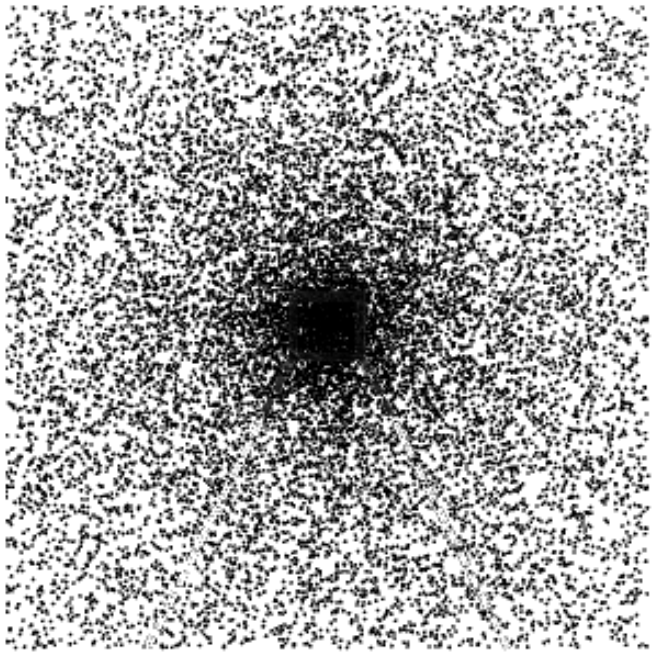
This Magic Moment: Frame Theory Meets Spherical Curves and Neurons

Emily J. King

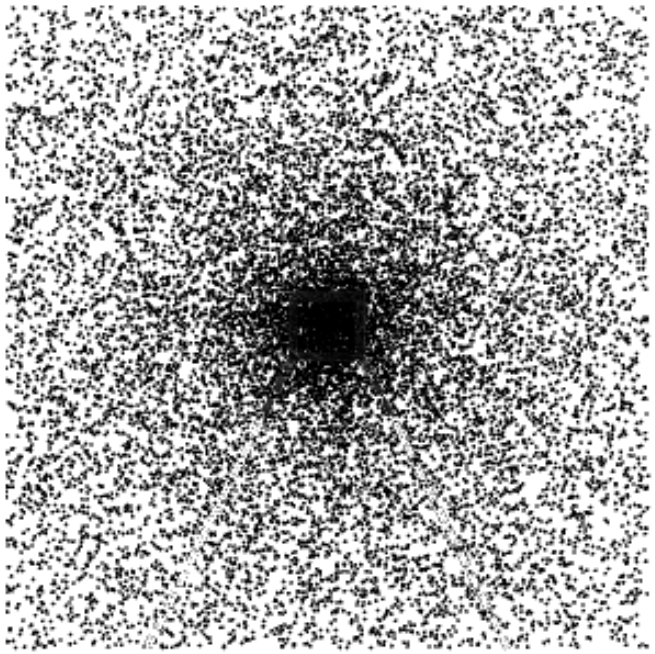
Colorado State University

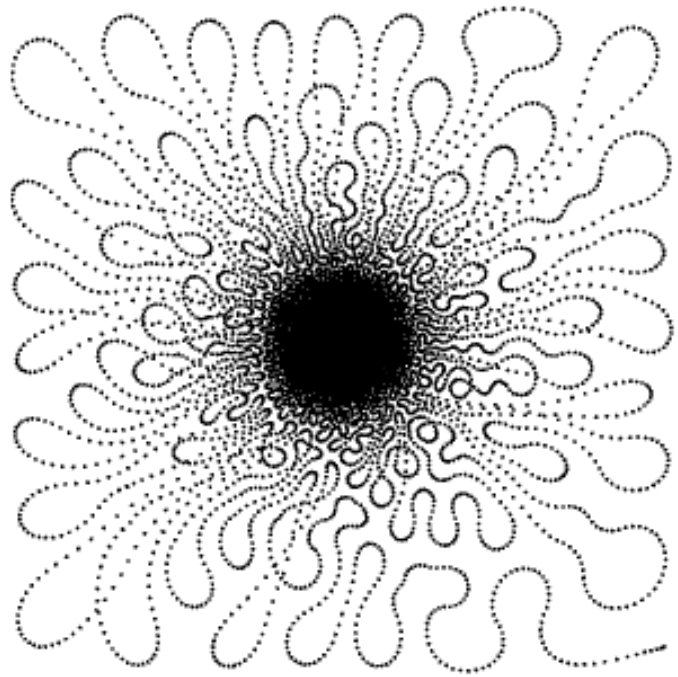
CodEx Seminar

SAMPLING



Mobile sampling





MRI / robotics / satellites : curve $>$ points

Ramamoorthy, Rajagopal, Wenzel 2008; Gröchenig et al. 2015; Boyer et al. 2016; Ehler, Gröchenig 2023

Goal 1:

Sample scene with short curve.

Jennifer Aniston neuron?



Getty Images

Quiroga, Reddy, Kreiman, Koch, Fried 2005

Jennifer Aniston neuron?



Getty Images

$\rightsquigarrow$ Bad generalization! High complexity!

Quiroga, Reddy, Kreiman, Koch, Fried 2005

Goal 2:

Efficiently pack concepts across neurons.

Main storyline

Approximating spherical moments

$\Rightarrow$ our goals

Main storyline

Approximating spherical moments
$\Rightarrow$ our goals

B plot

AI for math vs. math for AI

TALK OUTLINE

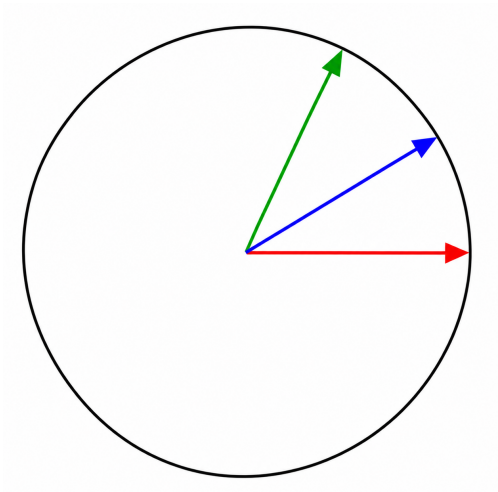
1. Spherical moments and frame theory

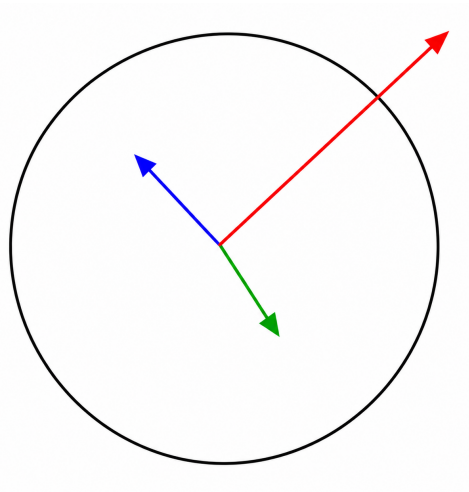
2. Spherical t-design curves

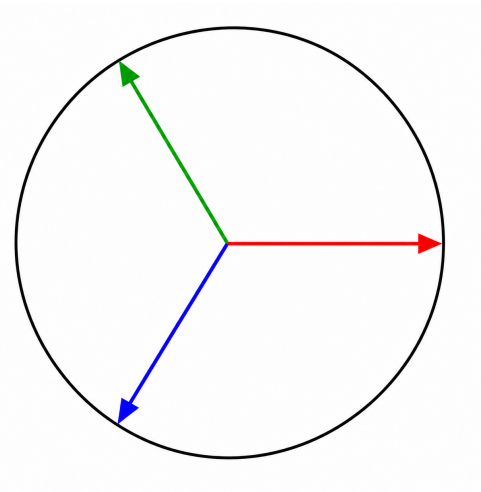
3. Superposition in neural networks

Spherical moments and frame theory

BEST 3 POINTS TO REP. S^1?



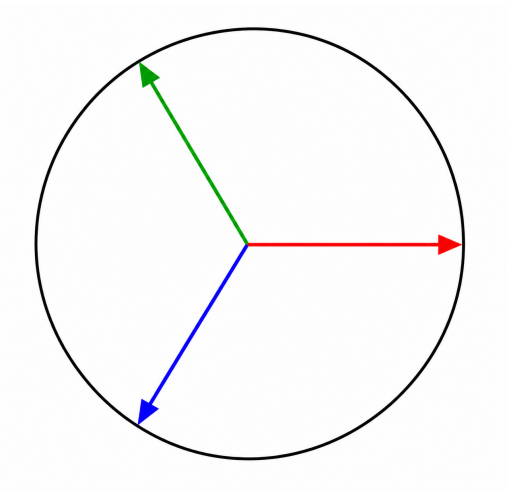




BEST 3 POINTS TO REP. S^1

$$\Phi = (\varphi_j)_{j=1}^3 = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix}$$



BEST 3 POINTS TO REP. S^1 VIA MOMENTS

$$\Phi = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix}$$

σ unif. meas. S^1

First moment of Φ

First moment of S^1

$$\frac{1}{3}\sum_{j=1}^{3} \varphi_j$$

$$= \frac{1}{3}\begin{pmatrix} 1 - 1/2 - 1/2 \\ 0 + \sqrt{3}/2 - \sqrt{3}/2 \end{pmatrix}$$

$$= 0$$

BEST 3 POINTS TO REP. S^1 VIA MOMENTS

$$\Phi = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix}$$

σ unif. meas. S^1

First moment of Φ

$$\frac{1}{3}\sum_{j=1}^{3} \varphi_j$$

$$= \frac{1}{3}\begin{pmatrix} 1 - 1/2 - 1/2 \\ 0 + \sqrt{3}/2 - \sqrt{3}/2 \end{pmatrix}$$

$$= 0$$

First moment of S^1

$$\int_{S^1} x \, d\sigma(x) = 0$$

BEST 3 POINTS TO REP. S^1 VIA MOMENTS

$$\mathbf{\Phi} = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix}$$

σ unif. meas. S^1

Second moment of $\mathbf{\Phi}$

Second moment of S^1

$$\frac{1}{3}\sum_{j=1}^{3} \varphi_j^{\otimes 2}$$

$$= \frac{1}{3}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{pmatrix} + \begin{pmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{pmatrix} \right)$$

$$= \frac{1}{2} I_2$$

BEST 3 POINTS TO REP. S^1 VIA MOMENTS

$$\Phi = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix}$$

σ unif. meas. S^1

Second moment of Φ

$$\frac{1}{3}\sum_{j=1}^{3} \varphi_j^{\otimes 2}$$

$$= \frac{1}{3}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{pmatrix} + \begin{pmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{pmatrix} \right)$$

$$= \frac{1}{2} I_2$$

Second moment of S^1

$$\int_{S^1} x_1 x_2 \, d\sigma(x) = -\int_{S^1} x_1 x_2 \, d\sigma(x)$$

$$\Rightarrow \int_{S^1} x_1 x_2 \, d\sigma(x) = 0$$

BEST 3 POINTS TO REP. S^1 VIA MOMENTS

$$\Phi = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix}$$

σ unif. meas. S^1

Second moment of Φ

$$\frac{1}{3}\sum_{j=1}^{3} \varphi_j^{\otimes 2}$$

$$= \frac{1}{3}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{pmatrix} + \begin{pmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{pmatrix} \right)$$

$$= \frac{1}{2} I_2$$

Second moment of S^1

$$\int_{S^1} (x_1^2 + x_2^2)\, d\sigma(x) = \int_{S^1} d\sigma(x) = 1$$

$$\Rightarrow \int_{S^1} x_i^2\, d\sigma(x) = \frac{1}{2}$$

BEST 3 POINTS TO REP. S^1 VIA MOMENTS

$$\Phi = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix}$$

σ unif. meas. S^1

Second moment of Φ

$$\frac{1}{3}\sum_{j=1}^{3} \varphi_j^{\otimes 2}$$

$$= \frac{1}{3}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{pmatrix} + \begin{pmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{pmatrix} \right)$$

$$= \frac{1}{2} I_2$$

Second moment of S^1

$$\int_{S^1} x^{\otimes 2}\, d\sigma(x) = \frac{1}{2} I_2$$

BEST 3 POINTS TO REP. S^1 VIA MOMENTS

$$\mathbf{\Phi} = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix}$$

σ unif. meas. S^1

$2k$th moment of $\mathbf{\Phi}$

$2k$th moment of S^1

$$\max_{i \neq j} \left| \langle \varphi_i, \varphi_j \rangle \right| \quad \Rightarrow \quad \frac{1}{3} \sum_{j=1}^{3} \varphi_j^{\otimes 2k} \approx \int_{S^1} x^{\otimes 2k} \, d\sigma(x)$$

min for 3 vecs

closest in Hilb.-Schm. for 3 vecs

Frame theory intro

$\Phi = (\varphi_j)_{j=1}^n \subset S^d \subset \mathbb{R}^{d+1}, n > d+1$ is ...

...centered if $\frac{1}{n}\sum_{j=1}^{n} \varphi_j = 0$

Frame theory intro

$\Phi = (\varphi_j)_{j=1}^n \subset S^d \subset \mathbb{R}^{d+1}, n > d+1$ is ...

...centered if $\frac{1}{n}\sum_{j=1}^n \varphi_j = 0$

...tight frame if $\frac{1}{n}\sum_{j=1}^n \varphi_j^{\otimes 2} = \frac{1}{d+1} I_{d+1}$

Frame theory intro

$\Phi = (\varphi_j)_{j=1}^n \subset S^d \subset \mathbb{R}^{d+1}, n > d+1$ is ...

...centered if $\frac{1}{n}\sum_{j=1}^{n} \varphi_j = 0$

...tight frame if $\frac{1}{n}\sum_{j=1}^{n} \varphi_j^{\otimes 2} = \frac{1}{d+1} I_{d+1}$

...equiangular if $\left|\langle \varphi_j, \varphi_k \rangle\right| = \mu$ for $j \neq k$

Frame theory facts

Centered tight frames exist for almost all $n > d + 1$

Benedetto, Fickus 2003; Mimura 1990

FRAME THEORY FACTS

Centered tight frames exist for almost all $n > d + 1$

Equiangular tight frames (ETFs) exist for fewer $n > d + 1$

Benedetto, Fickus 2003; Mimura 1990; Sustik et al. 2007

FRAME THEORY FACTS

Centered tight frames exist for almost all $n > d + 1$

Equiangular tight frames (ETFs) exist for fewer $n > d + 1$

ETFs $\Rightarrow$ minimal coherence $\max_{i \neq j} \left| \langle \varphi_i, \varphi_j \rangle \right|$ for n vectors

---

Benedetto, Fickus 2003; Mimura 1990; Sustik et al. 2007; Strohmer, Heath 2003

Frame theory facts

Centered tight frames exist for almost all $n > d + 1$

Equiangular tight frames (ETFs) exist for fewer $n > d + 1$

ETFs $\Rightarrow$ minimal coherence $\max_{i \neq j} \left| \langle \varphi_i, \varphi_j \rangle \right|$ for n vectors

$\Rightarrow$ best approx $2k$th spherical moments for n vectors

---

Benedetto, Fickus 2003; Mimura 1990; Sustik et al. 2007; Strohmer, Heath 2003; Ehler, Okoudjou 2012

FRAME THEORY FACTS

Centered tight frames exist for almost all $n > d + 1$

Equiangular tight frames (ETFs) exist for fewer $n > d + 1$

ETFs $\Rightarrow$ minimal coherence $\max_{i \neq j} \left| \langle \varphi_i, \varphi_j \rangle \right|$ for n vectors

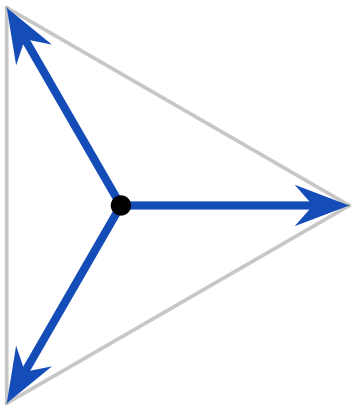
$\Rightarrow$ best approx $2k$th spherical moments for n vectors

(Centered) ETFs $\subsetneq$ strongly regular graphs (SRGs)

Benedetto, Fickus 2003; Mimura 1990; Sustik et al. 2007; Strohmer, Heath 2003; Ehler, Okoudjou 2012; Seidel 1966+; Fickus et al. 2018

Some real ETFs

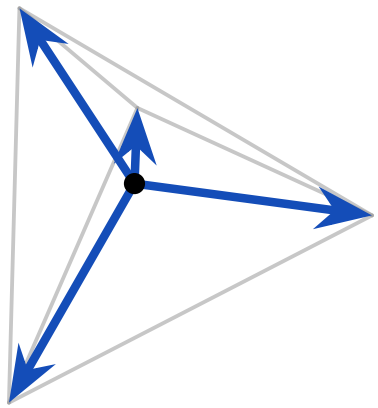
$(2,3)$



regular triangle

centered

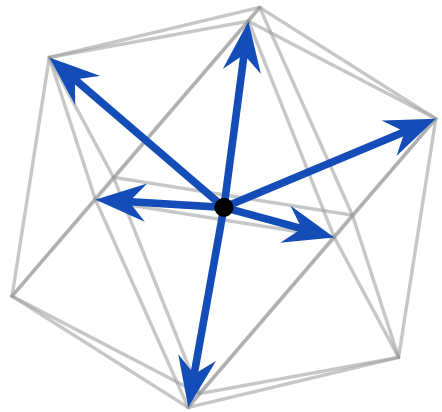
$(3,4)$



regular tetrahedron

centered

$(3,6)$



regular icosahedron

not centered

SPHERICAL t-DESIGNS

$\Pi_d^t =$ polys over S^d w/ deg $\leq t$

$\Phi = (\varphi_j)_{j=1}^n \subset S^d =$ spherical t-design $\Leftrightarrow$

$$\frac{1}{n}\sum_{j=1}^{n} f(\varphi_j) = \int_{S^d} f(x)\, d\sigma(x) \qquad \forall f \in \Pi_d^t$$

McLaren 1963; Delsarte, Goethals, Seidel 1977; Holmes, Paulsen 2004

SPHERICAL t-DESIGNS

$\mathbf{\Pi}_d^t = $ polys over S^d w/ deg $\leq t$

$\mathbf{\Phi} = (\varphi_j)_{j=1}^n \subset S^d =$ spherical t-design $\Leftrightarrow$

$$\frac{1}{n}\sum_{j=1}^{n} f(\varphi_j) = \int_{S^d} f(x)\, d\sigma(x) \qquad \forall f \in \mathbf{\Pi}_d^t$$

$\mathbf{\Phi}$ spherical $\mathbf{1}$-design $\Leftrightarrow$ centered

$\mathbf{\Phi}$ spherical $\mathbf{2}$-design $\Leftrightarrow$ centered tight frame

---

McLaren 1963; Delsarte, Goethals, Seidel 1977; Holmes, Paulsen 2004

Think:
global average temperature with fewest measurements

Think:
global average temperature with fewest measurements

What are the smallest spherical t-designs?

Delsarte, Goethals, Seidel 1977; Bannai et al. 1979+

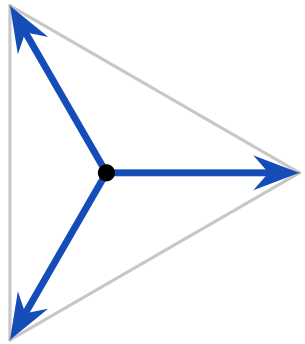
Think:
global average temperature with fewest measurements

What are the smallest spherical t-designs?

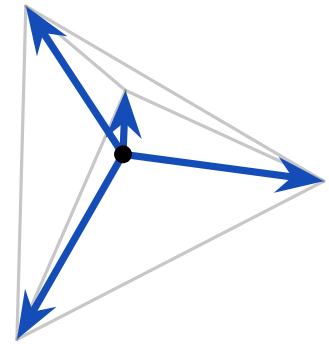
Answer typically asymptotic in t.

Delsarte, Goethals, Seidel 1977; Bannai et al. 1979+; Bondarenko, Radchenko, Viazovska 2013

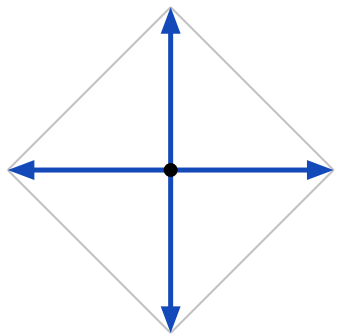
MINIMAL SPHERICAL t-DESIGNS



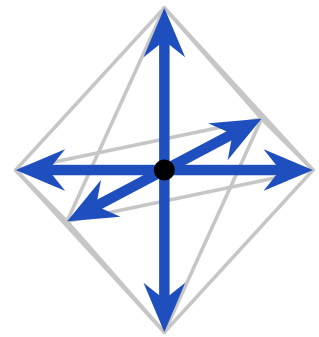
regular triangle
tight 2-design



regular tetrahedron
tight 2-design



square
tight 3-design



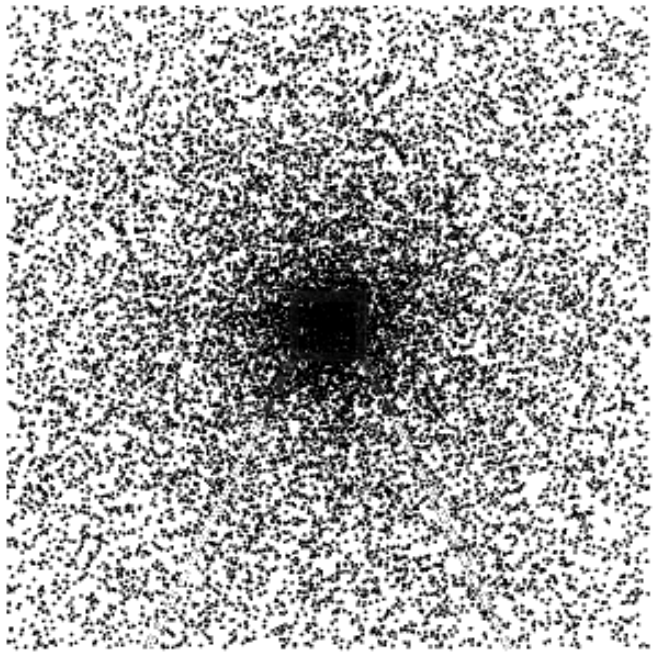
regular octahedron
tight 3-design

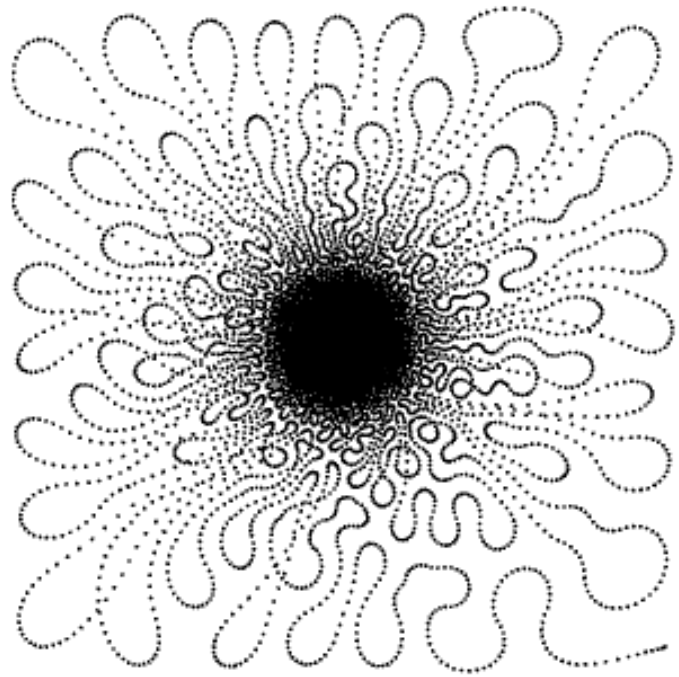
Spherical t-design curves

generalizing centered tight frames

AI $\Rightarrow$ math

Reminder: mobile sampling





Goal: Sample scene with short curve.

Ramamoorthy, Rajagopal, Wenzel 2008; Gröchenig et al. 2015; Boyer et al. 2016; Ehler, Gröchenig 2023

From points to curves

length-L closed rectifiable curve γ in S^d
$=$ spherical t-design curve $\Leftrightarrow$

$$\frac{1}{L}\int_\gamma f(x)\,ds = \int_{S^d} f(x)\,d\sigma(x) \qquad \forall f \in \Pi_d^t.$$

Ehler, Gröchenig 2023

From points to curves

length-L closed rectifiable curve γ in S^d
$=$ spherical t-design curve $\Leftrightarrow$

$$\frac{1}{L}\int_\gamma f(x)\,ds = \int_{S^d} f(x)\,d\sigma(x) \qquad \forall f \in \Pi_d^t.$$

γ spherical $\mathbf{1}$-design curve $\Leftrightarrow \frac{1}{L}\int_\gamma x\,ds = \mathbf{0}$

γ spherical $\mathbf{2}$-design curve $\Leftrightarrow \frac{1}{L}\int_\gamma x^{\otimes 2}\,ds = \frac{1}{d+1} I_{d+1}$, too

---

Ehler, Gröchenig 2023

Think:
global average temperature with shortest satellite path

Think:
global average temperature with shortest satellite path

What are the shortest spherical t-design curves?

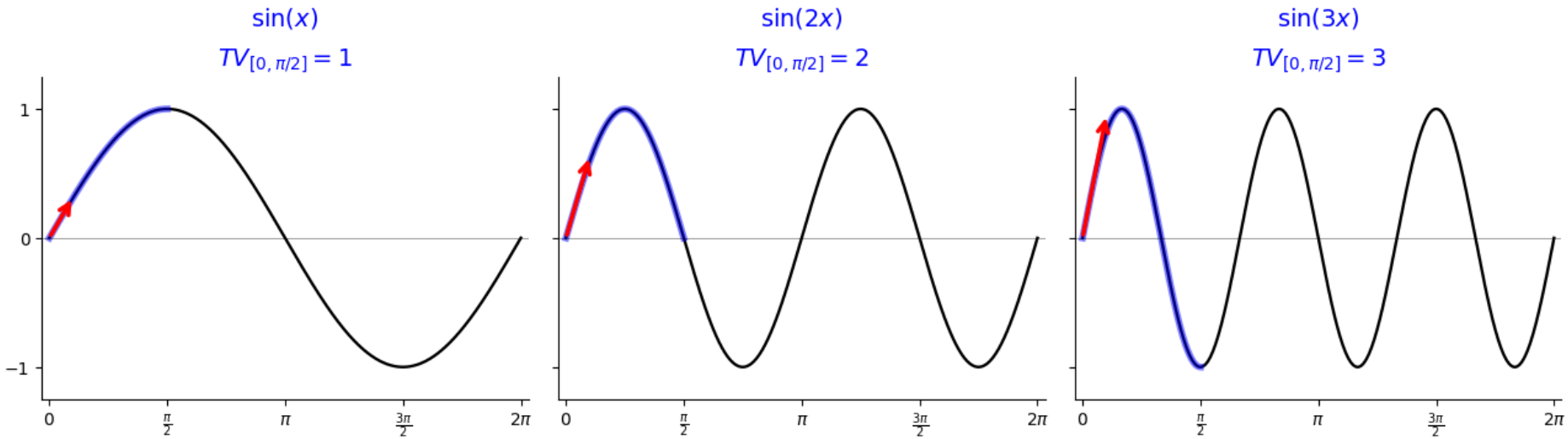
Think:
global average temperature with shortest satellite path

What are the shortest spherical t-design curves?

Answer until now asymptotic in t.

Ehler, Gröchenig 2023; Lindblad 2024; Ehler, Gröchenig, Karner 2025

An observation



high frequency $\Rightarrow$ large derivative $\Rightarrow$ long length

LOWER BOUND $t = 2$, d ODD

$\gamma = 2$-design curve in S^{2k-1}:

K, Mixon 2026

LOWER BOUND $t = 2$, d ODD

$\gamma = 2$-design curve in S^{2k-1}:

$2k\langle x_i, x_j \rangle_{S^{2k-1}} = \delta_{i,j}$ orthogonal invariance

K, Mixon 2026

LOWER BOUND $t = 2$, d ODD

$\gamma = 2$-design curve in S^{2k-1}:

$$2k\langle x_i, x_j\rangle_{S^{2k-1}} = \delta_{i,j} \qquad \text{orthogonal invariance}$$

$$\Leftrightarrow \quad 2k\langle x_i, x_j\rangle_{\gamma} = \delta_{i,j} \qquad \text{2-design property}$$

K, Mixon 2026

LOWER BOUND $t = 2$, d ODD

$\gamma = 2$-design curve in S^{2k-1}:

$$2k\langle x_i, x_j\rangle_{S^{2k-1}} = \delta_{i,j} \qquad \text{orthogonal invariance}$$

$$\Leftrightarrow \quad 2k\langle x_i, x_j\rangle_{\gamma} = \delta_{i,j} \qquad \text{2-design property}$$

$$\Rightarrow \quad \text{coords of } \gamma \text{ orth}$$

K, Mixon 2026

LOWER BOUND $t = 2$, d ODD

$\gamma = 2$-design curve in S^{2k-1}:

$2k\langle x_i, x_j\rangle_{S^{2k-1}} = \delta_{i,j}$ orthogonal invariance

$\Leftrightarrow \quad 2k\langle x_i, x_j\rangle_{\gamma} = \delta_{i,j}$ 2-design property

$\Rightarrow$ coords of γ orth

$\Rightarrow$ at least k diff freq in coords of γ Fourier

K, Mixon 2026

LOWER BOUND $t = 2$, d ODD

$\gamma = 2$-design curve in S^{2k-1}:

$$2k\langle x_i, x_j\rangle_{S^{2k-1}} = \delta_{i,j} \qquad \text{orthogonal invariance}$$

$$\Leftrightarrow \quad 2k\langle x_i, x_j\rangle_{\gamma} = \delta_{i,j} \qquad \text{2-design property}$$

$$\Rightarrow \quad \text{coords of } \gamma \text{ orth}$$

$$\Rightarrow \quad \text{at least } k \text{ diff freq in coords of } \gamma \qquad \text{Fourier}$$

$$\Rightarrow \quad \gamma \text{ is shortest when} \qquad \text{water-filling}$$

$$\gamma(\theta) = \frac{1}{\sqrt{k}}\big(\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots, \cos k\theta, \sin k\theta\big)$$

K, Mixon 2026

A SPECTRAL LOWER BOUND

Every t-design curve in S^d has

$$L(\gamma) \geq C_{t,d} \quad \longleftarrow \quad \text{explicit}$$

K, Mixon 2026

A SPECTRAL LOWER BOUND

Every t-design curve in S^d has

$$L(\gamma) \geq C_{t,d} \quad \longleftarrow \quad \text{explicit}$$

Proof idea:

1. t-design property
   $\Rightarrow$ low-deg orth polys on S^d are orth on γ

K, Mixon 2026

A SPECTRAL LOWER BOUND

Every t-design curve in S^d has

$$L(\gamma) \geq C_{t,d} \quad \longleftarrow \quad \text{explicit}$$

Proof idea:

1. t-design property
   $\Rightarrow$ low-deg orth polys on S^d are orth on γ

2. water-filling estimate
   $\Rightarrow$ total Dirichlet energy $\geq \sum S^1$ Laplacian evalues

---

K, Mixon 2026

A SPECTRAL LOWER BOUND

Every t-design curve in S^d has

$$L(\gamma) \geq C_{t,d} \quad \longleftarrow \quad \text{explicit}$$

Proof idea:

1. t-design property
   $\Rightarrow$ low-deg orth polys on S^d are orth on γ

2. water-filling estimate
   $\Rightarrow$ total Dirichlet energy $\geq \sum S^1$ Laplacian evalues

3. spherical harmonics
   $\Rightarrow$ total Dirichlet energy = explicit function of $L(\gamma)$

K, Mixon 2026

SHORTEST t-DESIGN CURVES

$t = 1$: great circles.

Lindblad 2024; Ehler, Gröchenig, Karner 2025; K, Mixon 2026

SHORTEST t-DESIGN CURVES

$t = 1$: great circles.

$t = 2$, $d = 2k - 1$:

$$\gamma(\theta) = \frac{1}{\sqrt{k}}\big(\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots, \cos k\theta, \sin k\theta\big)$$

Lindblad 2024; Ehler, Gröchenig, Karner 2025; K, Mixon 2026

SHORTEST t-DESIGN CURVES

$t = 1$: great circles.

$t = 2$, $d = 2k - 1$:

$$\gamma(\theta) = \frac{1}{\sqrt{k}}\big(\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots, \cos k\theta, \sin k\theta\big)$$

$t = 3$, $d = 2k - 1$ (conjectured):

$$\gamma(\theta) = \frac{1}{\sqrt{k}}\big(\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \ldots, \sin(2k-1)\theta\big)$$

Lindblad 2024; Ehler, Gröchenig, Karner 2025; K, Mixon 2026

SHORTEST t-DESIGN CURVES

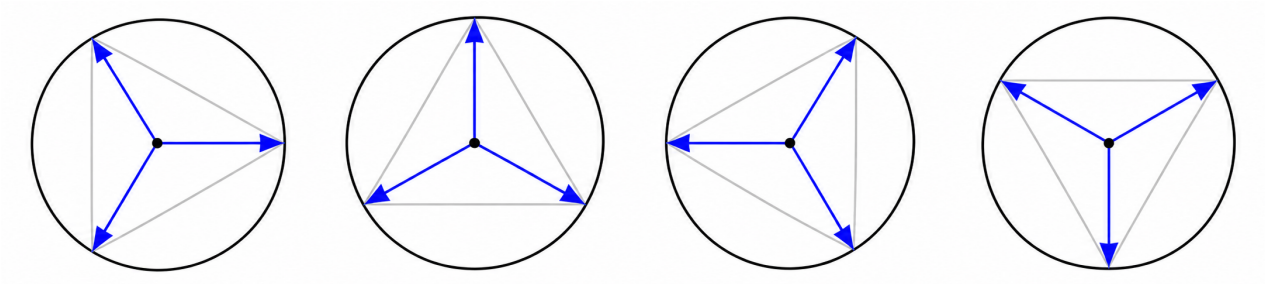
$t = 1$: great circles.

$t = 2$, $d = 2k - 1$:

$$\gamma(\theta) = \frac{1}{\sqrt{k}}\big(\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots, \cos k\theta, \sin k\theta\big)$$

$t = 3$, $d = 2k - 1$ (conjectured):

$$\gamma(\theta) = \frac{1}{\sqrt{k}}\big(\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \ldots, \sin(2k-1)\theta\big)$$



Lindblad 2024; Ehler, Gröchenig, Karner 2025; K, Mixon 2026

Minimal curves in even-dimensional spheres

$d = 2k$, min curve length 2-design = $L_{\min}(2, d)$

$$1.44 \cdot d^{-1} \quad \leq \quad L_{\min}(2, d) - \left(\frac{\pi}{\sqrt{3}} \cdot d + \frac{5\pi}{2\sqrt{3}} \right) \quad \leq \quad 5.86 \cdot d^{-1}$$

K, Mixon 2026

Minimal curves in even-dimensional spheres

$$d = 2k, \quad \text{min curve length } 2\text{-design} = L_{\min}(2, d)$$

$$1.44 \cdot d^{-1} \underset{\text{explicit bound}}{\leq} L_{\min}(2, d) - \left(\frac{\pi}{\sqrt{3}} \cdot d + \frac{5\pi}{2\sqrt{3}} \right) \underset{\text{construction}}{\leq} 5.86 \cdot d^{-1}$$

K, Mixon 2026

Minimal curves in even-dimensional spheres

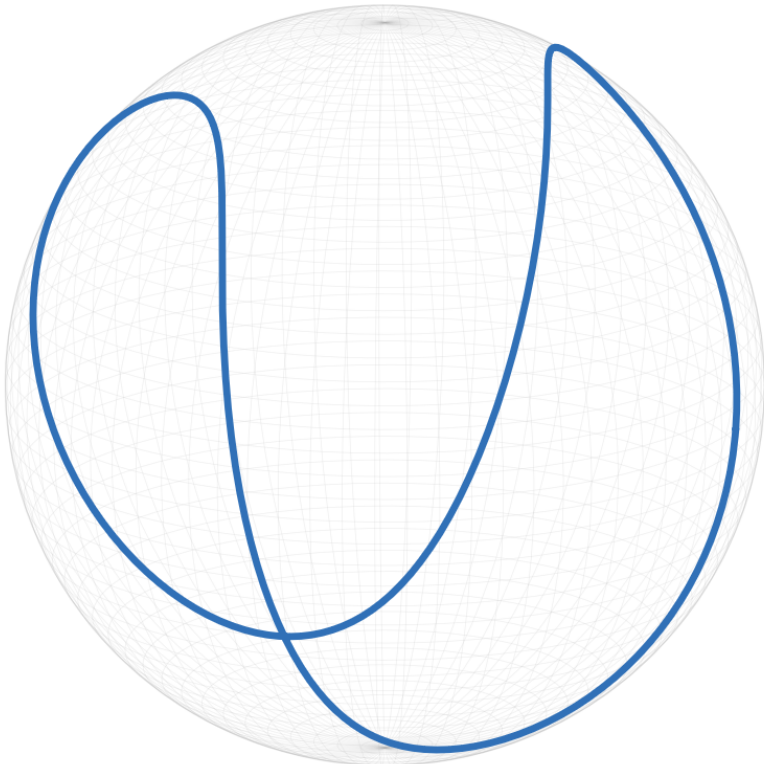
$d = 2k$, min curve length 2-design = $L_{\min}(2, d)$

$$1.44 \cdot d^{-1} \underset{\text{explicit bound}}{\leq} L_{\min}(2, d) - \left(\frac{\pi}{\sqrt{3}} \cdot d + \frac{5\pi}{2\sqrt{3}} \right) \underset{\text{construction}}{\leq} 5.86 \cdot d^{-1}$$

(Asymptotic optimality in d)

K, Mixon 2026

The first difficult case: S^2



provable lower bound

$2\pi\sqrt{2} \approx 8.886$

putatively shortest

≈ 9.743

D_4 symmetry contains regular tetrahedron

K, Mixon 2026

ChatGPT was used throughout the project to

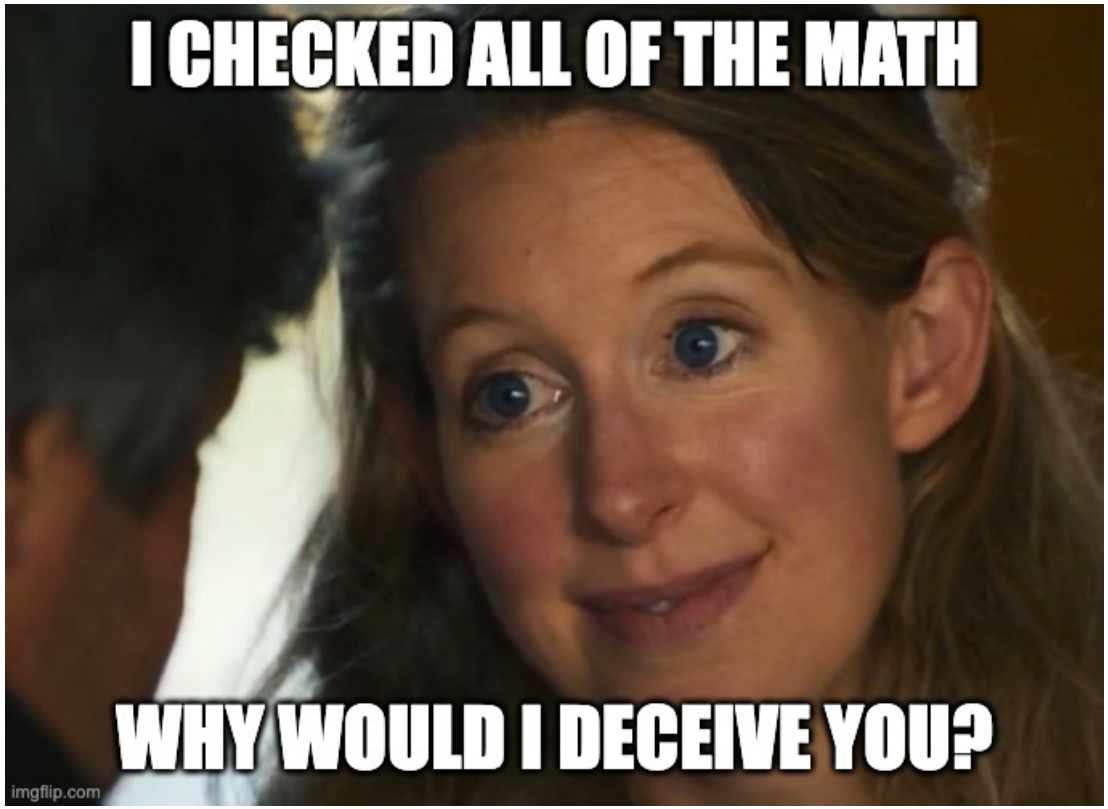
find literature

write and run exploratory code

explore possible proof strategies

give feedback on writing

Every mathematical claim, citation, and computation was checked by the authors.



Superposition

approximating centered tight frames

Math $\Rightarrow$ AI

Reminder: Jennifer Aniston Neuron



Getty Images

Quiroga, Reddy, Kreiman, Koch, Fried 2005

Superposition:

represent data with

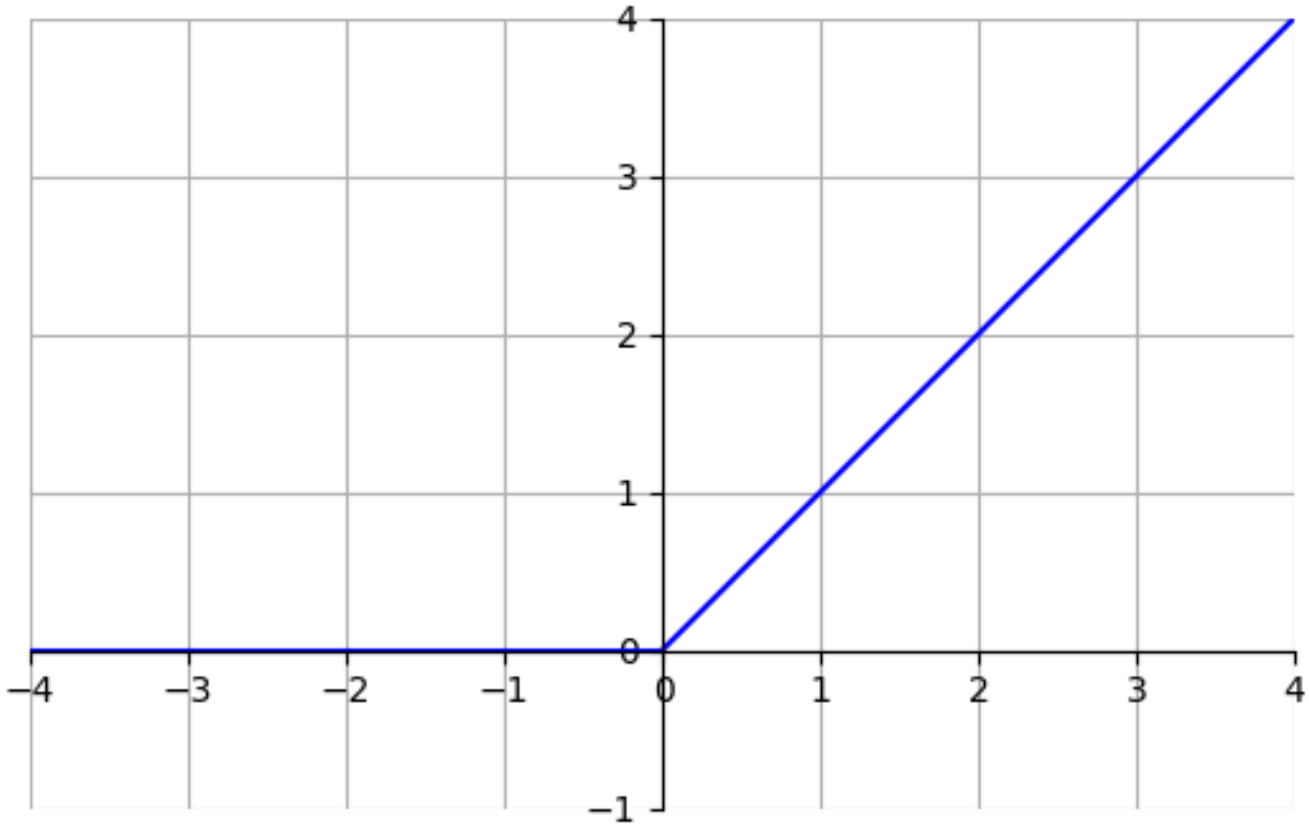
more features than dimensions (neurons)

Arora et al. 2018, Goh 2016, Olah et al. 2020, Elhage et al. 2022

What is a Neural Network?

affine linear map	$f_j : \mathbb{R}^{N_{j-1}} \to \mathbb{R}^{N_j}$
non-linear activation function	$\sigma : \mathbb{R} \to \mathbb{R}$
jth layer	$L_j := \sigma \text{ “}\circ\text{” } f_j$
(feedforward) neural network	$L_\ell \circ L_{\ell-1} \circ \cdots \circ L_2 \circ L_1$

ReLU Activation

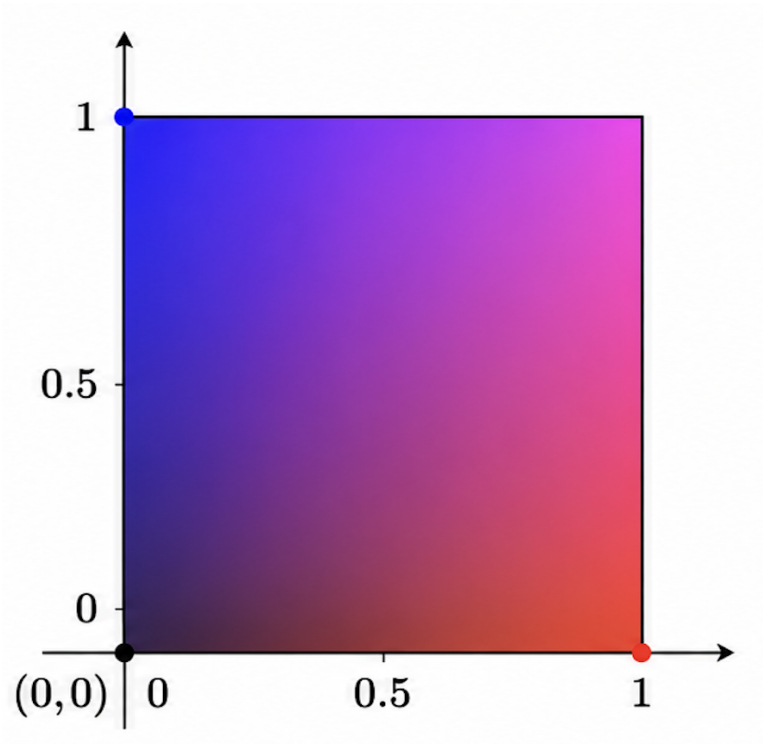


"keeps" positive numbers "unalives" negative numbers

Say you only had 2 dimensions to represent colors.

Say you only had 2 dimensions to represent colors.

Option: non-negative linear combinations of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



Better(?) option: non-negative linear combinations of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}$$

Better(?) option: non-negative linear combinations of

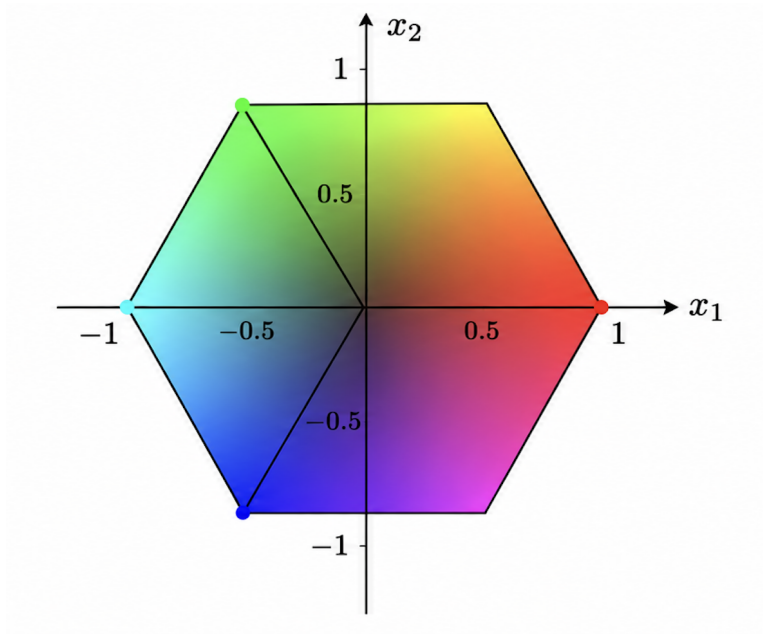
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}$$

$$\underbrace{1/3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} + 0 \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}}_{\text{rep'n of dark red}} = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$$

$$\underbrace{1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2/3 \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} + 2/3 \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}}_{\text{rep'n of medium salmon}} = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}.$$

Best(?) option: non-negative **2**-sparse[1] linear combinations of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}$$

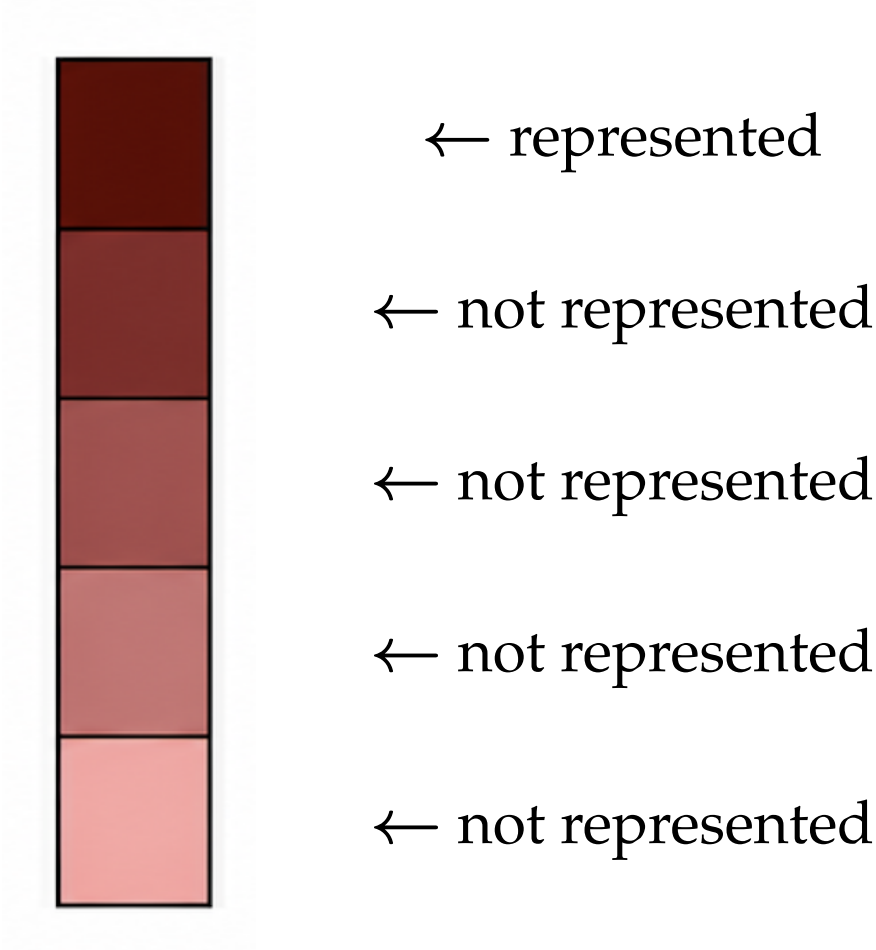


[1] i.e., at least one of r, g, b is equal to **0**

2-sparsity

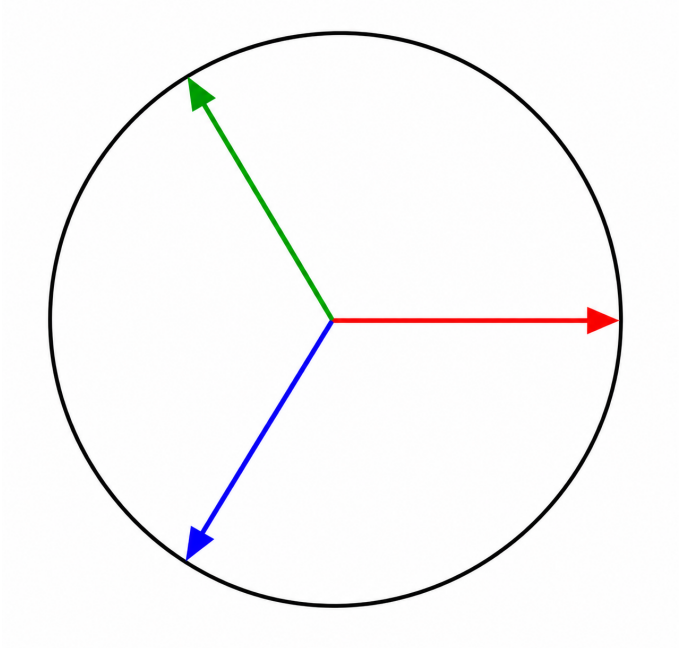
$\Rightarrow$ unique representation,

but only “maximum saturation” within a hue



Our old friend

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \qquad \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}$$



centered equiangular tight frame

Key idea:

Sparse linear combinations of

vectors that approx. spherical moments

allow superposition.

Toy model of superposition:

For each sparse $\boldsymbol{x}$, want

$$\mathbf{ReLU}(W^\top W\boldsymbol{x} + \boldsymbol{b}) \approx \boldsymbol{x}$$

for some $\boldsymbol{b}$.

Columns of $\boldsymbol{W} \approx$ features Concepts $\approx \boldsymbol{W}\boldsymbol{x}$ for $\boldsymbol{x}$ sparse

---

Empirical: Elhage et al. 2022; Cowsik, Dolev, Infanger 2024; Liu, Liu, Gore 2025;
Theoretical: Ivanitskiy, Jasper, K, Mixon 2026

Toy model of superposition:

For each sparse x, want

$$\mathbf{supp}\Big(\mathbf{ReLU}(W^\top Wx + b)\Big) = \mathbf{supp}(x)$$

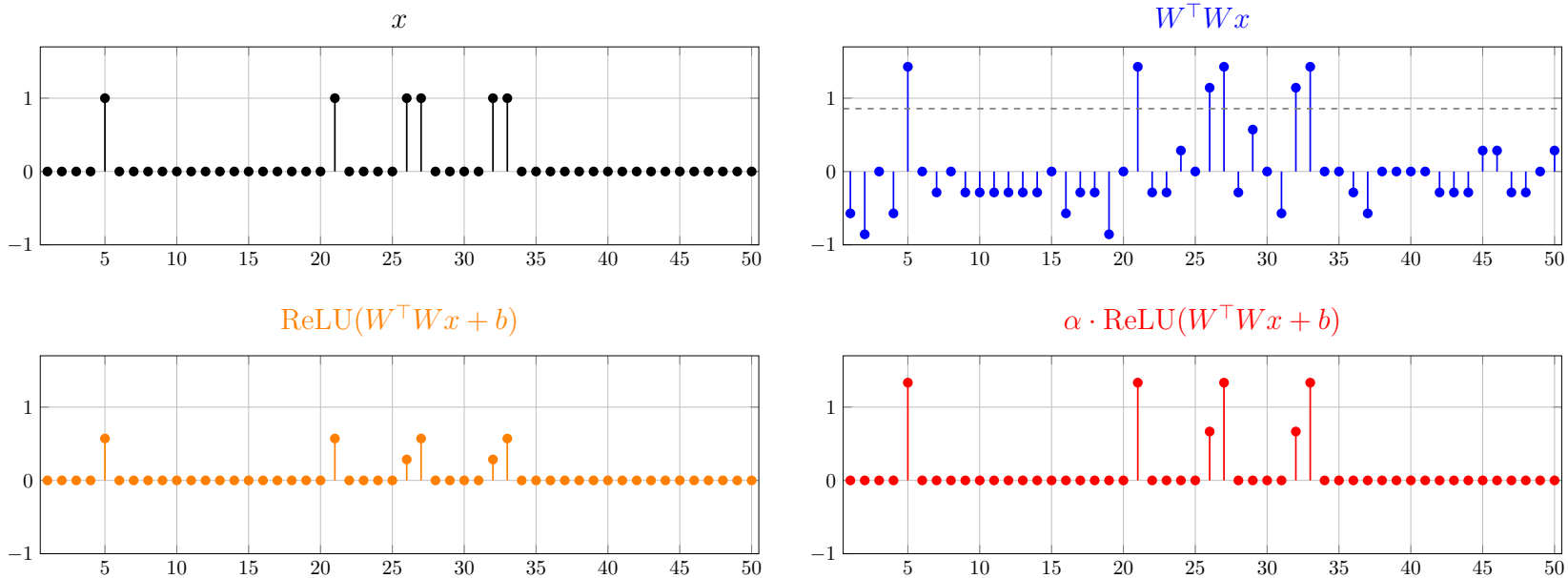
for some b.

Columns of $W \approx$ features Concepts $\approx Wx$ for x sparse

---

Empirical: Elhage et al. 2022; Cowsik, Dolev, Infanger 2024; Liu, Liu, Gore 2025;
Theoretical: Ivanitskiy, Jasper, K, Mixon 2026

$W \approx$ **1**st and **2***k*th spherical moments



WHY CENTERED + TIGHT + LOW COHERENCE?

$x \sim \mathbf{Bern}(p), p \ll \frac{d+1}{n \ln n}, \quad W \approx \mathbf{1}$st & $\mathbf{2}k$th spherical moments

$$\Rightarrow \qquad \left\| \frac{\mathbf{1}}{\mathbf{2}} \mathbf{ReLU}(W^\top W x + b) - x \right\|_\infty \quad \text{small} \quad \text{w.h.p.}$$

Ivanitskiy, Jasper, K, Mixon 2026

WHY CENTERED + TIGHT + LOW COHERENCE?

$x \sim \mathbf{Bern}(p), p \ll \frac{d+1}{n \ln n}, \quad W \approx$ **1**st & **2**kth spherical moments

$$\Rightarrow \qquad \left\| \frac{\mathbf{1}}{\mathbf{2}} \mathbf{ReLU}(W^\top W x + b) - x \right\|_\infty \quad \text{small} \quad \text{w.h.p.}$$

Proof idea:

1. $\frac{\mathbf{1}}{n} \sum_{j=1}^{n} w_j = \mathbf{0}$
   $\Rightarrow$ can use Bernstein inequalities
   (mean crosstalk vanishes)

Ivanitskiy, Jasper, K, Mixon 2026

WHY CENTERED + TIGHT + LOW COHERENCE?

$x \sim \mathbf{Bern}(p), p \ll \frac{d+1}{n \ln n}, \quad W \approx \mathbf{1}$st & $\mathbf{2}k$th spherical moments

$$\Rightarrow \qquad \left\| \frac{\mathbf{1}}{\mathbf{2}} \mathbf{ReLU}(W^\top W x + b) - x \right\|_\infty \quad \text{small} \quad \text{w.h.p.}$$

Proof idea:

1. $\frac{\mathbf{1}}{n} \sum_{j=1}^{n} w_j = \mathbf{0}$
   $\Rightarrow$ can use Bernstein inequalities
   (mean crosstalk vanishes)

2. $\frac{\mathbf{1}}{n} \sum_{j=1}^{n} w_j^{\otimes \mathbf{2}} \approx \frac{\mathbf{1}}{d+\mathbf{1}} I_{d+1}$ and $\left|\langle w_j, w_k \rangle\right|$ small
   $\Rightarrow$ Bernstein inequalities give good bounds
   (crosstalk variance and tails controlled)

Ivanitskiy, Jasper, K, Mixon 2026

If $\boldsymbol{W}$ is a $(\boldsymbol{d+1}, \boldsymbol{n})$-ETF with $\boldsymbol{n > d+2}$, then for all

$$\#\,\mathbf{supp}(x) \lesssim \frac{1}{2}\sqrt{d+1}$$

there is a $\boldsymbol{b}$ with

$$\mathbf{supp}\Big(\mathbf{ReLU}(W^\top Wx + b)\Big) = \mathbf{supp}(x).$$

If W is a $(d+1, n)$-ETF with $n > d+2$, then for all

$$\#\,\mathrm{supp}(x) \lesssim \frac{1}{2}\sqrt{d+1}$$

there is a b with

$$\mathrm{supp}\Big(\mathrm{ReLU}(W^\top W x + b)\Big) = \mathrm{supp}(x).$$

Proof idea:
Switching of strongly regular graphs
$\Rightarrow$ all real ETFs $\approx$ centered

If W is a $(d+1, n)$-ETF with $n > d+2$, then for all

$$\#\,\mathrm{supp}(x) \lesssim \frac{1}{2}\sqrt{d+1}$$

there is a b with

$$\mathrm{supp}\Big(\mathrm{ReLU}(W^\top W x + b)\Big) = \mathrm{supp}(x).$$

Proof idea:
Switching of strongly regular graphs
$\Rightarrow$ all real ETFs $\approx$ centered

Similar when W is Gaussian $(0, 1)$ i.i.d.

Spherical moments

appear in unexpected places

Spherical moments

appear in unexpected places

Humans and AI

make a good math team

Thanks for your time!

Towards a mathematical theory of superposition, arXiv:2608.27540

Short spherical t-design curves, arXiv:2607.15386

For papers and software:

`https://www.math.colostate.edu/~king`

CodEx Seminar:
An international remote seminar on the theory and applications of harmonic analysis, combinatorics, and algebra

`https://www.math.colostate.edu/~king/codex`