Applications of Symplectic Geometry to Frame Theory

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joint work with Clayton Shonkwiler (Colorado State University)

CODEX Seminar
September 14, 2021

Outline

- Frames are collections of vectors $\{f_i\}_{i=1}^N \subset \mathbb{C}^d$ used to give robust representations of signals $v \in \mathbb{C}^d$ via

$$v \mapsto \left(\left\langle v, f_j \right\rangle \right)_{j=1}^N \in \mathbb{C}^N$$

- We are interested in the geometry and topology of subsets of frames with prescribed vector norms and spectral data

- Certain properties of these frame spaces can be seen through the lens of symplectic geometry

- We will introduce the main ideas of symplectic geometry, illustrate them in the context of frames and use them to derive new results on frame spaces

Frames

Definition. A frame is a spanning collection of N vectors in $\mathbb{C}^d$.
We pick an ordering of these vectors f_i and represent the frame as

$$F = \left[f_1 \mid f_2 \mid \cdots \mid f_N \right] \in \mathbb{C}^{d \times N}$$

Let $\langle f, g \rangle = g^* f$ denote the standard Hermitian product on $\mathbb{C}^d$.

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A frame determines several operators.

- Analysis operator: $v \mapsto F^* v$

A frame F gives a redundant representation of a signal $v \in \mathbb{C}^d$ via the analysis operator

$$v \mapsto \left(\langle v, f_1 \rangle, \ldots, \langle v, f_N \rangle \right) = F^* v$$

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A frame determines several operators.

- Analysis operator: $v \mapsto F^* v$
- Synthesis operator: $w \mapsto Fw$

A collection of measurements $w = [w_1, \ldots, w_N]^T \in \mathbb{C}^N$ determines a signal via the synthesis operator

$$w \mapsto w_1 f_1 + \cdots + w_N f_N = Fw$$

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- Analysis operator: $v \mapsto F^* v$
- Synthesis operator: $w \mapsto F w$
- Frame operator: $v \mapsto F F^* v$

The composition of the two is called the frame operator

$$v \mapsto F F^* v$$

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A frame is called tight if

$$FF^* = \frac{N}{d} \mathrm{I}_d$$

(convenient normalization)

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A frame determines several operators.

- Analysis operator: $v \mapsto F^* v$
- Synthesis operator: $w \mapsto F w$
- Frame operator: $v \mapsto F F^* v$

Let $\mathscr{F}^{d,N}$ denote the collection of all frames of size $d \times N$.

Standard Geometric Structures

Let
$$f = [x_1 + iy_1, \ldots, x_d + iy_d]^T$$
$$g = [u_1 + iv_1, \ldots, u_d + iv_d]^T$$

The standard inner product on $\mathbb{R}^{2d} \approx \mathbb{C}^d$:

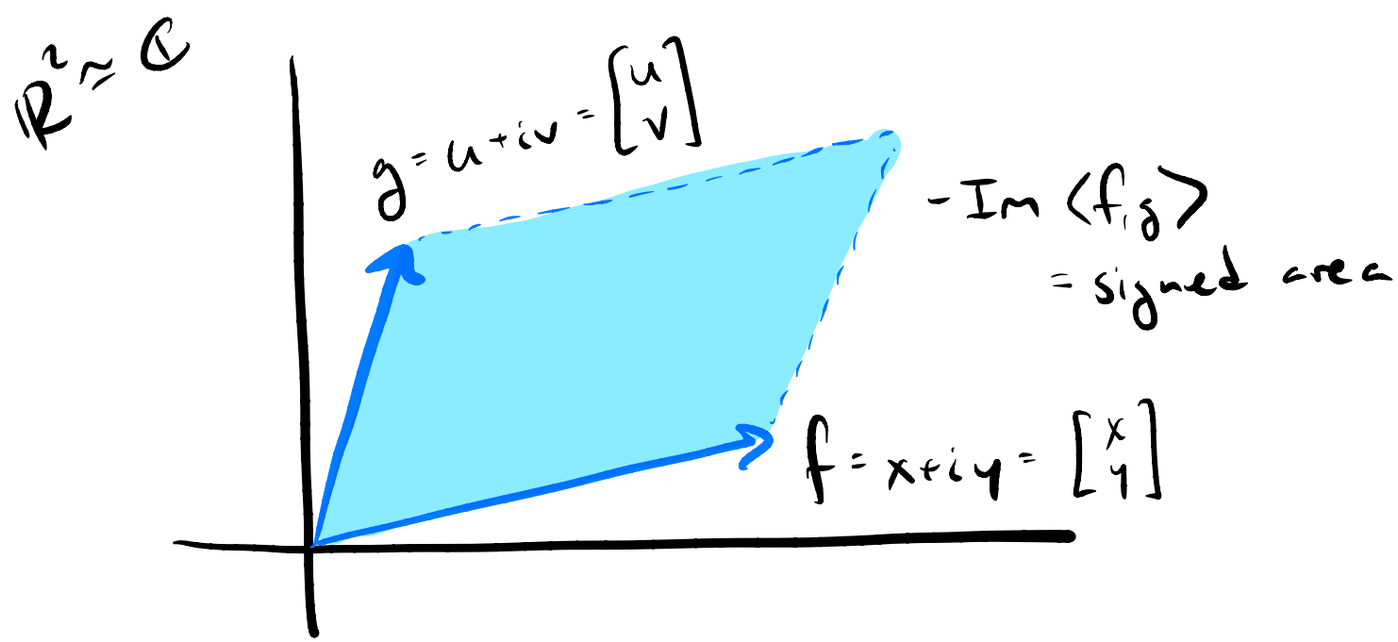
$$\mathrm{Re}\langle f, g\rangle = x_1 u_1 + y_1 v_1 + \cdots + x_d u_d + y_d v_d$$

Standard Geometric Structures

Let
$$f = [x_1 + iy_1, \ldots, x_d + iy_d]^T$$
$$g = [u_1 + iv_1, \ldots, u_d + iv_d]^T$$

The standard symplectic structure on $\mathbb{R}^{2d} \approx \mathbb{C}^d$:

$$-\mathrm{Im}\langle f, g\rangle = x_1 v_1 - y_1 u_1 + \cdots + x_d v_d - y_d u_d$$

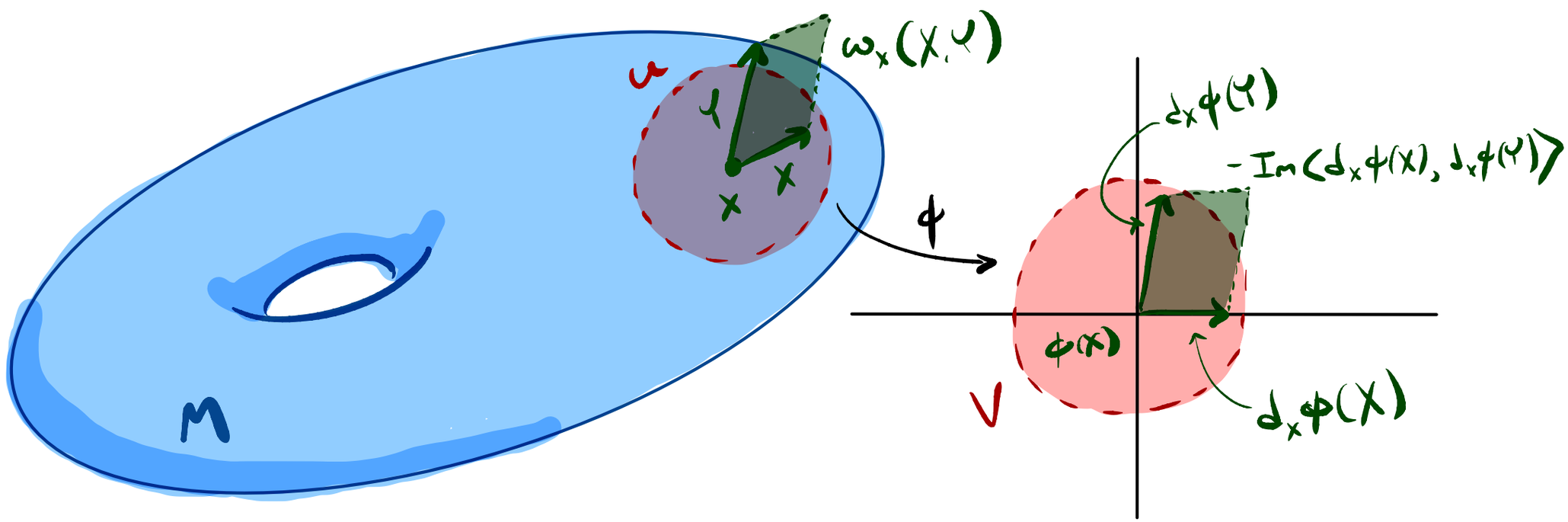


Symplectic Structures

Let M be a smooth manifold.

Definition (Local Coords). A symplectic structure ω on M is a smooth assignment of a map $\omega_x : T_xM \times T_xM \to \mathbb{R}$ such that $(M, \omega) \approx (\mathbb{R}^{2d}, -\operatorname{Im}\langle \cdot\,, \cdot \rangle)$, locally.

That is, near each point in M there exists a local diffeomorphism $\phi : M \supset U \to V \subset \mathbb{R}^{2d}$ with $\omega = \phi^* \left(-\operatorname{Im}\langle \cdot\,, \cdot \rangle\right)$.



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Definition (2-Form). A symplectic structure ω on M is a closed, nondegenerate 2-form.

That is, ω:

- is bilinear
- is non-degenerate: $\omega_x(u, v) = 0 \ \forall v \Rightarrow u = 0$
- is skew-symmetric: $\omega_x(u, u) = 0$
- is closed: $d\omega = 0$

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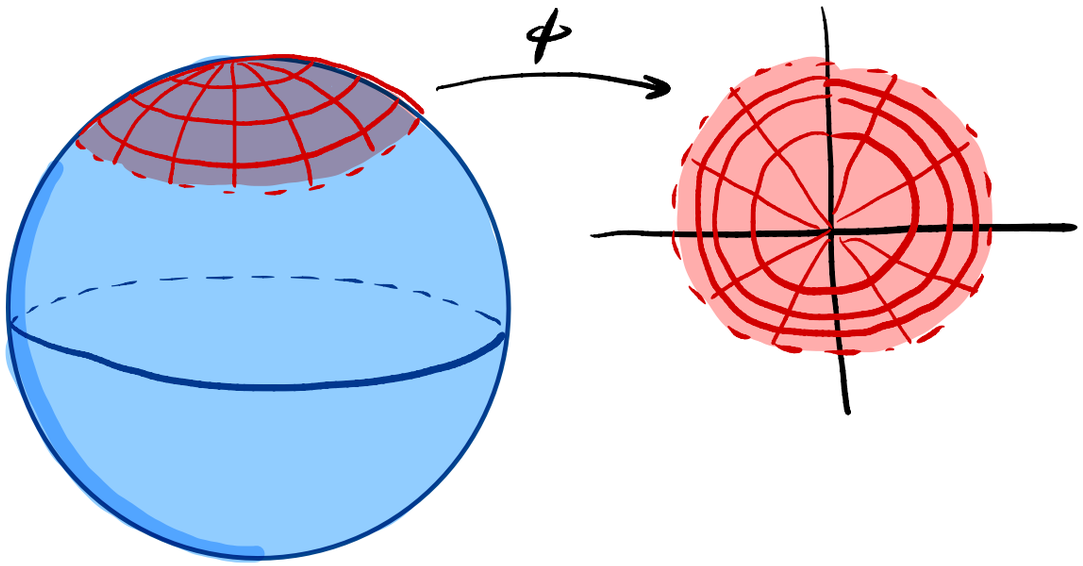
Definition (2-Form). A symplectic structure ω on M is a closed, nondegenerate 2-form.

Darboux's Theorem. These two definitions are equivalent.

All symplectic manifolds are locally the same!
Cf. Riemannian structures: no local iso of the 2-sphere to $\mathbb{R}^2$ with dot product.

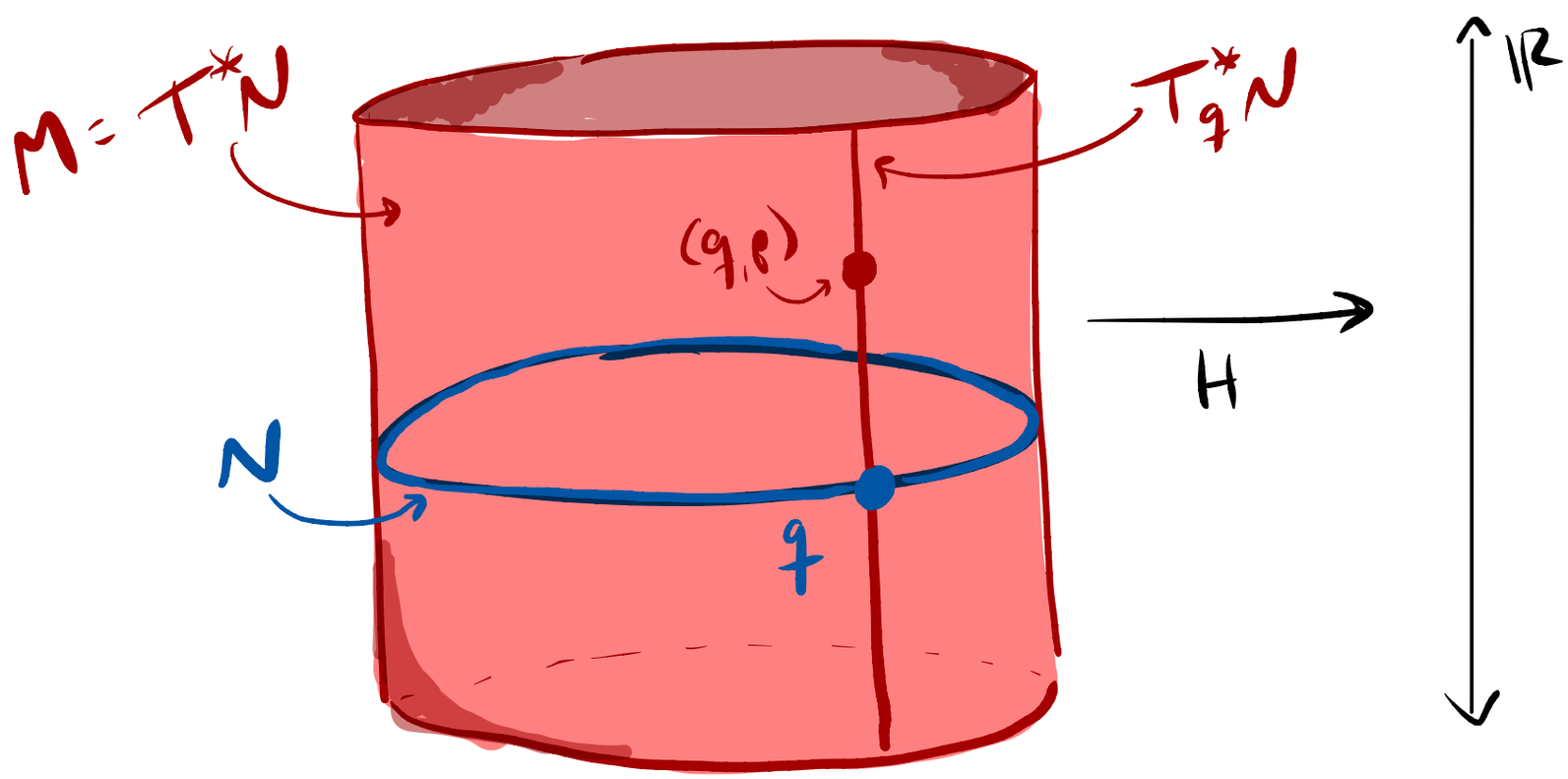


Symplectic Structures: Physical Interpretation

Definition. Let M be a smooth manifold. A symplectic structure ω on M is a closed, nondegenerate 2-form.

Intuition: Consider M as the phase space of a physical system. E.g., $M = T^*N$ with $(q, p) \in T^*N$ representing position and momentum.

Consider a smooth function $H : M \to \mathbb{R}$ as total energy of a phase.



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- ω is a bilinear form: turns H into a vector field V_H by solving

$$d_x H(v) = \omega_x(V_H(x), v).$$

- ω is non-degenerate: this equation can be solved for V_H.

Symplectic form turns an energy function into phase space dynamics.

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$$d_x H(v) = \omega_x(V_H(x), v).$$

- ω is non-degenerate: this equation can be solved for V_H.

- ω is skew-symmetric: phase space trajectories are constant energy

$$d_x H(V_H(x)) = \omega_x(V_H(x), V_H(x)) = 0$$

Symplectic Structures: Physical Interpretation

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Intuition: Consider M as the phase space of a physical system. E.g., $M = T^*N$ with $(q, p) \in T^*N$ representing position and momentum.

Consider a smooth function $H : M \to \mathbb{R}$ as total energy of a phase.

- ω is closed: laws of physics are time-invariant

$$\begin{aligned}\mathscr{L}_{V_H}\omega &= d\iota_{V_H}\omega + \iota_{V_H}d\omega \\ &= d^2H(\,\cdot\,) + d\omega(V_H, \cdot\,, \cdot\,) \\ &= 0.\end{aligned}$$

Examples

- $\left(\mathbb{C}^d, -\operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle\right)$

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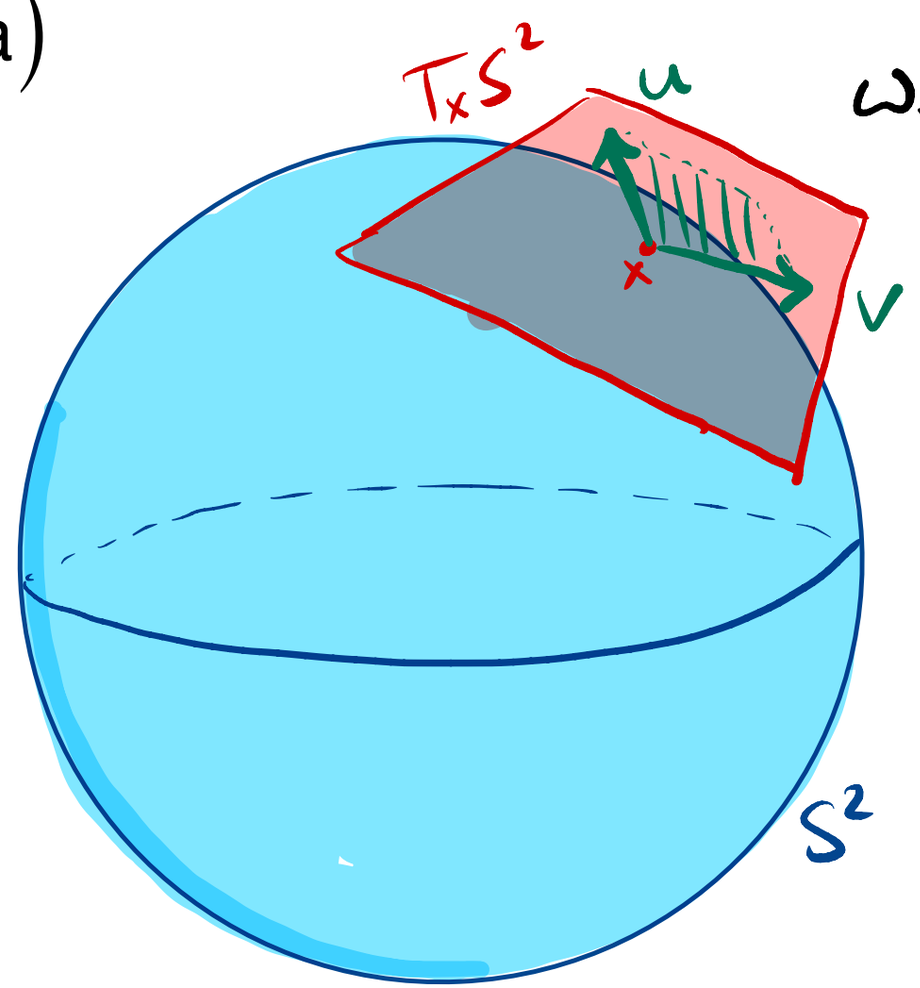
- $\left(\mathbb{C}^d, -\operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle\right)$
- $\left(\mathscr{F}^{d,N}, -\operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle\right) \subset \left(\mathbb{C}^{d\times N}, -\operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle\right) \approx \left(\mathbb{C}^{d\cdot N}, -\operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle\right)$

Frobenius inner product

Examples

- $\left(\mathbb{C}^d, -\operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle\right)$
- $\left(\mathscr{F}^{d,N}, -\operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle\right)$
- $\left(S^2, \omega = \text{signed area}\right)$



$\omega_x(u,v)$

$=$ signed area

$= x\cdot(u\times v)$

Examples

- $\left(\mathbb{C}^d, -\operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle\right)$
- $\left(\mathscr{F}^{d,N}, -\operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle\right)$
- $\left(S^2, \omega = \text{signed area}\right)$

$H : S^2 \to \mathbb{R}$

$H(x,y,z) = z$

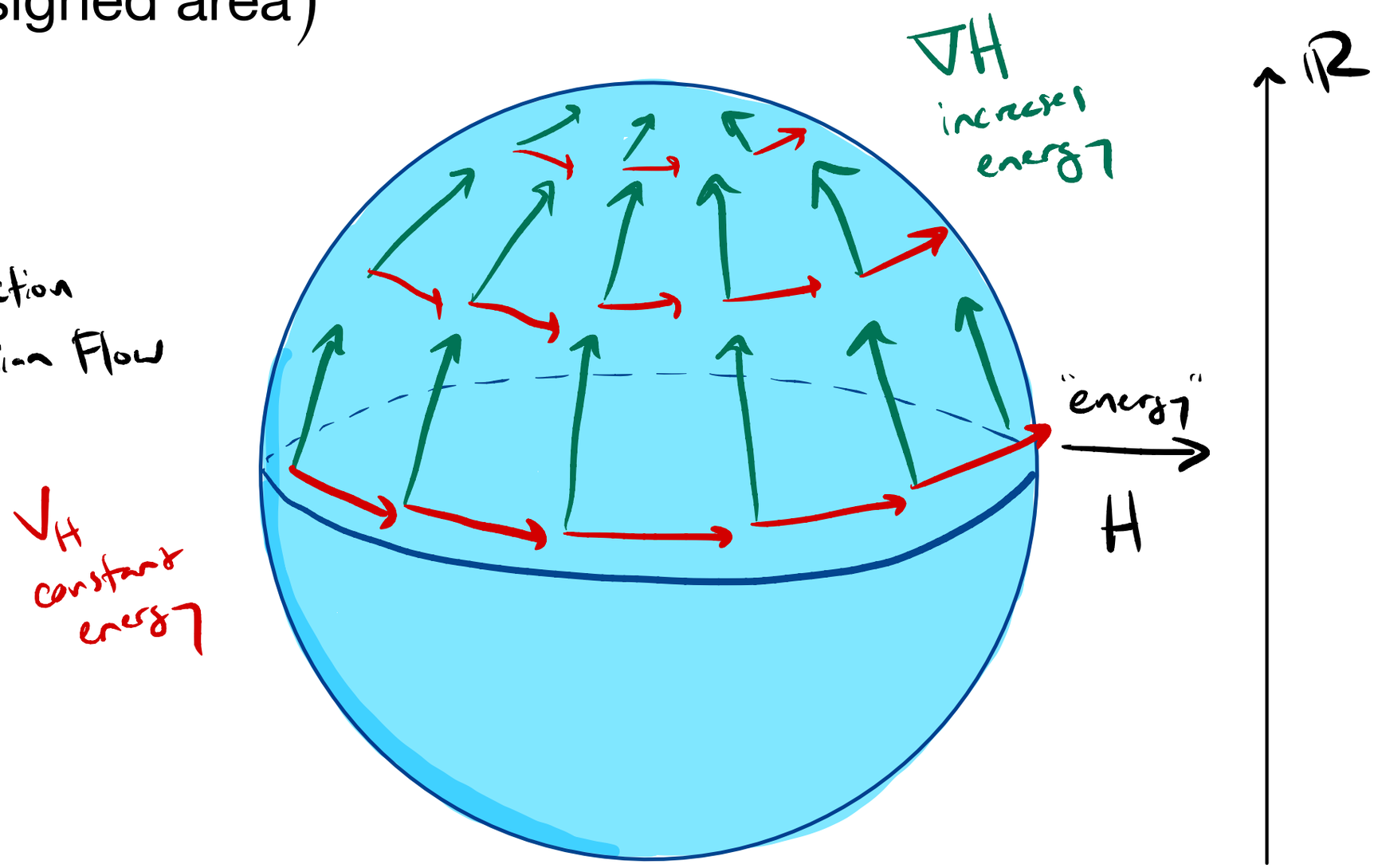
$d_{(x,y,z)}H(u,v,w) = w$

$V_H(x,y,z) = (-y, x, 0)$

$\omega_{(x,y,z)}\big((-y,x,0),(u,v,w)\big) = w$

Ex

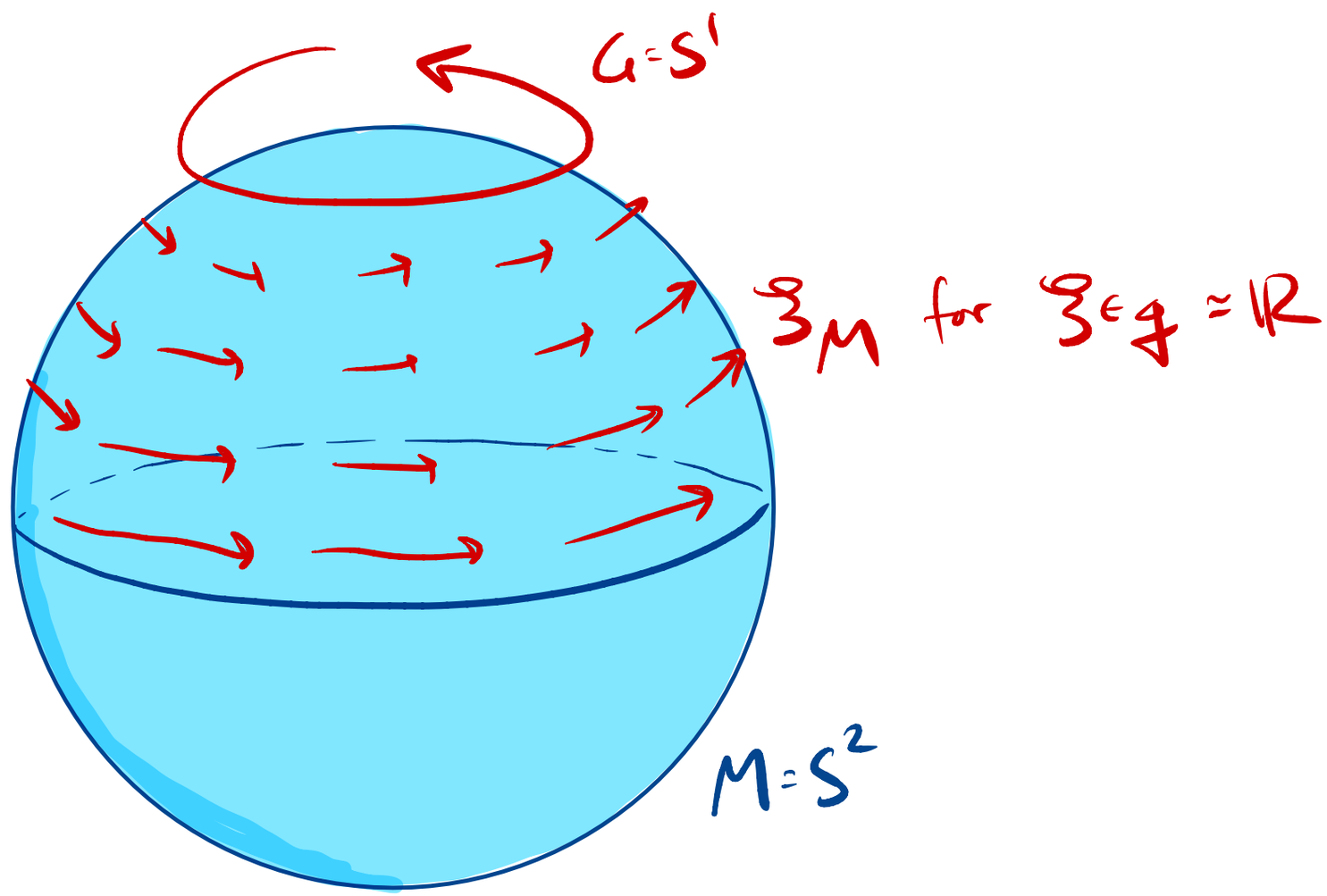
Energy Function and Hamiltonian Flow



Hamiltonian Group Actions

Let (M, ω) be a symplectic manifold with action by a Lie group G.
$\forall \xi \in \mathfrak{g}$, $\exists$ associated vector field ξ_M called the infinitesimal action:

$$\xi_M(x) = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot x$$



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The action is called Hamiltonian if it admits a momentum map:

$$\mu : M \to \mathfrak{g}^*$$

such that μ is equivariant with respect to the coadjoint action,

$$\mu(g \cdot x) = \mathrm{Ad}_g^* \cdot \mu(x),$$

and

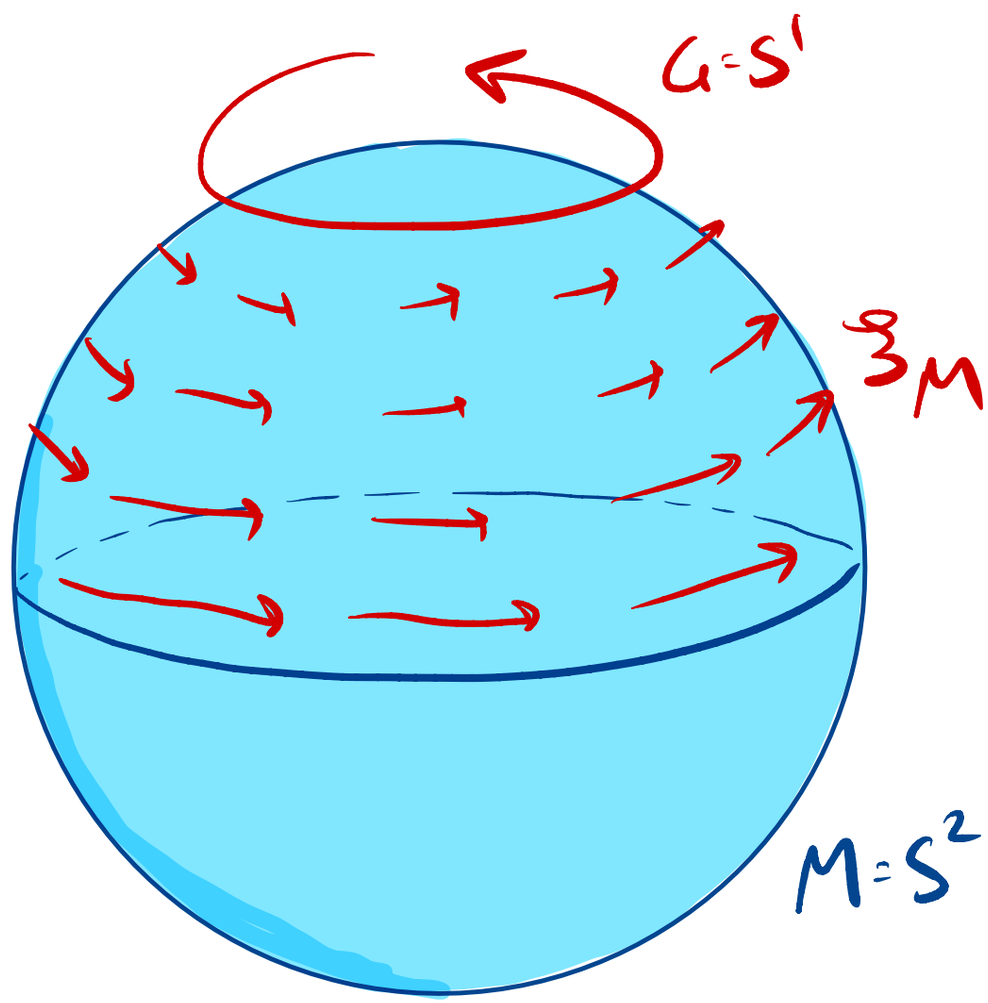
$$d_x\mu(v)(\xi) = \omega_x(\xi_M(x), v)\,.$$

Examples of Momentum Maps

Consider S^2 with $G = S^1$-action given by rotating around the z-axis.

Identifying $\mathfrak{g}^* \approx \mathbb{R}$, this action has a momentum map

$$\mu(x) = x \cdot [0,0,1]^T$$



$$d_{(x,y,z)}\mu(u,v,w)(\xi) = \xi w$$

$$\xi_M(x,y,z) = \xi \cdot (-y, x, 0)$$

$$\omega_{(x,y,z)}(\xi_M, (u,v,w)) = \xi w$$

Examples of Momentum Maps

Consider frame space $\mathscr{F}^{d,N} \subset \mathbb{C}^{d\times N}$ with $G = \mathrm{U}(d)$-action given by left multiplication.

Identifying $\mathfrak{g}^* \approx \mathscr{H}^{d\times d}$, the vector space of $d \times d$ Hermitian matrices, this action has momentum map

$$\mu(F) = FF^* = \text{frame operator}$$

Proof

$$\mathcal{H}^{d\times d} \xrightarrow{\approx} \mathfrak{u}(d)^* : \xi \longmapsto \left(\zeta \longmapsto \frac{i}{2}\operatorname{tr}(\xi\zeta)\right)$$

$$d_F\mu(G)(\zeta) = \frac{i}{2}\operatorname{tr}\left((FG^* + GF^*)\zeta\right)$$

$$\omega_F(\zeta_M, G) = \omega_F(\zeta F, G) = -\operatorname{Im}\operatorname{tr}(G^*\zeta F)$$

$$= \frac{i}{2}\operatorname{tr}(\zeta FG^* + \zeta GF^*)$$

$$= \frac{i}{2}\operatorname{tr}\left((FG^* + GF^*)\zeta\right)$$

Reduction of Symmetry

Marsden-Weinstein Reduction Theorem. Let (M, ω) be a symplectic manifold with Hamiltonian G-action and momentum map $\mu : M \to \mathfrak{g}^*$. Then the space

$$M//G := \mu^{-1}(0)/G$$

has a natural symplectic structure, provided 0 is a regular value and G acts freely on the level set.

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, hook, "\iota"] \arrow[d, two heads, "q"'] & (M, \omega) \\ (M//G, \omega_{red}) & \end{tikzcd}$$

$$\boxed{\iota^*\omega = q^*\omega_{red}}$$

Marsden and Weinstein, Reports on Mathematical Physics, 1974

Reduction of Symmetry and Frame Spaces

Marsden-Weinstein Theorem. $M//G := \mu^{-1}(0)/G$ is symplectic.

Proposition (N-Shonkwiler). The space of unitary orbits of tight frames is a symplectic manifold.

Proof.

$$\left\{ F \in \mathbb{C}^{d\times N} \mid FF^* = \frac{N}{d}\mathrm{I}_d \right\}/\mathrm{U}(d) = \mu^{-1}(0)/\mathrm{U}(d)$$

$$\mu : \mathscr{F}^{d,N} \to \mathfrak{u}(d)^* \approx \mathscr{H}^{d\times d}$$

where

$$F \mapsto FF^* - \frac{N}{d}\mathrm{I}_d$$

is a momentum map for the $\mathrm{U}(d)$-action. ∎

Reduction of Symmetry and Frame Spaces

Marsden-Weinstein Reduction Theorem, Refined. Let (M, ω) be a symplectic manifold with Hamiltonian G-action and momentum map $\mu : M \to \mathfrak{g}^*$. For each $\chi \in \mathfrak{g}^*$ with coadjoint orbit $\mathcal{O}_\chi$, the space

$$M//_\chi G := \mu^{-1}(\mathcal{O}_\chi)/G$$

has a natural symplectic structure, provided $\mathcal{O}_\chi$ consists of regular values and G acts freely on the level set.

$$\begin{tikzcd} \mu^{-1}(\mathcal{O}_\chi) \arrow[r, hook, "\iota"] \arrow[d, "q"'] & (M, \omega) \\ (M//_\chi G, \omega_{red}) & \end{tikzcd}$$

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Marsden and Weinstein, Reports on Mathematical Physics, 1974

Reduction of Symmetry and Frame Spaces

Marsden-Weinstein Theorem. $M//_{\chi}G := \mu^{-1}(\mathcal{O}_{\chi})/G$ is symplectic.

Proposition (N-Shonkwiler). The space of unitary orbits of frames with prescribed frame operator spectrum is a symplectic manifold.

Proof.

For $\Lambda = (\lambda_j)_{j=1}^{d}$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d > 0$, let

$$\mathcal{F}_{\Lambda}^{d,N} := \left\{ F \in \mathcal{F}^{d,N} \mid FF^* \text{ has spectrum } \Lambda \right\}$$

Then

$$\mathcal{F}_{\Lambda}^{d,N}/\mathrm{U}(d) = \mu^{-1}(\mathcal{O}_S)/\mathrm{U}(d),$$

$S \in \mathcal{H}^{d\times d}$ with spectrum Λ

where

$$\mu : \mathcal{F}^{d,N} \to \mathfrak{u}(d)^* \approx \mathcal{H}^{d\times d}$$
$$F \mapsto FF^*$$

is a momentum map for the $\mathrm{U}(d)$-action. ∎

Application: Frame Homotopy

Definition. A finite unit norm tight frame (FUNTF) is a frame $F = [f_1 \mid f_2 \mid \cdots \mid f_N] \in \mathbb{C}^{d \times N}$ with $\|f_j\| = 1$ for all j and $FF^* = \frac{N}{d}\mathrm{I}_d$.

Frame Homotopy Conjecture (Larson, 2002). The space of FUNTFs is path connected.

Proved by Cahill, Mixon and Strawn in 2017. Proof is constructive. Also applies to the space of real FUNTFs.

Cahill, Mixon and Strawn, SIAGA, 2017

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Theorem (N-Shonkwiler). Let $(\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$ and S an invertible $d \times d$ Hermitian matrix. The space of frames F with $f_j = \gamma_j$ and $FF^* = S$ is path connected.

Proof follows “easily” from basic symplectic principles (but doesn’t apply to real frames).

Cahill, Mixon and Strawn, SIAGA, 2017

Torus Action and Frame Homotopy

Let $\mathbb{T} := \mathrm{U}(1)^N/\mathrm{U}(1)$

$$\approx \{\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_{N-1}}, 1) \mid \theta_j \in [0, 2\pi)\}$$

$$\approx \mathrm{U}(1)^{N-1}$$

$\mathbb{T}$ acts on $\mathscr{F}_{\Lambda}^{d,N}/\mathrm{U}(d)$ by right multiplication, with momentum map

$$\mu_{\mathbb{T}} : \mathscr{F}_{\Lambda}^{d,N}/\mathrm{U}(d) \to \mathfrak{t}^* \approx \mathbb{R}^{N-1}$$

$$[F] \mapsto \left(-\frac{1}{2}\|f_j\|^2\right)_{j=1}^{N-1}$$

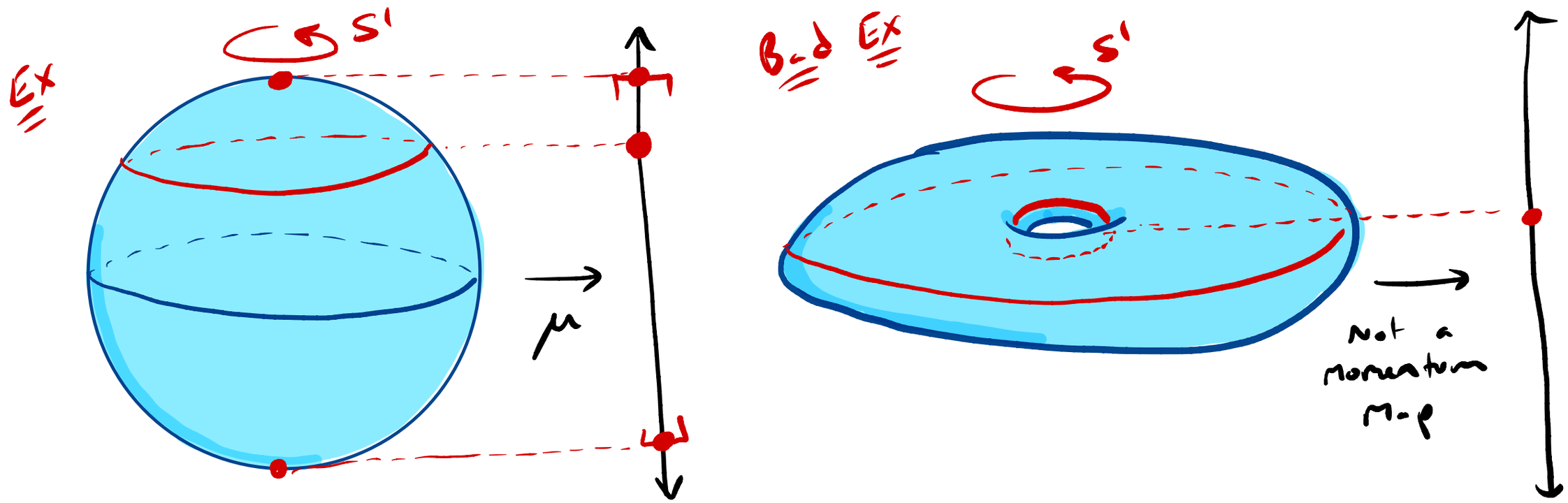
Image is a convex polytope. A norm vector Γ lies in the image $\Leftrightarrow$ Γ is Λ-admissible: assuming for simplicity that Γ is sorted,

$$\sum_{j=1}^{k} \gamma_j^2 \leq \sum_{j=1}^{k} \lambda_j \ \forall\, k \text{ and } \sum_{j=1}^{N} \gamma_j^2 = \sum_{j=1}^{d} \lambda_j$$

Casazza and Leon, International Journal of Pure and Applied Mathematics, 2010

Torus Action and Frame Homotopy

Theorem (Atiyah). The level sets of a momentum map for a Hamiltonian torus action are connected.



Atiyah, Bulletin of the London Mathematical Society, 1982

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Proof.

For $\Gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$ and $\Lambda = (\lambda_1, \ldots, \lambda_d)$ with $\lambda_1 \geq \cdots \geq \lambda_d > 0$,

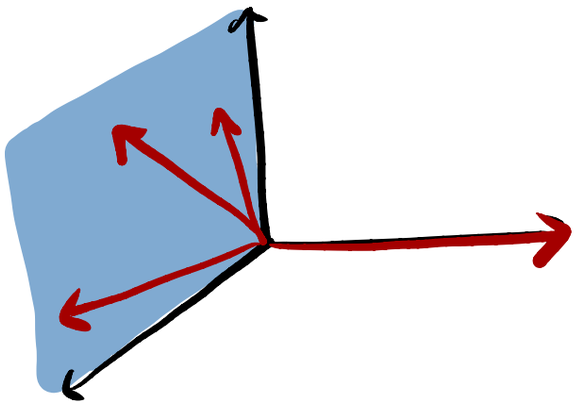
let

$$\mathscr{F}_{\Lambda}^{d,N}(\Gamma) := \{F \in \mathbb{C}^{d \times N} \mid FF^* \text{ has spectrum } \Lambda \text{ and } \|f_j\| = \gamma_j \ \forall\, j\}\,.$$

Then $\mathscr{F}_{\Lambda}^{d,N}(\Gamma)/\mathrm{U}(d) \approx \mu_{\mathbb{T}}^{-1}\left(-\frac{1}{2}\Gamma^{\odot 2}\right)$ is connected by Atiyah's Theorem. The proof follows by a few simple topological arguments. ∎

Atiyah, Bulletin of the London Mathematical Society, 1982

Application: Singularities of Frame Spaces

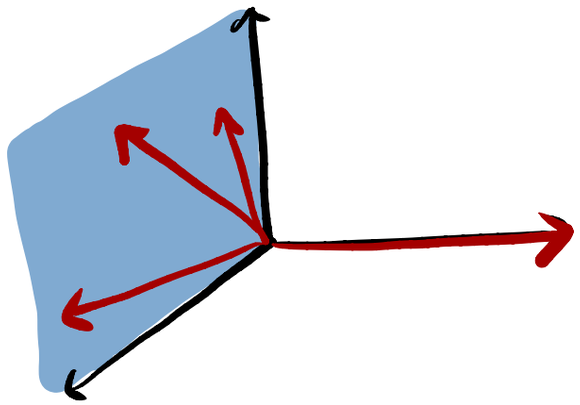


Definition. A frame is orthodecomposable if there is a partition of its vectors into frames spanning orthogonal subspaces.

Theorem (Dykema and Strawn). The space of FUNTFs is a stratified space with singular points occuring only at orthodecomposable frames. If d and N are relatively prime, then the space is a smooth manifold.

Dykema and Strawn, International Journal of Pure and Applied Mathematics, 2006
Arms, Marsden and Moncrief, Communications in Mathematical Physics, 1981

Application: Singularities of Frame Spaces



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Theorem (N-Shonkwiler). Let Γ be a vector of norms and Λ a spectrum. Singularities of $\mathscr{F}_{\Lambda}^{d,N}(\Gamma)$ occur only at orthodecomposable frames.

If there are no proper partitions $\Gamma = \Gamma_1 \sqcup \Gamma_2$ and $\Lambda = \Lambda_1 \sqcup \Lambda_2$ such that Γ_j is Λ_j-admissible and Γ_1 and Λ_1 have the same sum, then $\mathscr{F}_{\Lambda}^{d,N}(\Gamma)$ is a smooth manifold.

Moreover, singularities locally look like quadratic cones crossed with manifolds (using result of Arms, Marsden and Moncrief).

Dykema and Strawn, International Journal of Pure and Applied Mathematics, 2006
Arms, Marsden and Moncrief, Communications in Mathematical Physics, 1981

Reducing by the Torus Action

For a given spectrum Λ, we have a symplectic manifold

$$\mathscr{F}_{\Lambda}^{d,N}/\mathrm{U}(d) = \{F \in \mathbb{C}^{d\times N} \mid FF^* \text{ has spectrum } \Lambda\}/\mathrm{U}(d)$$

and a Hamiltonian action by the torus $\mathbb{T}$ with momentum map

$$\mu_{\mathbb{T}}([F]) = \left(-\frac{1}{2}\|f_j\|^2\right)_{j=1}^{N-1}$$

Due to singularities in level sets that we just described, Marsden-Weinstein reduction can't be directly applied to reduce by $\mathbb{T}$.

Reducing by the Torus Action

Sjamaar-Lerman Reduction Theorem. Let (M, ω) be a symplectic manifold with Hamiltonian G-action with momentum map μ. For each $\chi \in \mathfrak{g}^*$, the reduced space $\mu^{-1}(\mathcal{O}_\chi)/G$ is a symplectic stratified space. Each connected component contains a unique open stratum which is connected and dense.

Sjamaar and Lerman, Annals of Mathematics, 1991

Reducing by the Torus Action

Sjamaar-Lerman Reduction Theorem. Let (M, ω) be a symplectic manifold with Hamiltonian G-action with momentum map μ. For each $\chi \in \mathfrak{g}^*$, the reduced space $\mu^{-1}(\mathcal{O}_\chi)/G$ is a symplectic stratified space. Each connected component contains a unique open stratum which is connected and dense.

Proposition (N-Shonkwiler). For each admissible Γ and Λ, the space

$$\mathscr{F}_{\Lambda}^{d,N}(\Gamma)/G = \{F \in \mathbb{C}^{d\times N} \mid FF^* \text{ has spectrum } \Lambda \text{ and } \|f_j\| = \gamma_j\}/G,$$

where $G := \mathrm{U}(d) \times \mathbb{T}$, is a symplectic stratified space and contains an open dense symplectic manifold.

Sjamaar and Lerman, Annals of Mathematics, 1991

A New Torus Action and Eigensteps

Definition. The k-th partial frame operator of a frame $F \in \mathbb{C}^{d \times N}$ is

$$\sum_{\ell=1}^{k} f_\ell f_\ell^* .$$

The $(k, j)-$eigenstep for F, denoted $\lambda_{kj} = \lambda_{kj}(F)$, is the j-th eigenvalue (descending order) of the k-th partial frame operator of F.

$$\begin{array}{ccccccc}
\lambda_{41} & & \lambda_{42} & & \lambda_{43} & & \\
\geq & \geq & & \geq & & \geq & \\
 & \lambda_{31} & & \lambda_{32} & & \lambda_{33} & \\
 & \geq & \lambda_{21} & \geq & \lambda_{22} & & \\
 & & \geq & \lambda_{11} & & &
\end{array}$$

$N=4,\ d=3$

"Gelfand-Tsetlin pattern"

Cahill, Fickus, Mixon, Poteet and Strawn, Applied and Computational Harmonic Analysis, 2013

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Consider the subset of frames F whose k-th partial frame operators have isolated j-th eigenvalues $\lambda_{kj}(F)$ with eigenvectors $u_{kj}(F)$.

There is a circle action on this set defined by

$$\theta \cdot F = [Af_1 \mid \ldots \mid Af_k \mid f_{k+1} \mid \cdots \mid f_N],$$

$$A := e^{i\theta u_{kj} u_{kj}^*}$$

Cahill, Fickus, Mixon, Poteet and Strawn, Applied and Computational Harmonic Analysis, 2013

A New Torus Action and Eigensteps

Proposition (N-Shonkwiler). The collection of circle actions of this form gives a well defined torus action on an open dense subset of

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The action is Hamiltonian with momentum map $[F] \mapsto (\lambda_{kj}(F))_{k,j}$.

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The action is Hamiltonian with momentum map $[F] \mapsto (\lambda_{kj}(F))_{k,j}$.

If Γ is strongly Λ-admissible, image is a convex polytope of dimension

$$M = N(d-1) + 1 - \frac{d^2}{2} - \frac{1}{2}\sum_{j=1}^{\ell} k_j^2 = \frac{1}{2}\dim\left(\mathscr{F}_{\Lambda}^{d,N}(\Gamma)/G\right),$$

the k_j's giving multiplicities in Λ. In this case, the open dense subset is a toric symplectic manifold.

Generalizes results of Flaschka and Millson and Haga and Pegel, both in the context of FUNTF space.

Flaschka and Millson, Canadian Journal of Mathematics, 2005
Haga and Pegel, Discrete and Computational Geometry, 2016

Application: Full Spark Frames

Definition. A frame $F \in \mathbb{C}^{d \times N}$ is full spark if any subset of d columns is linearly independent.

Theorem (Cahill, Mixon and Strawn). In the space of FUNTFs, the set of full spark frames is open and dense.

Cahill, Mixon and Strawn, SIAGA, 2017

Application: Full Spark Frames

Definition. A frame $F \in \mathbb{C}^{d \times N}$ is full spark if any subset of d columns is linearly independent.

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For each spectrum Λ and vector of norms Γ the space of frames $\mathscr{F}_{\Lambda}^{d,N}(\Gamma)$ is a compact subset of $\mathbb{C}^{d \times N}$ and admits a uniform probability measure.

Theorem (N-Shonkwiler). There are three possibilities for a given Λ and Γ:

(1) $\mathscr{F}_{\Lambda}^{d,N}(\Gamma)$ is empty;

(2) $\mathscr{F}_{\Lambda}^{d,N}(\Gamma)$ is nonempty, but contains no full spark frames;

(3) the set of full spark frames is full measure in $\mathscr{F}_{\Lambda}^{d,N}(\Gamma)$.

Cahill, Mixon and Strawn, SIAGA, 2017

Full Spark Frames: Proof Sketch

(1) If Γ is not Λ-admissible, then $\mathscr{F}_{\Lambda}^{d,N}(\Gamma) = \emptyset$, by Casazza and Leon.

Duistermaat and Heckman, Inventiones Mathematicae, 1982

Full Spark Frames: Proof Sketch

(1) If Γ is not Λ-admissible, then $\mathscr{F}_{\Lambda}^{d,N}(\Gamma) = \varnothing$, by Casazza and Leon.

(2) If Γ is Λ-admissible, then $\mathscr{F}_{\Lambda}^{d,N}(\Gamma)$ is nonempty. If there is an equality

$$\sum_{j=1}^{k} \gamma_j^2 = \sum_{j=1}^{k} \lambda_j,$$

can show any frame is spark-deficient using eigenstep conditions.

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can show any frame is spark-deficient using eigenstep conditions.

(3) Assume all admissibility inequalities are strict. The set of points in the eigenstep polytope whose eigenstep $\lambda_{dd} \neq 0$ is full measure.

Now use the toric symplectic structure: by the Duistermaat-Heckman theorem, the set of frames whose first d columns are linearly independent is full measure in the frame space.

The result for a general subset of columns follows by permutation-invariance of the uniform measure.

Duistermaat and Heckman, Inventiones Mathematicae, 1982

Future Directions

- Sampling algorithms for spaces of frames with prescribed norms and frame operator spectra, using toric symplectic structure
- Experiments and hypothesis testing for distributions of eigenvalues of random frames
- Probabilistic guarantees for other frame-theoretic properties; e.g., RIP for frames with prescribed data
- Generalize to other spaces of frames; e.g., fusion frames

Thanks for Listening!

This research is supported by NSF Grant DMS - 2107808

References

Tom Needham and Clayton Shonkwiler, *Symplectic Geometry and Connectivity of Spaces of Frames*, Advances in Computational Mathematics, 2021.

Tom Needham and Clayton Shonkwiler, *Symplectic Geometry and Full Spark Frames (working title…)*, Preprint forthcoming!, 2021.