Applications of Symplectic Geometry to Frame Theory

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Outline

• **Frames** are collections of vectors \( \{ f_i \}_{i=1}^N \subset \mathbb{C}^d \) used to give robust representations of signals \( \nu \in \mathbb{C}^d \) via

\[
\nu \mapsto \left( \langle \nu, f_j \rangle \right)_{j=1}^N \in \mathbb{C}^N
\]

• We are interested in the geometry and topology of subsets of frames with prescribed vector norms and spectral data

• Certain properties of these **frame spaces** can be seen through the lens of symplectic geometry

• We will introduce the main ideas of symplectic geometry, illustrate them in the context of frames and use them to derive new results on frame spaces
Frames

Definition. A frame is a spanning collection of $N$ vectors in $\mathbb{C}^d$. We pick an ordering of these vectors $f_i$ and represent the frame as

$$F = \begin{bmatrix} f_1 & f_2 & \cdots & f_N \end{bmatrix} \in \mathbb{C}^{d \times N}$$

Let $\langle f, g \rangle = g^*f$ denote the standard Hermitian product on $\mathbb{C}^d$. 

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A frame determines several operators.

- **Analysis operator**: $\nu \mapsto F^* \nu$

A frame $F$ gives a redundant representation of a signal $\nu \in \mathbb{C}^d$ via the analysis operator

$$\nu \mapsto (\langle \nu, f_1 \rangle, \ldots, \langle \nu, f_N \rangle) = F^* \nu$$
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A frame determines several operators.

- Analysis operator: $v \mapsto F^* v$
- Synthesis operator: $w \mapsto Fw$

A collection of measurements $w = [w_1, \ldots, w_N]^T \in \mathbb{C}^N$ determines a signal via the synthesis operator

$$w \mapsto w_1 f_1 + \cdots + w_N f_N = Fw$$
Frames

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A frame determines several operators.

- Analysis operator: $v \mapsto F^* v$
- Synthesis operator: $w \mapsto Fw$
- Frame operator: $v \mapsto FF^* v$

The composition of the two is called the frame operator

$$v \mapsto FF^* v$$
Frames

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- Synthesis operator: $w \mapsto Fw$
- Frame operator: $\nu \mapsto FF^* \nu$

A frame is called tight if

$$FF^* = \frac{N}{d} I_d$$

(convenient normalization)
Frames

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F = \begin{bmatrix} f_1 & f_2 & \cdots & f_N \end{bmatrix} \in \mathbb{C}^{d \times N}
\]

Let \( \langle f, g \rangle = g^*f \) denote the standard Hermitian product on \( \mathbb{C}^d \).

A frame determines several operators.

- **Analysis operator:** \( \nu \mapsto F^*\nu \)
- **Synthesis operator:** \( \omega \mapsto F\omega \)
- **Frame operator:** \( \nu \mapsto FF^*\nu \)

Let \( \mathcal{F}^{d,N} \) denote the collection of all frames of size \( d \times N \).
Standard Geometric Structures

Let

\[ f = [x_1 + iy_1, \ldots, x_d + iy_d]^T \]
\[ g = [u_1 + iv_1, \ldots, u_d + iv_d]^T \]

The standard inner product on \( \mathbb{R}^{2d} \approx \mathbb{C}^d \):

\[ \text{Re}\langle f, g \rangle = x_1u_1 + y_1v_1 + \cdots + x_du_d + y_dv_d \]
Standard Geometric Structures

Let

\[ f = [x_1 + iy_1, \ldots, x_d + iy_d]^T \]
\[ g = [u_1 + iv_1, \ldots, u_d + iv_d]^T \]

The standard symplectic structure on \( \mathbb{R}^{2d} \approx \mathbb{C}^d \):

\[ -\text{Im} \langle f, g \rangle = x_1 v_1 - y_1 u_1 + \cdots + x_d v_d - y_d u_d \]
Symplectic Structures

Let $M$ be a smooth manifold.

**Definition (Local Coords).** A symplectic structure $\omega$ on $M$ is a smooth assignment of a map $\omega_x : T_x M \times T_x M \to \mathbb{R}$ such that $(M, \omega) \approx (\mathbb{R}^{2d}, -\text{Im}\langle \cdot, \cdot \rangle)$, locally.

That is, near each point in $M$ there exists a local diffeomorphism $\phi : M \supset U \to V \subset \mathbb{R}^{2d}$ with $\omega = \phi^* (-\text{Im}\langle \cdot, \cdot \rangle)$.
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Definition (2-Form). A symplectic structure $\omega$ on $M$ is a closed, nondegenerate 2-form.

That is, $\omega$:

- is bilinear
- is non-degenerate: $\omega_x(u, v) = 0 \forall v \Rightarrow u = 0$
- is skew-symmetric: $\omega_x(u, u) = 0$
- is closed: $d\omega = 0$
Symplectic Structures

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**Definition (2-Form).** A symplectic structure $\omega$ on $M$ is a closed, nondegenerate 2-form.

**Darboux's Theorem.** These two definitions are equivalent.

All symplectic manifolds are locally the same! Cf. Riemannian structures: no local iso of the 2-sphere to $\mathbb{R}^2$ with dot product.
**Symplectic Structures: Physical Interpretation**

**Definition.** Let $M$ be a smooth manifold. A symplectic structure $\omega$ on $M$ is a closed, nondegenerate 2-form.

**Intuition:** Consider $M$ as the phase space of a physical system. E.g., $M = T^*N$ with $(q, p) \in T^*N$ representing position and momentum.

Consider a smooth function $H : M \to \mathbb{R}$ as total energy of a phase.
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Consider a smooth function $H : M \to \mathbb{R}$ as total energy of a phase.

- $\omega$ is a bilinear form: turns $H$ into a vector field $V_H$ by solving
  \[ d_x H(v) = \omega_x(V_H(x), v). \]
- $\omega$ is non-degenerate: this equation can be solved for $V_H$.

Symplectic form turns an energy function into phase space dynamics.
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  \[
  d_x H(v) = \omega_x(V_H(x), v).
  \]
- $\omega$ is non-degenerate: this equation can be solved for $V_H$.
- $\omega$ is skew-symmetric: phase space trajectories are constant energy
  \[
  d_x H(V_H(x)) = \omega_x(V_H(x), V_H(x)) = 0
  \]
Symplectic Structures: Physical Interpretation

Definition. Let $M$ be a smooth manifold. A symplectic structure $\omega$ on $M$ is a closed, nondegenerate 2-form.

Intuition: Consider $M$ as the phase space of a physical system. E.g., $M = T^*N$ with $(q, p) \in T^*N$ representing position and momentum.

Consider a smooth function $H : M \to \mathbb{R}$ as total energy of a phase.

- $\omega$ is closed: laws of physics are time-invariant

\[ \mathcal{L}_{V_H} \omega = d\iota_{V_H} \omega + \iota_{V_H} d\omega = d^2 H(\cdot) + d\omega(V_H, \cdot, \cdot) = 0. \]
Examples

- \((\mathbb{C}^d, - \text{Im}\langle \cdot, \cdot \rangle)\)
Examples

- \((\mathbb{C}^d, - \text{Im} \langle \cdot, \cdot \rangle)\)

- \((\mathcal{F}^{d,N}, - \text{Im} \langle \cdot, \cdot \rangle) \subset (\mathbb{C}^{d \times N}, - \text{Im} \langle \cdot, \cdot \rangle) \approx (\mathbb{C}^{d \cdot N}, - \text{Im} \langle \cdot, \cdot \rangle)\)
Examples

- $\left( \mathbb{C}^d, - \text{Im} \langle \cdot, \cdot \rangle \right)$
- $\left( \mathcal{F}^{d,N}, - \text{Im} \langle \cdot, \cdot \rangle \right)$
- $\left( S^2, \omega = \text{signed area} \right)$
Examples

- \((\mathbb{C}^d, - \text{Im} \langle \cdot, \cdot \rangle)\)
- \((\mathbb{F}^d, - \text{Im} \langle \cdot, \cdot \rangle)\)
- \((S^2, \omega = \text{signed area})\)

\[
H : S^2 \to \mathbb{R}
\]
\[
H(x, y, z) = z
\]
\[
d_x \text{H}(u, v, w) = w
\]
\[
V_H(x, y, z) = (-y, x, 0)
\]
\[
\omega((x, y, z), (u, v, w)) = w
\]

\(\nabla H\) increases energy

\(E = \text{Energy Function and Hamiltonian Flow}\)

\(V_H\) constant energy
Hamiltonian Group Actions

Let $(M, \omega)$ be a symplectic manifold with action by a Lie group $G$. \( \forall \xi \in \mathfrak{g}, \exists \) associated vector field $\xi_M$ called the infinitesimal action:

$$\xi_M(x) = \left. \frac{d}{dt} \exp(t\xi) \cdot x \right|_{t=0}$$
Hamiltonian Group Actions

Let $(M, \omega)$ be a symplectic manifold with action by a Lie group $G$. 
\( \forall \xi \in \mathfrak{g}, \exists \) associated vector field $\xi_M$ called the infinitesimal action:

$$\xi_M(x) = \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi) \cdot x$$

The action is called **Hamiltonian** if it admits a momentum map:

$$\mu : M \to \mathfrak{g}^*$$

such that $\mu$ is equivariant with respect to the coadjoint action,

$$\mu(g \cdot x) = \operatorname{Ad}^*_g \cdot \mu(x),$$

and

$$d_x \mu(v)(\xi) = \omega_x(\xi_M(x), v).$$
Examples of Momentum Maps

Consider $S^2$ with $G = S^1$-action given by rotating around the $z$-axis. Identifying $\mathfrak{g}^* \cong \mathbb{R}$, this action has a momentum map

$$\mu(x) = x \cdot [0,0,1]^T$$
Examples of Momentum Maps

Consider frame space $\mathcal{F}_{d,N} \subset \mathbb{C}^{d \times N}$ with $G = U(d)$-action given by left multiplication.

Identifying $\mathfrak{g}^* \approx \mathcal{H}^{d \times d}$, the vector space of $d \times d$ Hermitian matrices, this action has momentum map

$$\mu(F) = FF^* = \text{frame operator}$$

\textbf{Proof}

$$\mathcal{H}^{d \times d} \cong U(d)^* : \mathfrak{g} \mapsto (\mathfrak{g} \mapsto \frac{i}{2} \text{tr}(\mathfrak{g} \mathfrak{g}))$$

$$d_F \mu(\mathfrak{g})(\mathfrak{g}) = \frac{i}{2} \text{tr}((FG^* + GF^*) \mathfrak{g})$$

$$\omega_F(\mathfrak{g}_M, \mathfrak{g}) = \omega_F(\mathfrak{g}_F, \mathfrak{g}) = -\text{Im} \text{tr}(G^* \mathfrak{g}_F)$$

$$= \frac{i}{2} \text{tr}(\mathfrak{g}_F G^* + \mathfrak{g} GF^*)$$

$$= \frac{i}{2} \text{tr}((FG^* + GF^*) \mathfrak{g})$$
Reduction of Symmetry

**Marsden-Weinstein Reduction Theorem.** Let \((M, \omega)\) be a symplectic manifold with Hamiltonian \(G\)-action and momentum map \(\mu : M \to \mathfrak{g}^*\). Then the space

\[ M//G := \mu^{-1}(0)/G \]

has a natural symplectic structure, provided 0 is a regular value and \(G\) acts freely on the level set.

\[ 2^*\omega = g^*\omega_{red} \]

Marsden and Weinstein, Reports on Mathematical Physics, 1974
Reduction of Symmetry and Frame Spaces

Marsden-Weinstein Theorem. $M//G := \mu^{-1}(0)/G$ is symplectic.

Proposition (N-Shonkwiler). The space of unitary orbits of tight frames is a symplectic manifold.

Proof.

\[
\left\{ F \in \mathbb{C}^{d \times N} \mid FF^* = \frac{N}{d} I_d \right\} / U(d) = \mu^{-1}(0)/U(d)
\]

where

\[
\mu : \mathcal{F}^{d, N} \rightarrow u(d)^* \approx \mathcal{H}^{d \times d}
\]

is a momentum map for the $U(d)$-action.
Reduction of Symmetry and Frame Spaces

Marsden-Weinstein Reduction Theorem, Refined. Let \((M, \omega)\) be a symplectic manifold with Hamiltonian \(G\)-action and momentum map \(\mu : M \to g^*\). For each \(\chi \in g^*\) with coadjoint orbit \(\mathcal{O}_\chi\), the space 

\[
M/\chi G := \mu^{-1}(\mathcal{O}_\chi)/G
\]

has a natural symplectic structure, provided \(\mathcal{O}_\chi\) consists of regular values and \(G\) acts freely on the level set.

\[
2^*\omega = q^*\omega_{red}
\]
Reduction of Symmetry and Frame Spaces

Marsden-Weinstein Theorem. \( M/\chi G := \mu^{-1}(\mathcal{O}_\chi)/G \) is symplectic.

Proposition (N-Shonkwiler). The space of unitary orbits of frames with prescribed frame operator spectrum is a symplectic manifold.

Proof. For \( \Lambda = (\lambda_j)_{j=1}^d \) with \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d > 0 \), let
\[
\mathcal{F}_{\Lambda}^{d,N} := \{ F \in \mathcal{F}^{d,N} \mid FF^* \text{ has spectrum } \Lambda \}
\]
Then
\[
\mathcal{F}_{\Lambda}^{d,N}/U(d) = \mu^{-1}(\mathcal{O}_S)/U(d),
\]
where
\[
\mu : \mathcal{F}^{d,N} \to \mathfrak{u}(d)^* \approx \mathcal{H}^{d\times d}
\]
is a momentum map for the \( U(d) \)-action.

\[ S \in \mathcal{H}^{d\times d} \text{ with spectrum } \Lambda \]
**Application: Frame Homotopy**

**Definition.** A finite unit norm tight frame (FUNTF) is a frame
\[ F = [f_1 \mid f_2 \mid \cdots \mid f_N] \in \mathbb{C}^{d \times N} \text{ with } \|f_j\| = 1 \text{ for all } j \text{ and } FF^* = \frac{N}{d} I_d. \]

**Frame Homotopy Conjecture (Larson, 2002).** The space of FUNTFs is path connected.

Proved by Cahill, Mixon and Strawn in 2017. Proof is constructive. Also applies to the space of real FUNTFs.
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Frame Homotopy Conjecture (Larson, 2002). The space of FUNTFs is path connected.

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Theorem (N-Shonkwiler). Let \((\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N\) and \(S\) an invertible \(d \times d\) Hermitian matrix. The space of frames \(F\) with \(f_j = \gamma_j\) and \(FF^* = S\) is path connected.

Proof follows “easily” from basic symplectic principles (but doesn’t apply to real frames).
Torus Action and Frame Homotopy

Let \( \mathbb{T} := U(1)^N / U(1) \)

\[ \approx \{ \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_{N-1}}, 1) \mid \theta_j \in [0, 2\pi) \} \]

\[ \approx U(1)^{N-1} \]

\( \mathbb{T} \) acts on \( \mathcal{F}_{\Lambda}^{d,N} / U(d) \) by right multiplication, with momentum map

\[ \mu_{\mathbb{T}} : \mathcal{F}_{\Lambda}^{d,N} / U(d) \to \mathfrak{t}^* \approx \mathbb{R}^{N-1} \]

\[ [F] \mapsto \left( -\frac{1}{2} \|f_j\|^2 \right)_{j=1}^{N-1} \]

Image is a convex polytope. A norm vector \( \Gamma \) lies in the image \( \iff \Gamma \) is \( \Lambda \)-admissible: assuming for simplicity that \( \Gamma \) is sorted,

\[ \sum_{j=1}^{k} \gamma_j^2 \leq \sum_{j=1}^{k} \lambda_j \forall k \] and \[ \sum_{j=1}^{N} \gamma_j^2 = \sum_{j=1}^{d} \lambda_j \]

**Torus Action and Frame Homotopy**

**Theorem (Atiyah).** The level sets of a momentum map for a Hamiltonian torus action are connected.
Torus Action and Frame Homotopy

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Theorem (N-Shonkwiler). Let \((\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N\) and \(S\) an invertible \(d \times d\) Hermitian matrix. The space of frames \(F\) with \(f_j = \gamma_j\) and \(FF^* = S\) is path connected.

Proof. For \(\Gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N\) and \(\Lambda = (\lambda_1, \ldots, \lambda_d)\) with \(\lambda_1 \geq \cdots \geq \lambda_d > 0\), let

\[
\mathcal{F}_{\Lambda}^{d,N}(\Gamma) := \{ F \in \mathbb{C}^{d \times N} \mid FF^* \text{ has spectrum } \Lambda \text{ and } \|f_j\| = \gamma_j \ \forall \ j \}.
\]

Then \(\mathcal{F}_{\Lambda}^{d,N}(\Gamma)/U(d) \approx \mu_T^{-1}\left(-\frac{1}{2} \Gamma \otimes 2\right)\) is connected by Atiyah’s Theorem. The proof follows by a few simple topological arguments.
Application: Singularities of Frame Spaces

**Definition.** A frame is **orthodecomposable** if there is a partition of its vectors into frames spanning orthogonal subspaces.

**Theorem (Dykema and Strawn).** The space of FUNTFs is a stratified space with singular points occurring only at orthodecomposable frames. If $d$ and $N$ are relatively prime, then the space is a smooth manifold.
Application: Singularities of Frame Spaces

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Theorem (N-Shonkwiler). Let $\Gamma$ be a vector of norms and $\Lambda$ a spectrum. Singularities of $\mathcal{F}_{\Lambda}^{d,N}(\Gamma)$ occur only at orthodecomposable frames. If there are no proper partitions $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Lambda = \Lambda_1 \cup \Lambda_2$ such that $\Gamma_j$ is $\Lambda_j$-admissible and $\Gamma_1$ and $\Lambda_1$ have the same sum, then $\mathcal{F}_{\Lambda}^{d,N}(\Gamma)$ is a smooth manifold.

Moreover, singularities locally look like quadratic cones crossed with manifolds (using result of Arms, Marsden and Moncrief).

Arms, Marsden and Moncrief, Communications in Mathematical Physics, 1981
Reducing by the Torus Action

For a given spectrum $\Lambda$, we have a symplectic manifold

$$\mathcal{F}_{\Lambda}^{d,N}/U(d) = \{ F \in \mathbb{C}^{d \times N} \mid FF^* \text{ has spectrum } \Lambda \}/U(d)$$

and a Hamiltonian action by the torus $\mathbb{T}$ with momentum map

$$\mu_{\mathbb{T}}([F]) = \left( -\frac{1}{2} \| f_j \|^2 \right)^{N-1}_{j=1}$$

Due to singularities in level sets that we just described, Marsden-Weinstein reduction can’t be directly applied to reduce by $\mathbb{T}$. 
Reducing by the Torus Action

Sjamaar-Lerman Reduction Theorem. Let \((M, \omega)\) be a symplectic manifold with Hamiltonian \(G\)-action with momentum map \(\mu\). For each \(\chi \in \mathfrak{g}^*\), the reduced space \(\mu^{-1}(\mathcal{O}_\chi)/G\) is a symplectic stratified space. Each connected component contains a unique open stratum which is connected and dense.
Reducing by the Torus Action

Sjamaar-Lerman Reduction Theorem. Let $(M, \omega)$ be a symplectic manifold with Hamiltonian $G$-action with momentum map $\mu$. For each $\chi \in \mathfrak{g}^*$, the reduced space $\mu^{-1}(\mathcal{O}_\chi)/G$ is a symplectic stratified space. Each connected component contains a unique open stratum which is connected and dense.

Proposition (N-Shonkwiler). For each admissible $\Gamma$ and $\Lambda$, the space

$$\mathcal{F}^{d,N}_\Lambda(\Gamma)/G = \{ F \in \mathbb{C}^{d \times N} | FF^* \text{ has spectrum } \Lambda \text{ and } \|f_j\| = \gamma_j \}/G,$$

where $G := U(d) \times \mathbb{T}$, is a symplectic stratified space and contains an open dense symplectic manifold.
A New Torus Action and Eigensteps

Definition. The $k$-th partial frame operator of a frame $F \in \mathbb{C}^{d \times N}$ is

$$\sum_{\ell=1}^{k} f_{\ell} f_{\ell}^*.$$ 

The $(k, j)$-eigenstep for $F$, denoted $\lambda_{kj} = \lambda_{kj}(F)$, is the $j$-th eigenvalue (descending order) of the $k$-th partial frame operator of $F$. 

N=4, d=3

"Cafland-Tsetlin pattern"
A New Torus Action and Eigensteps

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$$

The $(k, j)$-eigenstep for $F$, denoted $\lambda_{kj} = \lambda_{kj}(F)$, is the $j$-th eigenvalue (descending order) of the $k$-th partial frame operator of $F$.

Consider the subset of frames $F$ whose $k$-th partial frame operators have isolated $j$-th eigenvalues $\lambda_{kj}(F)$ with eigenvectors $u_{kj}(F)$.

There is a circle action on this set defined by

$$
\theta \cdot F = \begin{bmatrix}
Af_1 & \cdots & Af_k & f_{k+1} & \cdots & f_N
\end{bmatrix},
$$

$$
A := e^{i\theta u_{kj} u_{kj}^*}.
$$
A New Torus Action and Eigensteps

**Proposition (N-Shonkwiler).** The collection of circle actions of this form gives a well-defined torus action on an open dense subset of

\[ \mathcal{F}_{\Lambda}^{d,N}(\Gamma)/G = \{ F \in \mathbb{C}^{d\times N} \mid FF^* \text{ has spectrum } \Lambda \text{ and } \|f_j\| = \gamma_j \}/G. \]

The action is Hamiltonian with momentum map \( [F] \mapsto (\lambda_{kj}(F))_{k,j}. \)
A New Torus Action and Eigensteps

Proposition (N-Shonkwiler). The collection of circle actions of this form gives a well defined torus action on an open dense subset of

$$\mathcal{F}^{d,N}(\Gamma)/G = \{ F \in \mathbb{C}^{d \times N} \mid FF^* \text{ has spectrum } \Lambda \text{ and } \|f_j\| = \gamma_j \}/G .$$

The action is Hamiltonian with momentum map $[F] \mapsto (\lambda_{kj}(F))_{k,j}$. If $\Gamma$ is strongly $\Lambda$-admissible, image is a convex polytope of dimension

$$M = N(d - 1) + 1 - \frac{d^2}{2} - \frac{1}{2} \sum_{j=1}^\ell k_j^2 = \frac{1}{2} \dim(\mathcal{F}^{d,N}(\Gamma)/G),$$

the $k_j$’s giving multiplicities in $\Lambda$. In this case, the open dense subset is a toric symplectic manifold.

Generalizes results of Flaschka and Millson and Haga and Pegel, both in the context of FUNTF space.

Flaschka and Millson, Canadian Journal of Mathematics, 2005
Haga and Pegel, Discrete and Computational Geometry, 2016
Application: Full Spark Frames

Definition. A frame $F \in \mathbb{C}^{d \times N}$ is full spark if any subset of $d$ columns is linearly independent.

Theorem (Cahill, Mixon and Strawn). In the space of FUNTFs, the set of full spark frames is open and dense.
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Theorem (Cahill, Mixon and Strawn). In the space of FUNTFs, the set of full spark frames is open and dense.

For each spectrum \( \Lambda \) and vector of norms \( \Gamma \) the space of frames \( \mathcal{F}^{d,N}_{\Lambda}(\Gamma) \) is a compact subset of \( \mathbb{C}^{d \times N} \) and admits a uniform probability measure.

Theorem (N-Shonkwiler). There are three possibilities for a given \( \Lambda \) and \( \Gamma \):

1. \( \mathcal{F}^{d,N}_{\Lambda}(\Gamma) \) is empty;
2. \( \mathcal{F}^{d,N}_{\Lambda}(\Gamma) \) is nonempty, but contains no full spark frames;
3. the set of full spark frames is full measure in \( \mathcal{F}^{d,N}_{\Lambda}(\Gamma) \).
Full Spark Frames: Proof Sketch

(1) If $\Gamma$ is not $\Lambda$-admissible, then $\mathcal{F}^{d,N}_\Lambda(\Gamma) = \emptyset$, by Casazza and Leon.
Full Spark Frames: Proof Sketch

(1) If $\Gamma$ is not $\Lambda$-admissible, then $\mathcal{F}_{\Lambda}^{d,N}(\Gamma) = \emptyset$, by Casazza and Leon.

(2) If $\Gamma$ is $\Lambda$-admissible, then $\mathcal{F}_{\Lambda}^{d,N}(\Gamma)$ is nonempty. If there is an equality

$$\sum_{j=1}^{k} \gamma_j^2 = \sum_{j=1}^{k} \lambda_j,$$

then any frame is spark-deficient using eigenstep conditions.

Duistermaat and Heckman, Inventiones Mathematicae, 1982
Full Spark Frames: Proof Sketch

(1) If $\Gamma$ is not $\Lambda$-admissible, then $\mathcal{F}_{\Lambda}^{d,N}(\Gamma) = \emptyset$, by Casazza and Leon.

(2) If $\Gamma$ is $\Lambda$-admissible, then $\mathcal{F}_{\Lambda}^{d,N}(\Gamma)$ is nonempty. If there is an equality

$$\sum_{j=1}^{k} \gamma_j^2 = \sum_{j=1}^{k} \lambda_j,$$

then any frame is spark-deficient using eigenstep conditions.

(3) Assume all admissibility inequalities are strict. The set of points in the eigenstep polytope whose eigenstep $\lambda_{dd} \neq 0$ is full measure.

Now use the toric symplectic structure: by the Duistermaat-Heckman theorem, the set of frames whose first $d$ columns are linearly independent is full measure in the frame space.

The result for a general subset of columns follows by permutation-invariance of the uniform measure.

Duistermaat and Heckman, Inventiones Mathematicae, 1982
Future Directions

• Sampling algorithms for spaces of frames with prescribed norms and frame operator spectra, using toric symplectic structure

• Experiments and hypothesis testing for distributions of eigenvalues of random frames

• Probabilistic guarantees for other frame-theoretic properties; e.g., RIP for frames with prescribed data

• Generalize to other spaces of frames; e.g., fusion frames
Thanks for Listening!

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References
