Stark units.
  Kopp: conjectural construction of SICs from Stark units.
- Appleby, Flammia, Kopp: general conjectural construction for all $M$.

# A reduction of Zauner's conjecture

Appleby–Flammia–Kopp built candidates for $A_{k,\ell} = \langle f, \mathbf{p}^k \mathbf{q}^\ell f \rangle$ from Stark units and reduced the problem to:

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1. **Quadratic identities**: reflection and twisted convolution,

$$A_{k,\ell} A_{-k,-\ell} = \frac{\zeta_M^{-k\ell}}{M+1}, \qquad (k,\ell) \neq (0,0), \qquad \sum_{i,j} A_{i,j} A_{r-i,s-j}\, \zeta_M^{-i(s-j)} = M A_{r,s}.$$

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2. **Unitary Galois conjugates**: the units have absolute value 1 under complex embeddings extending the non-trivial automorphism of $K$.

(2) follows from the reciprocity laws for Stark units (Stark, Appleby–Flammia–Kopp).

# Explicit solution of the twisted convolution

Let $M \geq 4$, $\alpha = \left(\begin{smallmatrix} M-1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\gamma = \alpha^3$, and let $\varepsilon > 1$ be the unit of $\alpha$: $\varepsilon + \varepsilon^{-1} = M - 1$. Then $\gamma \equiv 1 \bmod M$, so

$$|G_\gamma| = M^2(M-3)\,, \qquad G_\gamma[M] = \tfrac{1}{M}\mathbb{Z}^2/\mathbb{Z}^2 \cong (\mathbb{Z}/M\mathbb{Z})^2\,.$$

$$P = \frac{1}{M}\sum_{i,j=0}^{M-1} E_\gamma\left(\tfrac{i}{M}, \tfrac{j}{M}\right)(-1)^{i+j}\,\mathbf{e}_{2M}(i^2+j^2)\,\mathbf{p}^i\mathbf{q}^j\,.$$

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For a unitary Galois conjugate, the 1-dimensional eigenspace is spanned by a fiducial $f$.

# Some other interesting constructions

- Beyond $\gamma^j$, $j = 1, 2, 3$: $r$-SICs from $\gamma^r$ (Appleby–Flammia–Kopp, 2026).

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- $E_{\gamma^2}$ solves Izumi's equations for the Drinfeld center of the near-group category of $G_\gamma$, giving its modular data (Garoufalidis–R.–Wheeler, 2026).
- Likely there is much more...

Thank you!