Recent developments in the discrete Fuglede conjecture

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Codes and Expansions (CodEx) Seminar
August 10th, 2021
Fourier Analysis on domains $\Omega \subseteq \mathbb{R}^n$

**Question**

On which measurable domains $\Omega \subseteq \mathbb{R}^n$ with $\mu(\Omega) > 0$ can we do Fourier analysis, that is, there is an orthonormal basis of exponential functions $\left\{ \frac{1}{\mu(\Omega)} e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda \right\}$ in $L^2(\Omega)$, where $\Lambda \subseteq \mathbb{R}^n$ discrete?

**Definition**

If $\Omega$ satisfies the above condition it is called *spectral*, and $\Lambda$ is the *spectrum* of $\Omega$. 

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Fuglede’s conjecture

Definition

A set $\Omega \subseteq \mathbb{R}^n$ of positive measure is called *tile* of $\mathbb{R}^n$ if there is $T \subseteq \mathbb{R}^n$ such that $\Omega \oplus T = \mathbb{R}^n$.

Conjecture (Fuglede, 1974)

A set $\Omega \subseteq \mathbb{R}^n$ of positive measure is spectral if and only if it tiles $\mathbb{R}^n$. 

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Fuglede’s conjecture

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Theorem (Fuglede, ’74)

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set of measure 1 and $\Lambda \subseteq \mathbb{R}^n$ be a lattice with density 1. then $\Omega \oplus \Lambda = \mathbb{R}^n$ if and only if $\Lambda^*$ is a spectrum of $\Omega$.

Theorem (Lev, Matolcsi, ’19)

Let $K \subseteq \mathbb{R}^n$ be a convex body; then $K$ is spectral if and only if it tiles $\mathbb{R}^n$. 

Recent developments in the discrete Fuglede conjecture
Special cases

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Recent developments in the discrete Fuglede conjecture
Tao’s counterexample

“A cataclysmic event in the history of this problem took place in 2004 when Terry Tao disproved the Fuglede Conjecture by exhibiting a spectral set in $\mathbb{R}^{12}$ which does not tile.”

The Fuglede Conjecture holds in $\mathbb{Z}_p \times \mathbb{Z}_p$, Iosevich, Mayeli, Pakianathan, 2017.
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**Theorem (Tao, ’04)**

There are spectral subsets of $\mathbb{R}^5$ of positive measure that do not tile $\mathbb{R}^5$. 
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Fuglede’s conjecture fails for $n \geq 3$ (both directions).
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Consider the standard basis of $\mathbb{Z}_3^6$, $e_1, \ldots, e_6$. Let $\omega = e^{2\pi i/3}$ and $\xi_1, \ldots, \xi_6$ be characters such that

$$[\xi_j(e_i)]_{1 \leq i,j \leq 6} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \omega^2 & \omega^2 \\ 1 & \omega & 1 & \omega^2 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & 1 & \omega \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 \end{pmatrix}$$
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The above matrix is Hadamard, hence $\Lambda = \{\xi_1, \ldots, \xi_6\}$ is a spectrum of $\Omega = \{e_1, \ldots, e_6\}$. $\Omega - e_1$ then is contained in a hyperplane, thus showing the existence of a counterexample in $\mathbb{Z}_3^5$. Obviously, such a set cannot tile, since $6 \nmid 3^5$. 

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Tao’s counterexample

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The above matrix is \textit{Hadamard}, hence \( \Lambda = \{\xi_1, \ldots, \xi_6\} \) is a spectrum of \( \Omega = \{e_1, \ldots, e_6\} \). \( \Omega - e_1 \) then is contained in a hyperplane, thus showing the existence of a counterexample in \( \mathbb{Z}_3^5 \). Obviously, such a set cannot tile, since \( 6 \nmid 3^5 \).
Definition

Let $G$ be an Abelian group. We write $(S-T(G))$ if every bounded spectral subset of $G$ is also a tile, and $(T-S(G))$ if every bounded tile of $G$ is spectral.

Theorem

The following hold:

$$(T-S(\mathbb{Z}_n)) \forall n \in \mathbb{N} \iff (T-S(\mathbb{Z})) \iff (T-S(\mathbb{R}))$$

and

$$(S-T(\mathbb{R})) \Rightarrow (S-T(\mathbb{Z})) \Rightarrow (S-T(\mathbb{Z}_n)) \forall n \in \mathbb{N}.$$
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**Theorem (Dutkay, Lai, ’14)**

*If Fuglede’s conjecture holds in $\mathbb{R}$, then every bounded spectral set has a rational spectrum.*
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The properties \((S-T(G))\) and \((T-S(G))\) are hereditary, that is, they hold for every subgroup of \(G\).

It suffices then to examine groups of the form \(\mathbb{Z}_N^d\). For \(d \geq 2\) we get the following results:

- There is a spectral subset of \(\mathbb{Z}_{32}^8\) that does not tile (Kolountzakis, Matolcsi, '06).
- There is a tile of \(\mathbb{Z}_{32}^{24}\) that is not spectral (Farkas, Matolcsi, Mora, '06).
- There are spectral subsets of \(\mathbb{Z}_{4p}^p\) that do not tile for \(p\) odd (Ferguson, Sothanaphan; independently Mattheus '20); the same holds for \(\mathbb{Z}_{10}^{2}\) (F-S, '20).
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Non-cyclic groups; positive results

- Fuglede’s conjecture holds in $\mathbb{Z}_p^2$, $p$ prime (Iosevich, Mayeli, Pakianathan, ’17).
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Let $A \subseteq \mathbb{Z}_N$ and $e_\lambda(a) = e^{2\pi i \lambda \cdot a/N}$. Inner product on $L^2(A)$:

$$\langle f, g \rangle_A = \sum_{a \in A} f(a)\overline{g(a)}.$$

It holds $\langle e_\lambda, e_{\lambda'} \rangle_A = \hat{1}_A(\lambda' - \lambda)$.

**Lemma**

\(\Lambda\) is a spectrum of $A \subseteq \mathbb{Z}_N$ if and only if

$$\hat{1}_A(\lambda' - \lambda) = 0, \ \forall \lambda \neq \lambda', \lambda, \lambda' \in \Lambda$$

and $|A| = |\Lambda|$. 

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The mask polynomial

**Definition (Coven-Meyerowitz, ’98)**

Let $A \subseteq \mathbb{Z}_N$. The mask polynomial $A$ is given by

$$\sum_{a \in A} X^a \in \mathbb{Z}[X]/(X^N - 1).$$

It holds

$$\hat{1}_A(d) = A(\zeta_N^d), \forall d \in \mathbb{Z}_N.$$

Λ is a spectrum of $A$ if and only if $|A| = |\Lambda|$ and

$$A(\zeta_{\text{ord}(\ell - \ell')}) = 0, \forall \ell, \ell' \in \Lambda, \ell \neq \ell'.$$
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Moreover, $A \oplus T = \mathbb{Z}_N$ if and only if

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A(X) T(X) \equiv 1 + X + X^2 + \cdots + X^{N-1} \mod (X^N - 1).
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The properties (T1) and (T2)

**Definition**

Let \( A(X) \in \mathbb{Z}[X]/(X^N - 1) \), and let

\[
S_A = \{ d \mid N : \text{d prime power}, A(\zeta_d) = 0 \}.
\]

We define the following properties:

- **(T1)** \( A(1) = \prod_{s \in S_A} \Phi_s(1) \)
- **(T2)** Let \( s_1, s_2, \ldots, s_k \in S_A \) be powers of different primes. Then \( \Phi_s(X) \mid A(X) \), where \( s = s_1 \cdots s_k \).

**Remark**

When \( N \) is a prime power, (T2) holds vacuously. If \( N = p^n q^m \), then (T2) is simply

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A(\zeta_{p^k}) = A(\zeta_{q^\ell}) = 0 \Rightarrow A(\zeta_{p^k q^\ell}) = 0
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When $N$ is a prime power, (T2) holds vacuously. If $N = p^n q^m$, then (T2) is simply

$$A(\zeta_{p^k}) = A(\zeta_{q^\ell}) = 0 \Rightarrow A(\zeta_{p^k q^\ell}) = 0$$
Let $A \subseteq \mathbb{Z}_N$, $N = p^4 q^4 r^3$, such that

$$A(\zeta_p) = A(\zeta_{p^3}) = A(\zeta_{q^2}) = A(\zeta_{r^3}) = 0,$$

and $A(X)$ has no other root of order a power of $p$, $q$, or $r$. Then,

- (T1) is $|A| = p^2 qr$.  

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- (T2): $A(\zeta_p) = A(\zeta_{q^2}) = 0 \Rightarrow A(\zeta_{pq^2}) = 0$. 
Let $A \subseteq \mathbb{Z}_N$, $N = p^4 q^4 r^3$, such that

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- (T2): $A(\zeta_{p^3}) = A(\zeta_{r^3}) = 0 \Rightarrow A(\zeta_{p^3r^3}) = 0$. 

R. D. Malikiosis

Recent developments in the discrete Fuglede conjecture
Let $A \subseteq \mathbb{Z}_N$, $N = p^4 q^4 r^3$, such that

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- We also have

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  $$A(\zeta_{p^3 q^2}) = A(\zeta_{pr^3}) = A(\zeta_{q^2 r^3}) = A(\zeta_{p^3 q^2 r^3}) = 0.$$
Theorem (Coven-Meyerowitz, ’98)

If \( A \subseteq \mathbb{Z}_N \) satisfies (T1) and (T2), then it tiles \( \mathbb{Z}_N \). If \( A \) tiles \( \mathbb{Z}_N \), then it satisfies (T1); if in addition \( N = p^n q^m \), then \( A \) satisfies (T2) as well.

Theorem (Laba, ’02)

If \( A \subseteq \mathbb{Z}_N \) satisfies (T1) and (T2), then it is spectral. If \( N = p^n \) and \( A \) is spectral, then it satisfies (T1).
Theorem (Coven-Meyerowitz, ’98)

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Theorem (Łaba, ’02)

If $A \subseteq \mathbb{Z}_N$ satisfies (T1) and (T2), then it is spectral. If $N = p^n$ and $A$ is spectral, then it satisfies (T1).
Let $A \subseteq \mathbb{Z}_N$ with spectrum $\Lambda$. The $N$th roots of unity on which $A(X)$ vanishes are precisely

$$
\zeta_{p^{\nu_1}}, \ldots, \zeta_{p^{\nu_k}}.
$$

Put $R = \{p^{\nu_1}, \ldots, p^{\nu_k}\}$. Therefore

$$
E(X) = \prod_{d \in R} \Phi_d(X) \mid A(X),
$$

whence $p^k \mid |A|$. Therefore, $E(X)$ is then the mask polynomial of a subset $E$ with $p^k$ elements, whose spectrum is $\Lambda$. Hence, $|A| = p^k$, so $A$ satisfies (T1).
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Let \( A \subseteq \mathbb{Z}_N \) with spectrum \( \Lambda \). The \( N \)th roots of unity on which \( A(X) \) vanishes are precisely

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Let $A \oplus T = \mathbb{Z}_N$, or equivalently

$$A(X) T(X) \equiv 1 + X + X^2 + \cdots + X^{N-1} \mod (X^N - 1).$$

As before, $\zeta_{p^{\nu_1}}, \ldots, \zeta_{p^{\nu_k}}$ are precisely the roots of $A(X)$, whence

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Lemma (Coven-Meyerowitz, ’98)

Suppose $A \oplus T = \mathbb{Z}_N$ and $p$ a prime such that $p \nmid |T|$. Then $A \oplus (pT) = \mathbb{Z}_N$.

Corollary

Suppose $A \oplus T = \mathbb{Z}_N$ and $M \in \mathbb{N}$ such that $\gcd(|T|, M) = 1$. Then $A \oplus (MT) = \mathbb{Z}_N$. 
(T-S(\(\mathbb{Z}_N\))), \(N = p_1^m p_2 \cdots p_n\)

**Lemma (Coven-Meyerowitz, ’98)**

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**Corollary**

Suppose \(A \oplus T = \mathbb{Z}_N\) with \(N\) square-free and \(M = |A|\). Then \(A \oplus (MT) = \mathbb{Z}_N\).

This was used by Łaba and Meyerowitz to prove (T-S(\(\mathbb{Z}_N\))), for \(N\) square-free (Tao’s blog, ’11).
Lemma (Coven-Meyerowitz, ’98)
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This was used by Łaba and Meyerowitz to prove \((T-S(\mathbb{Z}_N))\), for \( N \) square-free (Tao's blog, ’11).
Let $A \oplus T = \mathbb{Z}_N$, with $|A| = M$. Suppose first $\gcd(M, |T|) = 1$. Then $|MT| = |T| = N/M$ and $MT \subseteq M\mathbb{Z}_N$, hence $MT = M\mathbb{Z}_N$. This shows that $A$ tiles by the subgroup $M\mathbb{Z}_N$, whence

$$A(X) \equiv 1 + X + \cdots + X^{M-1} \mod (X^M - 1),$$

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Let $A \oplus T = \mathbb{Z}_N$, with $\gcd(|A|, |T|) > 1$. Then

$$\Phi_{\ell_1}(X) \cdots \Phi_{\ell_r}(X) \Phi_{p_2}(X) \cdots \Phi_{p_k}(X) \mid A(X)$$
Let $A \oplus T = \mathbb{Z}_N$, with $\gcd(|A|, |T|) > 1$. Then

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where $\{\ell_1, \ldots, \ell_r\}$ and $\{m_1, \ldots, m_s\}$ form a partition of $\{1, 2, \ldots, m\}$. 
Let $A \oplus T = \mathbb{Z}_N$, with $\gcd(|A|, |T|) > 1$. Then

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where $\{\ell_1, \ldots, \ell_r\}$ and $\{m_1, \ldots, m_s\}$ form a partition of $\{1, 2, \ldots, m\}$. Let $M = p_2 \cdots p_k$, so that $A \oplus (MT) = \mathbb{Z}_N$ and

$$\Phi_{p_1^{m_1}}(X) \cdots \Phi_{p_1^{m_s}}(X) \Phi_{p_{k+1}}(X) \cdots \Phi_{p_n}(X) \mid T(X^M).$$
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$$\Phi_{p_1^{m_1}}(X) \cdots \Phi_{p_1^{m_s}}(X) \Phi_{p_{k+1}}(X) \cdots \Phi_{p_n}(X) \mid T(X^M).$$
Let $d \mid M$ be composite. Then $A(\zeta_d)T(\zeta_d^M) = 0$ and $T(\zeta_d^M) = T(1) \neq 0$, hence $A(\zeta_d) = 0$, confirming (T2) for any set of primes dividing $M$.

Next, consider $p_1^j d$, where $d \mid M$, $d > 1$. 

(\text{T-S}(\mathbb{Z}_N)), \ N = p_1^m p_2 \cdots p_n
Let \( d \mid M \) be composite. Then \( A(\zeta_d) T(\zeta_d^M) = 0 \) and 
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Next, consider \( p_1^{\ell_j} d \), where \( d \mid M, d > 1 \). We have

\[
T(\zeta_{p_1^{\ell_j} d}^M) = T(\zeta_{p_1^{\ell_j} / d}^M) = \sigma(T(\zeta_{p_1^{\ell_j}})) \neq 0,
\]

for some \( \sigma \in \text{Gal} (\mathbb{Q}(\zeta_{p_1^{\ell_j}})/\mathbb{Q}) \), hence \( A(\zeta_{p_1^{\ell_j}}) = 0 \), confirming (T2) for \( A \) completely.
Let $d \mid M$ be composite. Then $A(\zeta_d) T(\zeta_d^M) = 0$ and 
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Next, consider $p_1^{\ell_j}d$, where $d \mid M$, $d > 1$. We have

$$T(\zeta_d^M_{p_1^{\ell_j}d}) = T(\zeta_d^M_{p_1^{\ell_j}}) = \sigma(T(\zeta_d^M_{p_1^{\ell_j}})) \neq 0,$$

for some $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_{p_1^{\ell_j}})/\mathbb{Q})$, hence $A(\zeta_{p_1^{\ell_j}}) = 0$, confirming (T2) for $A$ completely.
Vanishing sums of roots of unity

**Lemma**

Let \( \text{rad}(N) = pq \) and \( A(X) \in \mathbb{Z}[X] \) with nonnegative coefficients, such that \( A(\zeta_N^d) = 0 \), for some \( d \mid N \). Then,

\[
A(X^d) \equiv P(X^d) \Phi_p(X^{N/p}) + Q(X^d) \Phi_q(X^{N/q}) \mod (X^N - 1),
\]

where \( P(X), Q(X) \in \mathbb{Z}[X] \) can be taken with nonnegative coefficients.

- The polynomial \( A(X^d) \) is the mask polynomial of the multiset \( d \cdot A \).
Lemma

Let $\text{rad}(N) = pq$ and $A(X) \in \mathbb{Z}[X]$ with nonnegative coefficients, such that $A(\zeta_N^d) = 0$, for some $d \mid N$. Then,

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- The polynomial $A(X^d)$ is the mask polynomial of the multiset $d \cdot A$.
- $\Phi_p(X^{N/p})$ is the mask polynomial of the subgroup $\frac{N}{p} \mathbb{Z}_N$. Its cosets are called $p$-cycles.
Lemma

Let \( \text{rad}(N) = pq \) and \( A(X) \in \mathbb{Z}[X] \) with nonnegative coefficients, such that \( A(\zeta_d^N) = 0 \), for some \( d \mid N \). Then,

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- The above Lemma shows that if \( A(\zeta_N) = 0 \), then \( A \) is the disjoint union of \( p \)- and \( q \)-cycles.
Lemma

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- The above Lemma shows that if \( A(\zeta_N) = 0 \), then \( A \) is the disjoint union of \( p \)- and \( q \)-cycles.
Let \((A, B)\) be a spectral pair in \(\mathbb{Z}_N\). Wlog, \(0 \in A \cap B\) and each of \(A, B\) generates \(\mathbb{Z}_N\).

**Lemma**
Let \(0 \in A \subseteq \mathbb{Z}_N\), such that \(A\) generates \(\mathbb{Z}_N\), \(N = p^mq^n\). Then,
\[
(A - A) \cap \mathbb{Z}_N^* \neq \emptyset.
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Let \((A, B)\) be a spectral pair in \(\mathbb{Z}_N\). Wlog, \(0 \in A \cap B\) and each of \(A, B\) generates \(\mathbb{Z}_N\).

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(A - A) \cap \mathbb{Z}_N^* \neq \emptyset.
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**Proof.**

There are \(a \notin p\mathbb{Z}_N\) and \(a' \notin q\mathbb{Z}_N\).
Let $(A, B)$ be a spectral pair in $\mathbb{Z}_N$. Wlog, $0 \in A \cap B$ and each of $A, B$ generates $\mathbb{Z}_N$.

**Lemma**

Let $0 \in A \subseteq \mathbb{Z}_N$, such that $A$ generates $\mathbb{Z}_N$, $N = p^m q^n$. Then,

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**Proof.**

There are $a \notin p\mathbb{Z}_N$ and $a' \notin q\mathbb{Z}_N$. If $a \notin q\mathbb{Z}_N$, then $a \in \mathbb{Z}_N^*$. 
Let \((A, B)\) be a spectral pair in \(\mathbb{Z}_N\). Wlog, \(0 \in A \cap B\) and each of \(A, B\) generates \(\mathbb{Z}_N\).

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Let \(0 \in A \subseteq \mathbb{Z}_N\), such that \(A\) generates \(\mathbb{Z}_N\), \(N = p^m q^n\). Then,

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There are \(a \notin p\mathbb{Z}_N\) and \(a' \notin q\mathbb{Z}_N\). If \(a \notin q\mathbb{Z}_N\), then \(a \in \mathbb{Z}_N^*\), and similarly, if \(a' \notin p\mathbb{Z}_N\), then \(a' \in \mathbb{Z}_N^*\).
Let \((A, B)\) be a spectral pair in \(\mathbb{Z}_N\). Wlog, \(0 \in A \cap B\) and each of \(A, B\) generates \(\mathbb{Z}_N\).

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Let \((A, B)\) be a spectral pair in \(\mathbb{Z}_N\). Wlog, \(0 \in A \cap B\) and each of \(A, B\) generates \(\mathbb{Z}_N\).

**Lemma**

Let \(0 \in A \subseteq \mathbb{Z}_N\), such that \(A\) generates \(\mathbb{Z}_N\), \(N = p^m q^n\). Then,

\[(A - A) \cap \mathbb{Z}_N^\star \neq \emptyset.\]

**Proof.**

There are \(a \notin p\mathbb{Z}_N\) and \(a' \notin q\mathbb{Z}_N\). If \(a \notin q\mathbb{Z}_N\), then \(a \in \mathbb{Z}_N^\star\), and similarly, if \(a' \notin p\mathbb{Z}_N\), then \(a' \in \mathbb{Z}_N^\star\). If \(a \in q\mathbb{Z}_N\) and \(a' \in p\mathbb{Z}_N\), then \(a - a' \in \mathbb{Z}_N^\star\).
Therefore,

\[(A - A) \cap \mathbb{Z}_N^* \neq \emptyset \neq (B - B) \cap \mathbb{Z}_N^*,\]

which implies

\[A(\zeta_N) = B(\zeta_N) = 0.\]
Therefore,

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**Theorem (Lam & Leung)**

If \(A \subseteq \mathbb{Z}_N\) with \(A(\zeta_N) = \sum_{a \in A} \zeta_N^a = 0\), \(N = p^m q^n\), then \(A\) is a disjoint union of cosets of the subgroups \(\frac{N}{p} \mathbb{Z}_N\) and \(\frac{N}{q} \mathbb{Z}_N\).

Any two cosets of \(p\mathbb{Z}_{pq}\) and \(q\mathbb{Z}_{pq}\) intersect, so \(A\) (and \(B\)) is a disjoint union of cosets of \(p\mathbb{Z}_{pq}\) (say).
Therefore,

$$(A - A) \cap \mathbb{Z}_N^* \neq \emptyset \neq (B - B) \cap \mathbb{Z}_N^*,$$

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$$A(\zeta_N) = B(\zeta_N) = 0.$$ 

**Theorem (Lam & Leung)**

If $A \subseteq \mathbb{Z}_N$ with $A(\zeta_N) = \sum_{a \in A} \zeta_N^a = 0$, $N = p^m q^n$, then $A$ is a disjoint union of cosets of the subgroups $\frac{N}{p} \mathbb{Z}_N$ and $\frac{N}{q} \mathbb{Z}_N$.

Any two cosets of $p\mathbb{Z}_{pq}$ and $q\mathbb{Z}_{pq}$ intersect, so $A$ (and $B$) is a disjoint union of cosets of $p\mathbb{Z}_{pq}$ (say).
Since $B$ is also a union of cosets of $p\mathbb{Z}_{pq}$, we have $p\mathbb{Z}_{pq} \subseteq B - B$, hence

$$A(\zeta_q) = 0.$$ 

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R. D. Malikiosis

Recent developments in the discrete Fuglede conjecture
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(S-T($\mathbb{Z}_N$)), $N = pq$
Suppose \((S\cdot T(\mathbb{Z}_N))\) fails, but holds for any proper subgroup of \(\mathbb{Z}_N\).

Let \(A \subseteq \mathbb{Z}_N\) be a maximal spectral non-tile, with spectrum \(B\).
1. Suppose \((S-T(\mathbb{Z}_N))\) fails, but holds for any proper subgroup of \(\mathbb{Z}_N\).

2. Let \(A \subseteq \mathbb{Z}_N\) be a maximal spectral non-tile, with spectrum \(B\).

3. Both \(A\) and \(B\) must be primitive, which implies

   \[ A(\zeta_N) = B(\zeta_N) = 0. \]

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(S-T(\mathbb{Z}_N)), \ N = p^m q^n \ (\text{sketch})

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The following hold for cyclic groups $G = \mathbb{Z}_N$:

- If $N = p_1^n p_2 \cdots p_m$, then $(T-S(\mathbb{Z}_N))$ (M, '20).

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- Let $p < q$; Fuglede's conjecture holds if $N = p^m q^n$ with $p^m - 2 < q^4$ (M, '20).

- Fuglede's conjecture holds if $N = pqr$ (Kiss, M, Somlai, Vizer, '20).

- If $N = (pqr)^2$, then $(T-S(\mathbb{Z}_N))$ (Laba, Londner, '21).
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Thank you