Schönberg's Theorem and Association Schemes

Joint work with Brian Kodalen

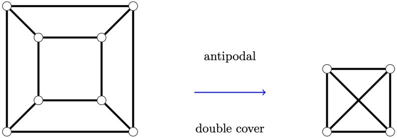
William J. Martin

Department of Mathematical Sciences
and
Department of Computer Science
Worcester Polytechnic Institute

Codes and Expansions (CodEx) Seminar
somewhere in the ether
September 15, 2020

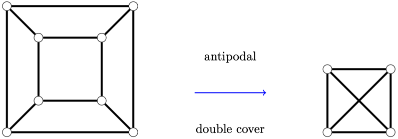
DRACKNs versus Covers of Strongly Regular Graphs

The cube is a DRACKN, a double cover of the complete graph K_4

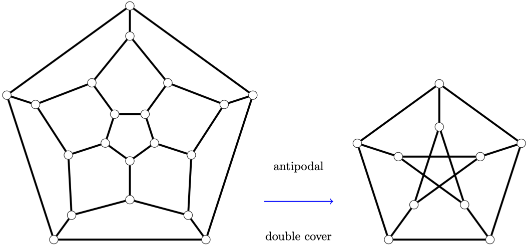


DRACKNs versus Covers of Strongly Regular Graphs

The cube is a DRACKN, a double cover of the complete graph K_4



The dodecahedron is an antipodal five-class (diameter 5) distance-regular double cover of the Petersen graph.



Jason Williford's Tables: feasible parameters for cometric schemes

`http://www.uwyo.edu/jwilliford/`

Here is a snapshot of Jason's table for $d = 4$, Q-bipartite (two angles, one of which is 90°)

Parameters	∃	v	m1	Krein Array	multiplicities	valencies	2nd Q	P	DRG	Quotient	Hyp	Comments
<42,6>	-	42	6	{6,5,27/7,12/5; 1,15/7,18/5,6}	1,6,14,15,6	1,10,20,10,1	-	01234	{10,6,3,1;1,3,6,10}	<21,10,3,6>		BCN Thm 4.4.11
<70,7>	!	70	7	{7,6,49/10,7/2; 1,21/10,7/2,7}	1,7,20,28,14	1,16,36,16,1	-	01234	{16,9,4,1;1,4,9,16}	<35,16,6,8>	FS	J(8,4)
<72,6>	+	72	6	{6,5,9/2,3; 1,3/2,3,6}	1,6,20,30,15	1,20,30,20,1	-	-		<36,15,6,6>		E6, Doubly Subtended Subquadrangles of GQ(3,9), Latin Square Type
<126,7>	+	126	7	{7,6,49/9,35/8; 1,14/9,21/8,7}	1,7,27,56,35	1,32,60,32,1	-	-		<63,30,13,15>		E7
<128,8>	!	128	8	{8,7,6,5; 1,2,3,8}	1,8,28,56,35	1,28,70,28,1	-	01234	{28,15,6,1;1,6,15,28}	<64,28,12,12>	FS	Halved 8-cube, Latin Square Type
<132,11>	+	132	11	{11,10,242/27,11/5; 1,55/27,44/5,11}	1,11,54,55,11	1,45,40,45,1	-	-		<66,20,10,4>	FS	Witt 5-(12,6,1)
<200,12>	-	200	12	{12,11,256/25,36/11; 1,44/25,96/11,12}	1,12,75,88,24	1,66,66,66,1	-	-		<100,33,14,9>		Gavrilyuk, Vidali [GV]
<240,8>	+	240	8	{8,7,32/5,6; 1,8/5,2,8}	1,8,35,112,84	1,56,126,56,1	-	-		<120,56,28,24>		E8
<240,15>	+	240	15	{15,14,25/2,5; 1,5/2,10,15}	1,15,84,105,35	1,63,112,63,1	-	-		<120,56,28,24>	FS	NO+(8,2)
<240,18>	+	240	18	{18,17,72/5,6; 1,18/5,12,18}	1,18,85,102,34	1,51,136,51,1	-	-		<120,51,18,24>	FS	Doubly Subtended Subquadrangles of GQ(4,16)
<252,21>	-	252	21	{21,20,49/3,7; 1,14/3,14,21}	1,21,90,105,35	1,45,160,45,1	-	01234	{45,32,9,1;1,9,32,45}	<126,45,12,18>		Jurisic and Koolen
<260,13>	?	260	13	{13,12,169/15,13/3; 1,26/15,26/3,13}	1,13,90,117,39	1,81,96,81,1	-	-		<130,48,20,16>		
<308,28>	?	308	28	{28,27,245/11,14/3; 1,63/11,70/3,28}	1,28,132,126,21	1,72,162,72,1	-	-		<154,72,26,40>		
<324,36>	-	324	36	{36,35,27,6; 1,9,30,36}	1,36,140,126,21	1,56,210,56,1	-	01234	{56,45,12,1;1,12,45,56}	<162,56,10,24>		BCN, Thm. 11.4.6
<378,21>	?	378	21	{21,20,147/8,7/2; 1,21/8,35/2,21}	1,21,160,168,28	1,128,120,128,1	-	-		<189,60,27,15>		
<380,15>	?	380	15	{15,14,250/19,45/7; 1,35/19,60/7,15}	1,15,114,175,75	1,105,168,105,1	-	-		<190,84,38,36>		
<392,21>	?	392	21	{21,20,35/2,9; 1,7/2,12,21}	1,21,120,175,75	1,75,240,75,1	-	-		<196,75,26,30>		

Sample Challenges: 4-class Q-Bipartite Association Schemes

Problem: Find 1288 lines in $\mathbb{R}^{23}$ with two angles, $\arccos(1/3)$ and $\pi/2$, in the configuration of the strongly regular graph $srg(1288, 495; 206, 180)$ coming from $M_{24}/2.M_{12}$

Problem: Find 2048 lines in $\mathbb{R}^{24}$ with two angles, $\arccos(1/3)$ and $\pi/2$, in the configuration of the strongly regular graph $srg(2048, 759; 310, 264)$ coming from $2^{11}.M_{24}/M_{24}$

Problem: Find 2232 lines in $\mathbb{R}^{24}$ with two angles, $\arccos(1/3)$ and $\pi/2$, in the configuration of a strongly regular graph $srg(2232, 828; 339, 288)$

Sample Challenges: 4-class Q-Bipartite Association Schemes

Problem: Find 1288 lines in $\mathbb{R}^{23}$ with two angles, $\arccos(1/3)$ and $\pi/2$, in the configuration of the strongly regular graph $srg(1288, 495; 206, 180)$ coming from $M_{24}/2.M_{12}$
Exists
Problem: Find 2048 lines in $\mathbb{R}^{24}$ with two angles, $\arccos(1/3)$ and $\pi/2$, in the configuration of the strongly regular graph $srg(2048, 759; 310, 264)$ coming from $2^{11}.M_{24}/M_{24}$
Open
Problem: Find 2232 lines in $\mathbb{R}^{24}$ with two angles, $\arccos(1/3)$ and $\pi/2$, in the configuration of a strongly regular graph $srg(2232, 828; 339, 288)$
Ruled out today

Double Covers of Strongly Regular Graphs

A graph Γ is *strongly regular* with parameters $(v, k; \lambda, \mu)$ if Γ is a k-regular graph on v vertices with the additional properties

- any two adjacent vertices share λ common neighbors
- any two non-adjacent vertices share μ common neighbors

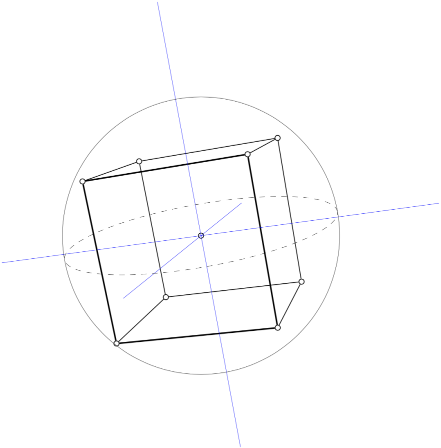
Example: Complete multipartite graph $\overline{wK_m}$: $srg(wm, (w-1)m; (w-2)m, (w-1)m)$.

We seek a set of lines through the origin with two angles "governed" by a strongly regular graph Γ in the sense that there is a bijection from the lines to the vertices of Γ such that a pair of lines form angle α iff the corresponding vertices are adjacent in Γ.

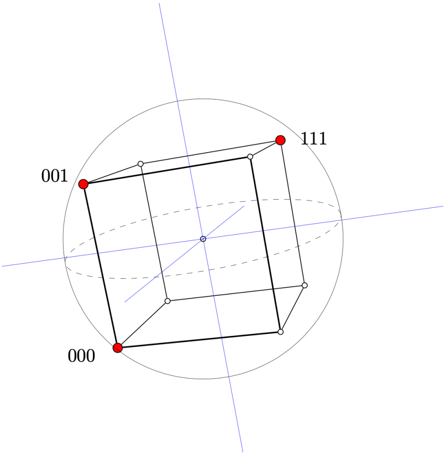
Theorem (LeCompte,WJM,Owens (2010))

When the underlying strongly regular graph is complete multipartite, Q-bipartite 4-class association schemes are (essentially) in one-to-one correspondence with real mutually unbiased bases.

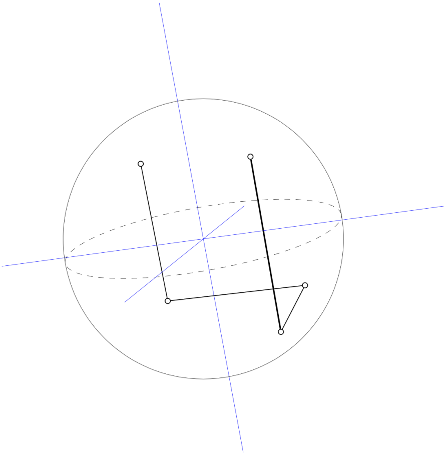
A Toy Spherical Code



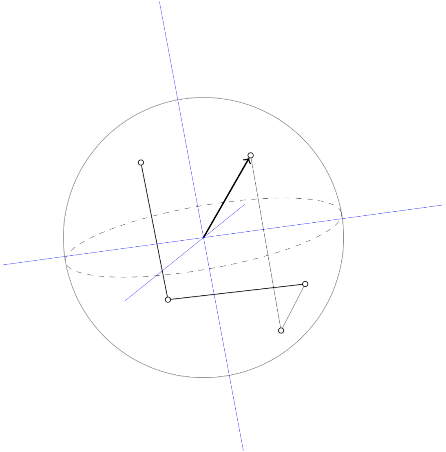
A Toy Spherical Code



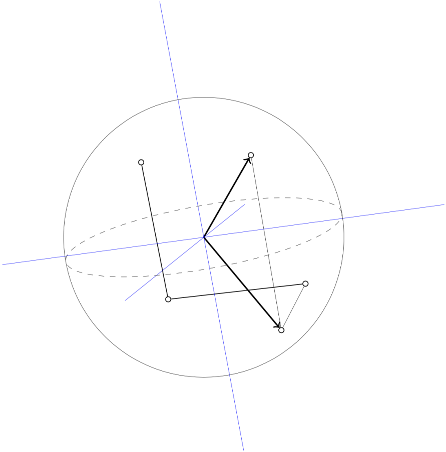
A Toy Spherical Code



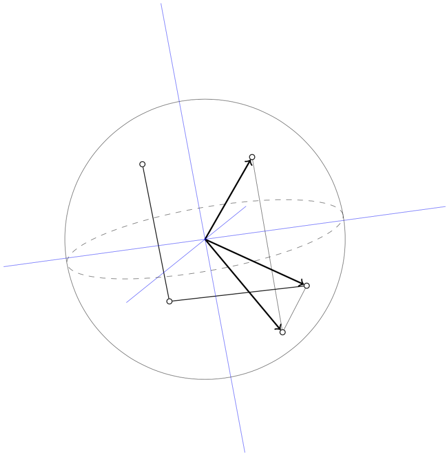
A Toy Spherical Code



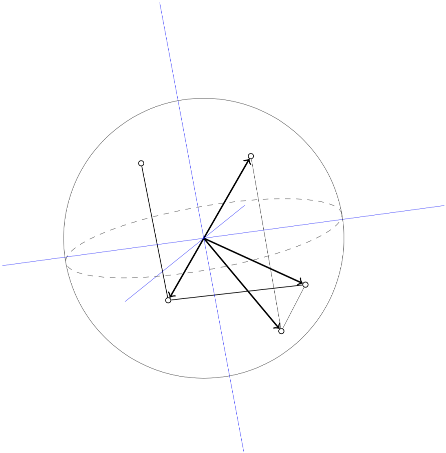
A Toy Spherical Code



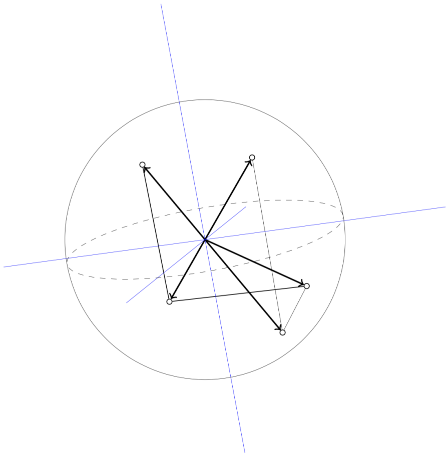
A Toy Spherical Code



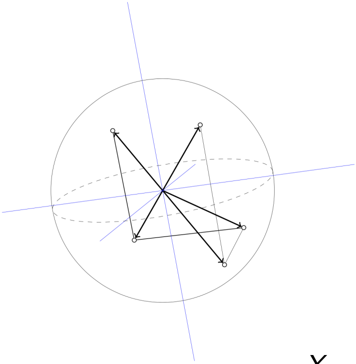
A Toy Spherical Code



A Toy Spherical Code



A Toy Spherical Code and its Gram Matrix



$$X = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

A Toy Spherical Code and its Gram Matrix

Matrix of inner products $G = XX^\top$

$$X = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \qquad G = \frac{1}{3} \begin{bmatrix} 3 & 1 & -1 & -3 & -1 \\ 1 & 3 & 1 & -1 & -3 \\ -1 & 1 & 3 & 1 & -1 \\ -3 & -1 & 1 & 3 & 1 \\ -1 & -3 & -1 & 1 & 3 \end{bmatrix}$$

A Toy Spherical Code and its Gram Matrix

We easily compute the entrywise square of the matrix G and its entrywise cube:

$$G \circ G = \frac{1}{9}\begin{bmatrix} 9 & 1 & 1 & 9 & 1 \\ 1 & 9 & 1 & 1 & 9 \\ 1 & 1 & 9 & 1 & 1 \\ 9 & 1 & 1 & 9 & 1 \\ 1 & 9 & 1 & 1 & 9 \end{bmatrix}, \quad G \circ G \circ G = \frac{1}{27}\begin{bmatrix} 27 & 1 & -1 & -27 & -1 \\ 1 & 27 & 1 & -1 & -27 \\ -1 & 1 & 27 & 1 & -1 \\ -27 & -1 & 1 & 27 & 1 \\ -1 & -27 & -1 & 1 & 27 \end{bmatrix}$$

Taking the Schur closure

This is a spherical 3-distance set. So the vector space

$$\mathbb{A} = \langle J, G, G^{\circ 2}, G^{\circ 3}, \ldots \rangle = \langle J, G, G^{\circ 2}, G^{\circ 3} \rangle$$

admits a basis of 01-matrices: $A_0, A_1, A_2, A_3 =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

But observe that

$$G^2 = \frac{1}{9} \begin{bmatrix} 21 & 11 & -7 & -21 & -11 \\ 11 & 21 & 7 & -11 & -21 \\ -7 & 7 & 13 & 7 & -7 \\ -21 & -11 & 7 & 21 & 11 \\ -11 & -21 & -7 & 11 & 21 \end{bmatrix}$$

does not belong to this space: $\mathbb{A}$ is not closed under multiplication.

Entrywise Operations on PSD Matrices

For a Hermitian matrix G, write $G \succeq 0$ to indicate that G is *positive semidefinite*: $\mathbf{x}^\top G\mathbf{x} \geq 0$ for all $\mathbf{x}$.

Since $G \succeq 0$, we know that $G \circ G \succeq 0$, $G \circ G \circ G \succeq 0$, etc.

Entrywise Operations on PSD Matrices

For a Hermitian matrix G, write $G \succeq 0$ to indicate that G is *positive semidefinite*: $\mathbf{x}^\top G\mathbf{x} \geq 0$ for all $\mathbf{x}$.

Since $G \succeq 0$, we know that $G \circ G \succeq 0$, $G \circ G \circ G \succeq 0$, etc.

If we apply $f(t) = \frac{1}{2}\left(3t^2 - 1\right)$ entrywise to G, the resulting matrix remains positive semidefinite.

Entrywise Operations on PSD Matrices

For a Hermitian matrix G, write $G \succeq 0$ to indicate that G is *positive semidefinite*: $\mathbf{x}^\top G \mathbf{x} \geq 0$ for all $\mathbf{x}$.

Since $G \succeq 0$, we know that $G \circ G \succeq 0$, $G \circ G \circ G \succeq 0$, etc.

If we apply $f(t) = \frac{1}{2}\left(3t^2 - 1\right)$ entrywise to G, the resulting matrix remains positive semidefinite.

If we instead apply $g(t) = t^2 - 2$ entrywise to G, we obtain a matrix with eigenvalues 0, 0, $16/9$ and

$$\frac{-61 \pm \sqrt{6473}}{18} \approx 1.080830908, \quad -7.858608686$$

Entrywise Operations on PSD Matrices

For a Hermitian matrix G, write $G \succeq 0$ to indicate that G is *positive semidefinite*: $\mathbf{x}^\top G\mathbf{x} \geq 0$ for all $\mathbf{x}$.

Since $G \succeq 0$, we know that $G \circ G \succeq 0$, $G \circ G \circ G \succeq 0$, etc.

If we apply $f(t) = \frac{1}{2}\left(3t^2 - 1\right)$ entrywise to G, the resulting matrix remains positive semidefinite.

If we instead apply $g(t) = t^2 - 2$ entrywise to G, we obtain a matrix with eigenvalues 0, 0, $16/9$ and

$$\frac{-61 \pm \sqrt{6473}}{18} \approx 1.080830908, \quad -7.858608686$$

(So what's special about $f(t) = \frac{1}{2}(3t^2 - 1)$?)

Gegenbauer Polynomials

For each dimension m, we have a basis $\{Q_\ell^m(t)\}_{\ell=0}^{\infty}$ for $\mathbb{R}[t]$ given by the three-term recurrence

$$Q_\ell^m(t) = \frac{(2\ell + m - 4)\ t\ Q_{\ell-1}^m(t) - (\ell - 1)\ Q_{\ell-2}^m(t)}{\ell + m - 3} \qquad \ell \geq 2,$$

$$Q_0^m(t) = 1 \qquad Q_1^m(t) = t.$$

These are the *Gegenbauer polynomials*.

Gegenbauer Polynomials

For each dimension m, we have a basis $\{Q_\ell^m(t)\}_{\ell=0}^\infty$ for $\mathbb{R}[t]$ given by the three-term recurrence

$$Q_\ell^m(t) = \frac{(2\ell + m - 4)\ t\ Q_{\ell-1}^m(t) - (\ell - 1)\ Q_{\ell-2}^m(t)}{\ell + m - 3} \qquad \ell \geq 2,$$

$$Q_0^m(t) = 1 \qquad Q_1^m(t) = t.$$

These are the *Gegenbauer polynomials*.
Note that $Q_\ell^m(1) = 1$ for all $\ell \geq 0$. We will suppress the superscript m if it is clear in the context.

Gegenbauer Polynomials

For each dimension m, we have a basis $\{Q_\ell^m(t)\}_{\ell=0}^\infty$ for $\mathbb{R}[t]$ given by the three-term recurrence

$$Q_\ell^m(t) = \frac{(2\ell+m-4)\ t\ Q_{\ell-1}^m(t) - (\ell-1)\ Q_{\ell-2}^m(t)}{\ell+m-3} \qquad \ell \geq 2,$$

$$Q_0^m(t) = 1 \qquad Q_1^m(t) = t.$$

These are the *Gegenbauer polynomials*.
Note that $Q_\ell^m(1) = 1$ for all $\ell \geq 0$. We will suppress the superscript m if it is clear in the context.
These are the zonal spherical harmonics: for each $\mathbf{y} \in \mathbb{R}^m$, $F : \mathbf{x} \mapsto Q_\ell^m(\langle \mathbf{x}, \mathbf{y} \rangle)$ satisfies

$$\Delta\ F = \frac{\partial^2 F}{\partial x_1^2} + \cdots + \frac{\partial^2 F}{\partial x_m^2} = 0$$

Gegenbauer Polynomials

Here are the first six Gegenbauer polynomials for spherical codes in dimension m.

$$Q_0(t) = 1, \qquad Q_1(t) = t, \qquad Q_2(t) = \frac{mt^2 - 1}{m-1},$$

$$Q_3(t) = \frac{(m+2)t^3 - 3t}{m-1},$$

$$Q_4(t) = \frac{(m+4)(m+2)t^4 - 6(m+2)t^2 + 3}{m^2-1},$$

$$Q_5(t) = \frac{(m+6)(m+4)t^5 - 10(m+4)t^3 + 15t}{m^2-1}.$$

Schönberg's Theorem (specialized)

Let m be a fixed positive integer. For a finite set of unit vectors $X \subset S^{m-1}$, let G_X denote the Gram matrix of X.
A function $f : [-1, 1] \to \mathbb{R}$ is *positive definite* on S^{m-1} if, for every finite subset X, f applied entrywise to G_X results in a positive semidefinite matrix; we write $f \circ (G_X) \succeq 0$.

Theorem (Schönberg (1942))

Fix $m \in \mathbb{Z}^+$. A polynomial $f : [-1, 1] \to \mathbb{R}$ of degree d is positive definite on S^{m-1} if and only if $f(t) = \sum_{\ell=0}^{d} c_\ell Q_\ell^m(t)$ for non-negative constants c_ℓ. □

In particular, $Q_\ell^m(t)$ is a positive definite function for any choice of m and ℓ.

Cones

Let $G = G_X$ be the Gram matrix of a finite subset X of S^{m-1}. Schönberg's Theorem implies that the map

$$\mathbb{R}[t] \to \mathbb{A} = \langle J, G, G \circ G, \ldots \rangle$$

given by $f(t) \mapsto f \circ (G)$

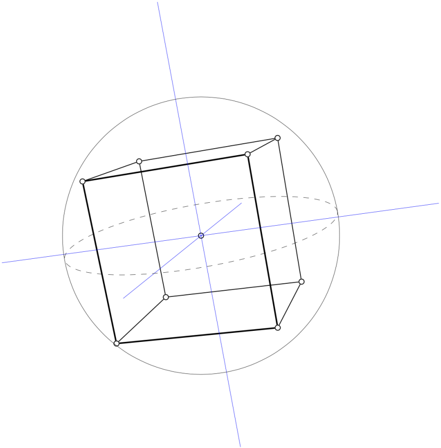
maps the cone generated by the Gegenbauer polynomials into the positive semidefinite cone of $\mathbb{A}$.

$$f(t) = \sum_{\ell=0}^{n} c_\ell Q_\ell^m(t)$$

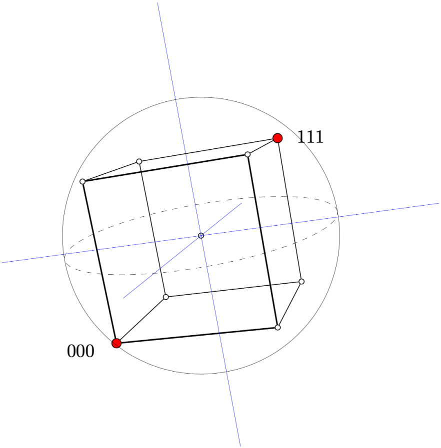
$$c_\ell \geq 0 \;\; \forall\, \ell \quad \Rightarrow \quad f \circ (G) \succeq 0$$

This can be used to give powerful constraints on spherical codes via semidefinite programming. (Wei-Hsuan Yu talked at WPI on this.)

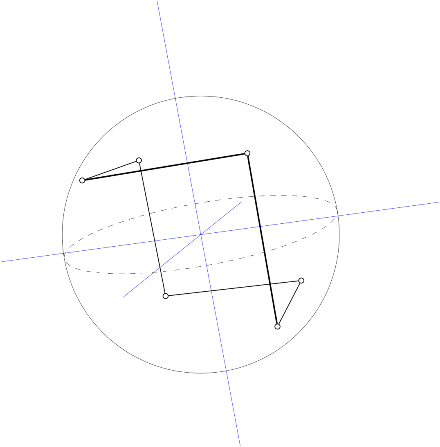
A More Interesting Spherical Code



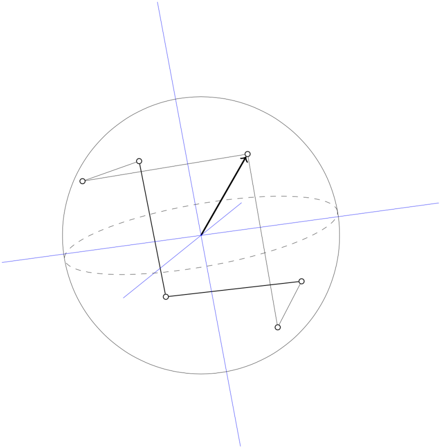
A More Interesting Spherical Code



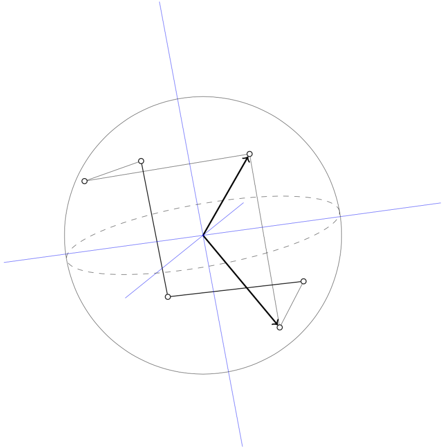
A More Interesting Spherical Code



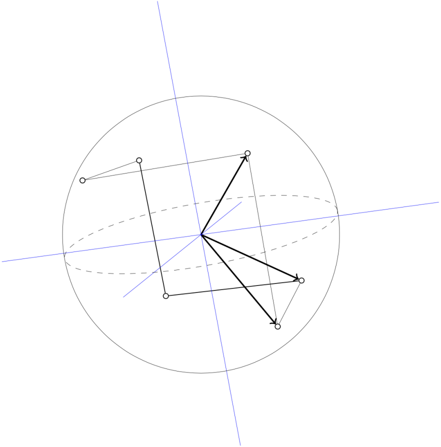
A More Interesting Spherical Code



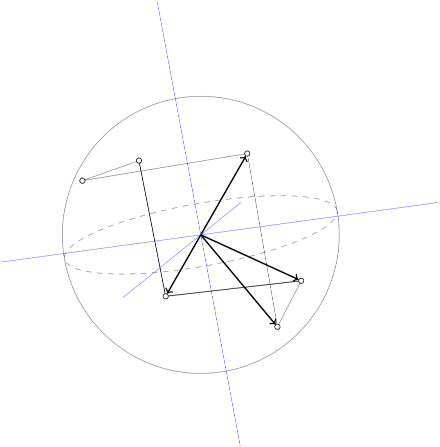
A More Interesting Spherical Code



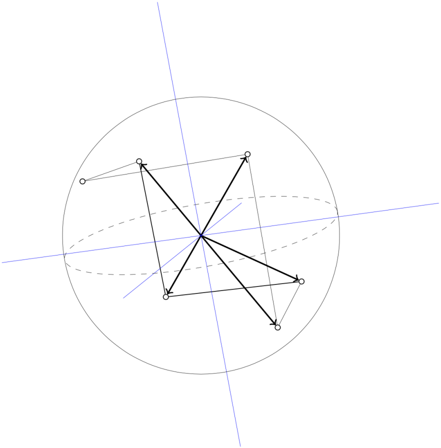
A More Interesting Spherical Code



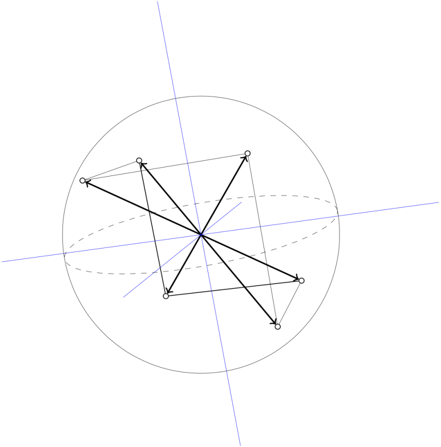
A More Interesting Spherical Code



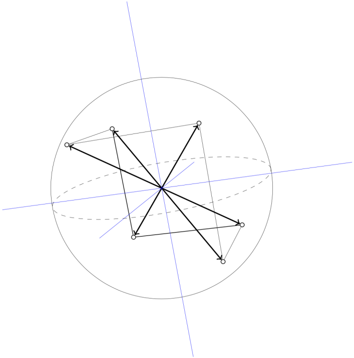
A More Interesting Spherical Code



A More Interesting Spherical Code



Our Second Spherical Code and its Gram Matrix



$$X = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Our Second Spherical Code and its Gram Matrix

$$X = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \qquad G = \frac{1}{3} \begin{bmatrix} 3 & 1 & -1 & -3 & -1 & 1 \\ 1 & 3 & 1 & -1 & -3 & -1 \\ -1 & 1 & 3 & 1 & -1 & -3 \\ -3 & -1 & 1 & 3 & 1 & -1 \\ -1 & -3 & -1 & 1 & 3 & 1 \\ 1 & -1 & -3 & -1 & 1 & 3 \end{bmatrix}$$

Our Second Spherical Code and its Gram Matrix

We easily compute the entrywise square of the matrix G and its entrywise cube:

$$G \circ G = \frac{1}{9}\begin{bmatrix} 9 & 1 & 1 & 9 & 1 & 1 \\ 1 & 9 & 1 & 1 & 9 & 1 \\ 1 & 1 & 9 & 1 & 1 & 9 \\ 9 & 1 & 1 & 9 & 1 & 1 \\ 1 & 9 & 1 & 1 & 9 & 1 \\ 1 & 1 & 9 & 1 & 1 & 9 \end{bmatrix}, \quad G \circ G \circ G = \frac{1}{27}\begin{bmatrix} 27 & 1 & -1 & -27 & -1 & 1 \\ 1 & 27 & 1 & -1 & -27 & -1 \\ -1 & 1 & 27 & 1 & -1 & -27 \\ -27 & -1 & 1 & 27 & 1 & -1 \\ -1 & -27 & -1 & 1 & 27 & 1 \\ 1 & -1 & -27 & -1 & 1 & 27 \end{bmatrix}$$

The Bose-Mesner Algebra of the Hexagon

This is a spherical 3-distance set. So the vector space

$$\mathbb{A} = \langle J, G, G^{\circ 2}, G^{\circ 3}, \ldots \rangle = \langle J, G, G^{\circ 2}, G^{\circ 3} \rangle$$

admits a basis of 01-matrices: $A_0 = I$, $A_1, A_2, A_3 =$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

This space is closed under matrix multiplication. So we have a *Bose-Mesner algebra*, an *association scheme*.

Bose-Mesner Algebras

A vector space of $v \times v$ matrices $\mathbb{A}$ is a *Bose-Mesner algebra* if

- it is closed under conjugate transpose (e.g., all matrices are symmetric)
- it is closed under (ordinary) multiplication and contains I
- it is closed under Schur/Hadamard (entrywise) multiplication and contains J

Two bases:

$$\{A_0, \ldots, A_d\} \qquad \{E_0, \ldots, E_d\}$$

$$A_i \circ A_j = \delta_{i,j} A_i \qquad E_i E_j = \delta_{i,j} E_i$$

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k \quad E_i \circ E_j = \frac{1}{v} \sum_{k=0}^{d} q_{ij}^k E_k$$

Association Schemes: The Spherical Code Viewpoint

For today, a (commutative) *association scheme* is a set X of distinct unit vectors in $\mathbb{C}^m$ for some m whose (Hermitian) Gram matrix $G = G_X$ has the property that the vector space

$$\mathbb{A} = \langle J, G, G^{\circ 2}, G^{\circ 3}, \ldots \rangle$$

is closed under matrix multiplication.

The association scheme is *cometric* (or Q-polynomial) with respect to X if, for each r and s ($r \leq s$),

$$G^{\circ r} G^{\circ s} \in \langle J, G, G^{\circ 2}, \ldots, G^{\circ r} \rangle$$

Note: every commutative association scheme arises in this way.

The structure constants of the structure constants are the structure constants

Bose-Mesner algebra $\mathbb{A}$ admits basis $\{E_0, \ldots, E_d\}$ with $E_i E_j = \delta_{ij} E_i$ and

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k E_k$$

Define

$$L_i^* = \begin{bmatrix} q_{i0}^0 & q_{i1}^0 & q_{i2}^0 & \cdots & q_{id}^0 \\ q_{i0}^1 & q_{i1}^1 & q_{i2}^1 & \cdots & q_{id}^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{i0}^d & q_{i1}^d & q_{i2}^d & \cdots & q_{id}^d \end{bmatrix}$$

Then

$$L_i^* L_j^* = \sum_{k=0}^{d} q_{ij}^k L_k^*$$

Two key isomorphisms

The linear map from $\mathbb{A}$ to the space of $(d+1)\times(d+1)$ matrices that sends A_i to $L_i = [p_{i,j}^k]_{k,j}$ is a ring (algebra) homomorphism

$$A_i A_j \mapsto L_i L_j$$

Two key isomorphisms

The linear map from $\mathbb{A}$ to the space of $(d+1)\times(d+1)$ matrices that sends A_i to $L_i = [p^k_{i,j}]_{k,j}$ is a ring (algebra) homomorphism

$$A_i A_j \mapsto L_i L_j$$

The map ϕ^* from $\mathbb{A}$ to the space of $(d+1)\times(d+1)$ matrices that sends

$$\phi^*(E_i) = \frac{1}{|X|} L_i^* \quad \text{where} \quad L_i^* = [q^k_{i,j}]_{k,j}$$

extended linearly, is an algebra monomorphism:

$$\phi^*(M \circ N) = \phi^*(M)\phi^*(N)$$

So $(\mathbb{A}, +, \circ)$ is isomorphic to the subalgebra $\langle L_0, \ldots, L_d \rangle$.

The Map ϕ^* and Cones

We are thinking about

$$\phi^*(E_h) = \frac{1}{v} L_h^* \quad \text{where} \quad L_h^* = [q_{h,j}^i]_{i,j}$$

extended linearly.

$$\phi^*(M \circ N) = \phi^*(M)\phi^*(N)$$

Lemma

The map ϕ^* sends the positive semidefinite cone of $\mathbb{A}$ bijectively to

$$\left\{ \sum_{i=0}^{d} c_i L_i^* \;\middle|\; c_0, c_1, \ldots, c_d \geq 0 \right\}$$

Schönberg's Theorem (specialized)

This leads to the following theorem:

Theorem

Let $(X, \mathcal{R})$ be an association scheme with minimal idempotents $E_0, \ldots, E_d$ and matrices of Krein parameters $L_0^*, \ldots, L_d^*$. Fix some E_i, $0 \le i \le d$, and let $m_i := rank(E_i)$. Then for any choice of $\ell > 0$, there exist non-negative constants $\theta_{\ell j}$, $0 \le j \le d$, such that

$$Q_\ell^{m_i} \circ \left(\frac{|X|}{m_i} E_i \right) = \sum_j \theta_{\ell j} E_j; \qquad Q_\ell^{m_i} \left(\frac{1}{m_i} L_i^* \right) = \frac{1}{|X|} \sum_j \theta_{\ell j} L_j^*. \tag{1}$$

The eigenvalues of $Q_\ell^{m_i} \circ \left(\frac{|X|}{m_i} E_i \right)$ are $\theta_{\ell 0}, \ldots, \theta_{\ell d}$ where $\theta_{\ell j}$ is non-zero only if E_j is contained in the Schur subalgebra generated by E_i.

Did we accomplish anything?

- We are looking for a spherical code X and want to apply Schönberg
- In the case X generates an association scheme, we must test if a matrix lies in the psd cone of a Bose-Mesner algebra $\mathbb{A}$
- Even though we don't know the matrices in $\mathbb{A}$, we know their entries from the parameters
- So we instead apply ϕ^* and check if we are in the cone of $\{L_0^*, \ldots, L_d^*\}$
- We only need to consider the first column (column "zero") of $\phi^*(f \circ (E_j))$
- We only need to consider $f(t) \in \{Q_0(t), Q_1(t), Q_2(t), \ldots\}$
- But how far out should we check?

A Really Nice Kodalen Theorem

View $\frac{|X|}{m_j} E_j$ as the Gram matrix of a spherical code.
Consider the $|X|$ (or $|X|/2$) lines spanned by these vectors and let $\lambda^* = \cos(\theta_{min})$, the cosine of the smallest angle formed. (Assume this is the smallest angle of the spherical code, for convenience.)
Define

$$\ell^* = \left\lceil \frac{\ln\left[(1+(\lambda^*)^2)|X|(|X|-1)\right]}{-2\ln(\lambda^*)} \right\rceil$$

As long as $(\lambda^*)^2 \geq \ell^*/(\ell^* + m_j - 2)$ we have

$$Q_\ell^{m_j}\left(\frac{1}{m_j} L_j^*\right) \geq 0$$

for all $\ell \geq \ell^*$.

The Krein Array

Suppose $\mathbb{A}$ is a Bose-Mesner algebra with Q-polynomial ordering

$$E_0, E_1, \ldots, E_d$$

of its primitive idempotents. Then L_1^* is irreducible tridiagonal. It is customary to write

$$L_1^* = \begin{bmatrix} 0 & b_0^* & & & \\ 1 & a_1^* & b_1^* & & \\ & c_2^* & a_2^* & b_2^* & \\ & & \ddots & \ddots & \ddots \\ & & & c_d^* & a_d^* \end{bmatrix}$$

This is recorded in the *Krein array*:
$\iota^*(X, \mathcal{R}) = \left\{ b_0^*, b_1^*, \ldots, b_{d-1}^*; 1, c_2^* \ldots, c_d^* \right\}$

New Feasibility Conditions for Cometric Association Schemes

Theorem

Suppose we have a feasible parameter set for a cometric association scheme with Krein array $\iota^*(X,\mathcal{R}) = \left\{m, b_1^*, \ldots, b_{d-1}^*; 1, c_2^* \ldots, c_d^*\right\}$ *where* $m > 2$. *Then the scheme is realizable only if*

(iii) $(a_1^*)^2 + b_1^* c_2^* \geq \frac{2m(m-1)}{m+2}$,

(iv) $(a_1^*)^2 + 2a_1^* a_2^* + c_2^* q_{22}^2 \geq \frac{4m(m-2)}{m+4}$,

(v) $\frac{6m(m-1)(m-4)}{(m+4)(m+6)} + \frac{\left(3a_1^*\left(a_1^*+a_2^*\right)+c_2^* q_{22}^2\right)b_1^* c_2^* + \left(a_1^*\right)^4}{m} \geq \frac{(7m-18)\left(\left(a_1^*\right)^2+b_1^* c_2^*\right)}{m+6}$,

$(\mathrm{v})_2$ $\sum_{i=1}^{3} \left(b_i^* c_{i+1}^* + a_i^* \sum_{j=i}^{3} a_j^*\right) \leq \frac{3(3m-2)}{m+6}$.

New Feasibility Conditions for Cometric Association Schemes

The conditions get more technical as we consider $Q_\ell^m(t)$ for $\ell = 5, 6$:

$$\frac{16m(m-1)}{(m+4)(m+8)} + \frac{\left(a_1^*\right)^4 + \left(3a_1^*\left(a_1^*+a_2^*\right)+c_2^* q_{22}^2\right)b_1^* c_2^*}{(m-2)m} \geq \frac{12\left(\left(a_1^*\right)^2 + b_1^* c_2^*\right)}{m+8}$$

If $a_1^* > 0$, then

$$(a_1^*)^2 + b_1^* c_2^* \left(2 + \frac{a_2^*}{a_1^*}\right) \geq \frac{4m(2m-3)}{m+6}$$

$$(a_1^*)^2 + 2a_1^* a_2^* - (a_2^*)^2 + 2c_2^* q_{22}^2 + \frac{b_2^* c_3^*\left(a_3^* - a_1^*\right) - m a_2^*}{a_1^* + a_2^*} \geq \frac{6m(m-4)}{m+6}$$

Non-Existence Results for Cometric Schemes

Using these lists, we find nine 3-class primitive cometric schemes and 11 4-class Q-bipartite schemes which are ruled out by these inequalities. For each, here are $(|X|, m)$ where $|X|$ is the number of points and $m = \text{rank}\, E_1$ is the dimension.

- 3-class primitive schemes ruled out

$$\{(441, 20), (576, 23), (729, 26), (1015, 28), (1240, 30),$$

$$(1548, 35), (1836, 35), (1944, 29), (1976, 25)\}.$$

- 4-class Q-bipartite schemes

$$\{(4464, 24), (4968, 27), (5280, 30), (5436, 27), (6148, 29),$$

$$(8432, 31), (9984, 32), (594, 9), (7776, 27), (8478, 27), (9984, 24)\}$$

Why Association Schemes?

- efficiency in statistical experiments and coding theory
- the center of the group algebra of any finite group is a commutative a.s.
- distance-regular graphs (cubes, Hamming, Johnson, Grassmann, dual polar spaces, cages, generalized polygons, DRACKNs, . . .)
- tight spherical designs and extremal codes
- every spin model for knot invariants comes from a Bose-Mesner algebra
- linked simplices, real mutually unbiased bases
- and more!

The End

Thank you for listening. I welcome questions.

Jennifer and I just ripped up some dead lawn to build a succulent garden (such is the "vacation week" in the age of Covid)