A Potpourri of Projective 2-Designs

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MY FAVORITE[1] SET OF 9 VECTORS IN $\mathbb{C}^3$

Let ζ be a primitive third root of unity. Define

$$\mathcal{G}(3) = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \zeta & \zeta^2 & 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta & 0 & 0 & 0 & 1 & \zeta^2 & \zeta \end{array} \right).$$

Why is $\mathcal{G}(3)$ so cool?

[1] Actually this is slightly different than my actual favorite set of 9 Vectors in $\mathbb{C}^3$.

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   (We can easily compute a change of basis.)

3. The inner products between the vectors are equal and optimally small in absolute value.
   (Vectors represent "different information" geometrically.)

How can we generalize these traits?

Change of Basis

Let $\Phi = \begin{pmatrix} \varphi_1 & \dots & \varphi_d \end{pmatrix}$, where the columns are an ONB for $\mathbb{F}^d$.

Then for all $x \in \mathbb{F}^d$

$$x = \sum_{j=1}^{d} \langle x, \varphi_j \rangle \varphi_j = \Phi \begin{pmatrix} \langle x, \varphi_1 \rangle \\ \vdots \\ \langle x, \varphi_d \rangle \end{pmatrix} = \Phi\Phi^* x = \sum_{j=1}^{n} \varphi_j \varphi_j^* x.$$

I.e., change of basis holds since …

- …the rows of Φ are orthonormal;
- …the outer products of the columns of Φ yield a decomposition of I.

Cubature via Orthonormal Rows, I

Let $\Phi = (\varphi_{j,k})_{j=1,k=1}^{d}$ be unitary (with $d \geq 2$).

Claim: the columns of Φ yield a cubature rule because the rows of Φ are orthonormal.

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(Exact) cubature rule:
For $\mathcal{F} = \{f : D \to C \,|\, \text{characteristics of } f\}$ with all f integrable w.r.t σ, $\{d_j\}_{j=1}^{n} \subset D$ is such that $\frac{1}{n}\sum_{j=1}^{n} f(d_j) = \int_D f(x)d\sigma(x)$ for all $f \in \mathcal{F}$.

Cubature via Orthonormal Rows, II

Let $\Phi = (\varphi_{j,k})_{j=1,k=1}^{d}$ be unitary (with $d \geq 2$).

Set $D = S(\mathbb{C}^d)$, $\mathcal{F}^{2} = \mathbb{C}[z_1, \ldots, z_d, \overline{z_1}, \ldots, \overline{z_d}]$, σ unif. meas., $d_j = \varphi_j$.

[2] Actually a subspace of this.

CUBATURE VIA ORTHONORMAL ROWS, II

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$$\int_{S(\mathbb{C}^d)} p(z_1, \ldots, z_d)d\sigma = \int_{S(\mathbb{C}^d)} z_1\overline{z_2}d\sigma = -\int_{S(\mathbb{C}^d)} z_1\overline{z_2}d\sigma = 0$$

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since

$$1 = \int_{S(\mathbb{C}^d)} 1 d\sigma = \int_{S(\mathbb{C}^d)} |z_1|^2 + |z_2|^2 + \ldots + |z_d|^2 \, d\sigma$$

Some Important Spaces, I

$$\mathrm{Hom}_{\mathbb{F}^d}(t,t) = \mathrm{span}_{\mathbb{F}} \left\{ z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d} \bar{z}_1^{\beta_1} \bar{z}_2^{\beta_2} \cdots \bar{z}_d^{\beta_d} \mid t = \sum_j \alpha_j = \sum_j \beta_j \right\}$$

E.g.: $z_1 \overline{z_2} \in \mathrm{Hom}_{\mathbb{F}^d}(1,1)$.

Note: $\mathrm{Hom}_{\mathbb{R}^d}(t,t)$ is just the space of homogeneous polynomials of degree $2t$.

Fact: Orthonormal bases for $\mathbb{F}^d$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ yield cubature rules for $\mathrm{Hom}_{\mathbb{F}^d}(1,1)$ over $S(\mathbb{F}^d)$.

Some Important Spaces, II

Let $(\mathbb{F}^d)^{\otimes t} = \operatorname{span}\{e_{j_1} \otimes \ldots \otimes e_{j_t} \,|\, j_1, \ldots, j_t \in [d]\}$, $\{e_j\}_{j=1}^d$ basis for $\mathbb{F}^d$.

The space of symmetric tensors of order t $\operatorname{Sym}^t(\mathbb{F}^d)$ is

$$\left\{ \sum_{j_1,\ldots,j_t=1}^{N} a_{j_1,\ldots,j_t} e_{j_1} \otimes \ldots \otimes e_{j_t} \,|\, a_{j_1,\ldots,j_t} = a_{j_{\pi(1)},\ldots,j_{\pi(t)}} \text{ for all } \pi \in S_t \right\},$$

with $\dim_{\mathbb{F}} \operatorname{Sym}^t(\mathbb{F}^d) = \begin{pmatrix} d+t-1 \\ t \end{pmatrix}$

Let $\Pi_d^{(t)}$ be the orthogonal projection $(\mathbb{F}^d)^{\otimes t} \to \operatorname{Sym}^t(\mathbb{F}^d)$.

E.g.: $x = 2e_1 \otimes e_2 + 4e_3 \otimes e_3 \in (\mathbb{F}^3)^{\otimes 2}$ and
$\Pi_3^{(2)}(x) = e_1 \otimes e_2 + e_2 \otimes e_1 + 4e_3 \otimes e_3 \in \operatorname{Sym}^2(\mathbb{F}^3)$

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$\Pi_d^{(1)} = I_{\mathbb{F}^d}$

Tight Frames

Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ and $\Phi = (\varphi_j)_{j=1}^n \in \mathbb{F}^{d \times n}$.

Φ is a tight frame if one of the following equivalent statements hold:

- There exists $A > 0$ s.t. for all $x \in \mathbb{F}^d$, $x = \frac{1}{A} \sum_{j=1}^n \langle x, \varphi_j \rangle \varphi_j$.

Duffin, Schaeffer 1952; Neumaier, Hoggar, Bannai 1980s; Benedetto, Fickus 2003; Waldron 2017

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- $\sum_{j=1}^n \sum_{k=1}^n \left|\langle \varphi_j, \varphi_k\rangle\right|^2 = \frac{1}{d}\left(\sum_{j=1}^n \left\|\varphi_j\right\|^2\right)^2$.

A is called the frame bound.

A tight frame is also known as a weighted projective 1-design.

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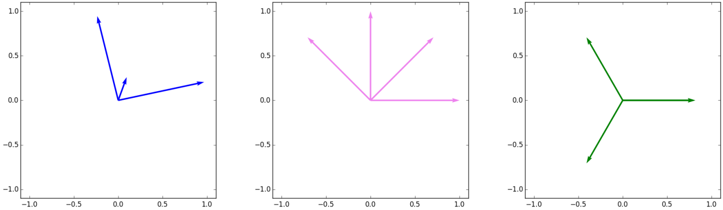
Tight Frame Examples



$$
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When the vectors are equal-length, a tight frame is a projective 1-design.

WHAT'S THE 't'?

Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, $\Phi = (\varphi_j)_{j=1}^n \in \mathbb{F}^{d\times n}$, $t \in \mathbb{N}$.

Further define:

$$c_t(d,\mathbb{C}) = \binom{d+t-1}{t}^{-1}, \quad c_t(d,\mathbb{R}) = \frac{1\cdot 3\cdot 5\cdots(2t-1)}{d(d+2)\cdots(d+2(t-1))}.$$

Consider the following statements:

1. $(\varphi_j)_{j=1}^n$ yields a cubature rule for $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$ over $S(\mathbb{F}^d)$.
2. $\sum_{j=1}^n \left(\varphi_j^{\otimes t}\right)\left(\varphi_j^{\otimes t}\right)^* = \left(c_t(d,\mathbb{F}) \sum_{j=1}^n \left\|\varphi_j\right\|^{2t}\right) \Pi_d^{(t)}$.
   (I.e., $\{\varphi_j^{\otimes t}\}_{j=1}^n$ is a tight frame for $\mathrm{Sym}^t(\mathbb{F}^d)$.)
3. $\sum_{j=1}^n \sum_{k=1}^n \left|\langle \varphi_j, \varphi_k\rangle\right|^{2t} = c_t(d,\mathbb{F}) \left(\sum_{j=1}^n \left\|\varphi_j\right\|^2\right)^2$.

Datta, Howard, Cochran 2012; Waldron 2017

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These statements are all equivalent when $t = 1$ or $\mathbb{F} = \mathbb{C}$.

When $t > 1$ and $\mathbb{F} = \mathbb{R}$, only 1. $\Leftrightarrow$ 3. holds. And 2. never holds.

Datta, Howard, Cochran 2012; Waldron 2017

Projective t-Designs

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, $\Phi = (\varphi_j)_{j=1}^n \in \mathbb{F}^{d \times n}$, $t \in \mathbb{N}$.

Φ is a weighted projective t-design if 1., or equivalently 3. (or equivalently 2. when $\mathbb{F} = \mathbb{C}$) holds.

If the vectors are equal norm, we call Φ a projective t-design.

Equiangular Projective 2-Designs

Let $\Phi = (\varphi_j)_{j=1}^n \subset S(\mathbb{C}^d)$.

Φ is equiangular if there exists α s.t. $\left|\langle \varphi_j, \varphi_k \rangle\right| = \alpha$ for all $j \neq k$.

If Φ is a projective 2-design, then $n \geq d^2$ with equality precisely when Φ is equiangular.

If Φ is equiangular, then $n \leq d^2$, with equality precisely when Φ is a projective 2-design.

If Φ is a weighted projective 2-design, then $n \geq d^2$ with equality only when Φ is a projective 2-design.

A projective 2-design with $n = d^2$ is a maximal equiangular tight frame (ETF) or a symmetric, informationally complete, positive operator-valued measure (SIC).

Gerzon; Bannai, Hoggar 1985; Scott 2006

$\mathcal{G}(3)$ is a Projective 2-Design

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$$\mathcal{G}(3) = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \zeta & \zeta^2 & 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta & 0 & 0 & 0 & 1 & \zeta^2 & \zeta \end{array} \right).$$

$\Rightarrow \mathcal{G}(3)\mathcal{G}(3)^* = \frac{9}{3} I_3,$

and $\left|\langle \varphi_j, \varphi_k \rangle\right| = 1/2$ for distinct $\varphi_j, \varphi_k \in \mathcal{G}(3)$.

There's also a neat way to generate another projective 2-design of 12 vectors in $\mathbb{C}^3$ called a maximal set of mutually unbiased bases (MUBs) from $\mathcal{G}(3)$[4]!

[4] Or rather the real $\mathcal{G}(3)$.

Open Problems

Zauner's (weak) conjecture:
For all d, there exists a projective 2-design in $\mathbb{C}^d$ of d^2 elements.

For any d not a prime power, does there exist a projective 2-design in $\mathbb{C}^d$ of $d(d+1)$-elements consisting of a union of orthonormal bases?

For any d not a prime power, does there exist a weighted projective 2-design in $\mathbb{C}^d$ of $d(d+1)$-elements consisting of a weighted union of orthonormal bases?

Zauner 1999; Klappenecker, Rötteler 2005; Roy, Scott 2007

Result 1

The smallest weighted projective 2-design for $\mathbb{C}^d$ has size equal to a type of rank of a certain quantum channel,

giving an alternate formulation of Zauner's weak conjecture.

Iverson, K, Mixon 2021; Pandey, Paulsen, Prakash, and Rahaman 2020; Paulsen CodEx Talk July 21, 2020

Entanglement Breaking Channels

A linear map $\mathfrak{X} : \mathbb{C}^{d\times d} \to \mathbb{C}^{m\times m}$ is entanglement breaking if it admits a entanglement breaking decomposition:

$$\mathfrak{X}(A) = \sum_{k=1}^{n} R_k A R_k^*, \quad \sum_{k=1}^{n} R_k^* R_k = I_d, \quad \text{rank}(R_k) = 1 \text{ for all } k \in [n]. \tag{1}$$

The entanglement breaking rank ebr of $\mathfrak{X}$, denoted by $\text{ebr}(\mathfrak{X})$ is the smallest n such that $\{R_k\}_{k=1}^{n} \subset \mathbb{C}^{m\times d}$ satisfying (1) exist.

The quantum depolarizing channel $\mathfrak{Z}_d : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$,

$$\mathfrak{Z}_d(A) = \frac{1}{d+1}\left(A + \text{tr}(A) I_d\right)$$

is entanglement breaking.

Watrous 2018; Pandey, Paulsen, Prakash, and Rahaman 2020; Paulsen CodEx Talk July 21, 2020

$\mathrm{ebr}(\mathfrak{Z}_d)$ and Zauner's Conjecture

$\mathrm{ebr}(\mathfrak{Z}_d) \geq d^2$ with equality if and only if there exists a SIC in $\mathbb{C}^d$.

$\mathrm{ebr}(\mathfrak{Z}_d) \leq d(d+1)$ whenever d is a prime power due to the existence of maximal MUBs.

ebr is lower semicontinuous, so if channels arbitrarily near to $\mathfrak{Z}_d$ have ebr bounded by d^2, then Zauner's (weak) conjecture is true.

Caves 1999; Pandey, Paulsen, Prakash, and Rahaman 2020; Paulsen CodEx Talk July 21, 2020

$\mathrm{ebr}(\mathfrak{Z}_d)$ AND WEIGHTED 2-DESIGNS?

Q: Wait, weren't d^2 and $d(d+1)$ (for d a prime power) the sizes of the complex projective 2-designs we cared about?

Could there be something deeper going on?

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A: Yes!

$\mathrm{ebr}(\mathfrak{Z}_d)$ AND WEIGHTED 2-DESIGNS

The smallest weighted projective 2-design for $\mathbb{C}^d$ has size $\mathrm{ebr}(\mathfrak{Z}_d)$.

This implies

$$\mathrm{ebr}(\mathfrak{Z}_d) \leq \begin{cases} kd^2 + 2d & kd + 1 \text{ a prime power, } k \in \mathbb{N} \\ d^2 + (p+1)d & d + 1 = p^k \text{ with } p \text{ prime and } k \in \mathbb{N} \\ d^2 + 1 & d - 1 \text{ prime power} \\ d^2 + d - 1 & d \text{ a prime power} \end{cases}$$

Iverson, K, Mixon 2021; Bodmann, Haas 2016; Roy, Scott 2007

RESULT 2

Each projective 2-design over the quaternions is associated with some "nice" configuration of $3d$ subspaces of some real Euclidean space.

Iverson, K, Mixon 2021

Hyper Quick Quaternion Review

The quaternions $\mathbb{H}$ are a 4-dim real vector space

$$\{a_0 + a_1\,\mathrm{i} + a_2\,\mathrm{j} + a_3\,\mathrm{k} \mid a_n \in \mathbb{R}\}$$

with an associative but non-commutative multiplication defined using

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{i}\,\mathrm{j}\,\mathrm{k} = -1$$

which has no zero-divisors.

Linear algebra over $\mathbb{H}$ is sometimes just like over $\mathbb{C}$, but sometimes it's $\mathbb{H}$airy.

Revisit: What's the 't'?

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, $\Phi = (\varphi_j)_{j=1}^n \in \mathbb{F}^{d\times n}$, $t \in \mathbb{N}$, $m = (\dim_{\mathbb{R}} \mathbb{F})/2$ and

$$c_t(d, \mathbb{F}) = \prod_{j=0}^{t-1} \frac{m+j}{d+j}.$$

Consider the following statements:

1. $(\varphi_j)_{j=1}^n$ yields a cubature rule for $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$ over $S(\mathbb{F}^d)$.
2. $\sum_{j=1}^n \left(\varphi_j^{\otimes t}\right)\left(\varphi_j^{\otimes t}\right)^* = \left(c_t(d,\mathbb{F}) \sum_{j=1}^n \left\|\varphi_j\right\|^{2t}\right) \Pi_d^{(t)}$.
3. $\sum_{j=1}^n \sum_{k=1}^n \left|\langle \varphi_j, \varphi_k \rangle\right|^{2t} = c_t(d,\mathbb{F}) \left(\sum_{j=1}^n \left\|\varphi_j\right\|^2\right)^2$.

Hoggar 1982 & 1984; Waldron 2020

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When $t > 1$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$, 1. $\Leftrightarrow$ 3. holds. And 2. never holds.

When $t = 2, d > 1, n \leq d(md - m + 1)$.

Hoggar 1982 & 1984; Waldron 2020

Complex Projective 2-Designs and Cousins

2. shows that projective 2-designs over $\mathbb{C}$ yield tight frames over the $\binom{d+1}{2} = c_2(d,\mathbb{C})^{-1}$-dimensional space $\mathrm{Sym}^2(\mathbb{C}^d)$.

Further, an equiangular complex projective 2-design (i.e., a SIC) yields an equiangular tight frame for $\mathrm{Sym}^2(\mathbb{C}^d)$ (a cousin ETF).

ETFs with those parameters are plentiful.

($\mathcal{G}(3)$ is both a SIC and has parameters of a cousin ETF. $\mathcal{G}(q)$ for other odd prime powers have parameters of cousin ETFs.)

Waldron 2017; Bodmann, K 2020; Iverson, Jasper, Mixon 2020

Quaternion Projective 2-Designs and Cousins?

Q: Do equiangular quaternion projective 2-designs yield some sort of cousins?

QUATERNION PROJECTIVE 2-DESIGNS AND COUSINS?

Q: Do equiangular quaternion projective 2-designs yield some sort of cousins?

A: Yes!

Quaternion Projective 2-Designs and Cousins

Note that $c_2(d, \mathbb{F})^{-1} = \frac{d(2d+1)}{3} \neq \dim_{\mathbb{H}} \mathrm{Sym}^2(\mathbb{H}^d)$.

So the symmetric tenors of quaternion projective 2-designs cannot form a tight frame for $\mathrm{Sym}^2(\mathbb{H}^d)$.

Iverson, K, Mixon 2021

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Every $x \in \mathbb{H}^d \setminus \{0\}$ yields a 3-dim $\mathbb{R}$ subspace of the anti-Hermitian operators $\mathfrak{u}(d, \mathbb{H})$ in $\mathbb{H}^{d \times d}$,

$$u(x) = \{xzx^* \mid z \in \mathbb{H},\ \ \mathrm{Re}(z) = 0\} \in \mathfrak{u}(d, \mathbb{H}).$$

Iverson, K, Mixon 2021

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If $(\varphi_j)_{j=1}^n$ is a quaternion projective 2-design, then $(u(\varphi_j))_{j=1}^n$ is a cousin tight fusion frame of 3-dimensional subspace of the $d(2d+1)$-dimensional $\mathfrak{u}(d, \mathbb{H})$.

Iverson, K, Mixon 2021

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If $(\varphi_j)_{j=1}^n$ is further equiangular, then $(u(\varphi_j))_{j=1}^n$ is a cousin equiisoclinic tight fusion frame.

Iverson, K, Mixon 2021

Result 3

One can, if careful, define projective 2-designs over finite fields.

Iverson, K, Mixon 2021

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One can, if careful, define projective 2-designs over finite fields.

A construction of an infinite class of such designs generalizes the construction of $\mathcal{G}(3)$.

Iverson, K, Mixon 2021

Why Finite Fields?

It's fun!

Greaves, Iverson, Jasper, Mixon 2020 x2

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The existence of complex ETFs (including SICs) implies the existence of ETFs over infinitely many finite fields.

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The existence of certain ETFs over finite fields implies the existence of certain real ETFs (new strongly regular graphs?)

Greaves, Iverson, Jasper, Mixon 2020 x2

WHY FINITE FIELDS?

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See the introductions to Greaves et al. for further ~~rationalization~~ motivation.

Greaves, Iverson, Jasper, Mixon 2020 x2

What's the Deal with Inner Products?

Let $\mathbb{F}$ be any field and σ any field automorphism such that $\sigma^2 = I$. E.g., $\mathbb{F} = \mathbb{F}_{q^2}$ for q a prime power, $\sigma(\alpha) = \alpha^q$.

Define for $x \in \mathbb{F}^d$, $x^* = \sigma(x)^\top$, where σ is applied component-wise, and
$\langle \cdot, \cdot \rangle : \mathbb{F}^d \times \mathbb{F}^d \to \mathbb{F}$, $\langle x, y \rangle = x^* y$.

$\langle \cdot, \cdot \rangle$ will in general not be positive definite!!!

See, e.g., Greaves, Iverson, Jasper, Mixon 2020 x2

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E.g., take $x = (1,1,1)^\top \in \mathbb{F}_3^3$.
$x \neq 0$, but $\langle x, x \rangle = 3 = 0$.

Define $V \leq \mathbb{F}^d$ to be non-degenerate if $V \cap V^\perp = \{0\}$, where

$$V^\perp = \{y \in \mathbb{F}^d \mid \langle y, x \rangle = 0 \text{ for all } x \in V\}.$$

See, e.g., Greaves, Iverson, Jasper, Mixon 2020 x2

Revisit Once More: What's the 't'?

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, $\Phi = (\varphi_j)_{j=1}^n \in \mathbb{F}^{d\times n}$, $t \in \mathbb{N}$. Recall the following statements:

1. $(\varphi_j)_{j=1}^n$ yields a cubature rule for $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$ over $S(\mathbb{F}^d)$.
2. $\sum_{j=1}^n \left(\varphi_j^{\otimes t}\right)\left(\varphi_j^{\otimes t}\right)^* = \left(c_t(d,\mathbb{F})\sum_{j=1}^n \left\|\varphi_j\right\|^{2t}\right)\Pi_d^{(t)}$.
   (I.e., $\{\varphi_j\}_{j=1}^n$ is a tight frame for $\mathrm{Sym}^t(\mathbb{F}^d)$.)
3. $\sum_{j=1}^n \sum_{k=1}^n \left|\langle \varphi_j, \varphi_k\rangle\right|^{2t} = c_t(d,\mathbb{F})\left(\sum_{j=1}^n \left\|\varphi_j\right\|^2\right)^2$.

Frame Theory over Finite Fields[5]

Let $V \leq \mathbb{F}_{q^2}^d$ be nondegenerate and $\Phi = (\varphi_j)_{j=1}^n \subset V$.

Channeling 2. for $t = 1$, we have:

Φ is a c-tight frame if

$$\operatorname{span}\{\varphi_j\}_{j=1}^n = V \quad \text{and} \quad \sum_{j=1}^n \varphi_j \varphi_j^* = cI_V.$$

A c-tight frame is an (a, b, c)-ETF if

$$\langle \varphi_j, \varphi_j \rangle = a \ \text{ for all } j \in [n] \text{ and}$$
$$\langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_j \rangle = b \ \text{ for all } j, k \in [n] \text{ with } j \neq k.$$

Greaves, Iverson, Jasper, Mixon 2020 x2; [5] Not quite coding theory.

Projective 2-Designs over Finite Fields[3]

Let $\Phi = (\varphi_j)_{j=1}^n \subset \mathbb{F}_{q^2}^d$ and q be an odd prime power.

Channeling 2. for $t = 1$ and $t = 2$, we have:

Φ is an (a, c_1, c_2)-projective 2-design if

- $\langle \varphi_j, \varphi_j \rangle = a$ for all $j \in [n]$;
- Φ is an c_1-tight frame for $\mathbb{F}_{q^2}^d$; and
- $(\varphi_j^{\otimes 2})_{j=1}^n$ is a c_2-tight frame for (the non-degenerate!) $\mathrm{Sym}^2(\mathbb{F}_{q^2}^d)$.

[3] Iverson, K, Mixon 2020

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Note: c_1 or c_2 could be zero!!

1. is junk over finite fields because integrating $\mathrm{Hom}_{\mathbb{F}_{q^2}^d}$ over $S(\mathbb{F}_{q^2}^d)$ always yields 0.

[3] Iverson, K, Mixon 2020

Equiangular Projective 2-Designs over $\mathbb{F}_{q^2}^d$?

Q: Are "small" enough projective 2-designs equiangular like in fields of characteristic 0?

EQUIANGULAR PROJECTIVE 2-DESIGNS OVER $\mathbb{F}_{q^2}^d$?

Q: Are "small" enough projective 2-designs equiangular like in fields of characteristic 0?

A: Yes!

Equiangular Projective 2-Designs over $\mathbb{F}_{q^2}^d$

Let q be an odd prime power and $\Phi = (\varphi_j)_{j=1}^{n} \subset \mathbb{F}_{q^2}^d$

Any two of the following statements imply the third:

1. Φ is a projective 2-design.
2. $n = d^2$.
3. There exist $a, b, c_1 \in \mathbb{F}_q$ s.t.
   3.1 (Some constraints on a, b, c_1), and
   3.2 Φ is an (a, b, c_1)-equiangular tight frame.

Iverson, K, Mixon 2020

Q: What about the promised construction?

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A: It’s coming right now, but we need to return to $\mathbb{C}$ for a moment.

Difference Sets

Let G be a finite abelian group of size v and $D \subseteq G$ a subset of size k. Finally consider the multiset $\Delta = \{d_i - d_j : d_i, d_j \in D, d_i \neq d_j\}$.

D is a (v, k, λ)-difference set if Δ contains each element of $G \cap \{0\}^c$ λ times.

$\{1, 2\}$ is a $(3, 2, 1)$-difference set in $\mathbb{Z}_3$.

$-$	1	2
1	0	1
2	2	0

$\{0, 1, 3\}$ is a $(7, 3, 1)$-difference set in $\mathbb{Z}_7$.

$-$	0	1	3
0	0	1	3
1	6	0	2
3	4	5	0

Note that any shift of a difference set is a difference set, e.g, $\{0, 1\}$ is a $(3, 2, 1)$-difference set.

Character Tables

Let G be a finite, abelian group, and let $\widehat{G}$ be the group of homomorphisms $G \to S^1 \subseteq \mathbb{C}$.

The character table of G is the (square) matrix with rows labeled by $g \in G$, columns labeled by $\chi \in \widehat{G}$, and entries $\chi(g)$.

Examples:

- $G = \mathbb{Z}_n$: $n \times n$ DFT (properly scaled)
- $G = \oplus_{r=1}^{R} \mathbb{Z}_2$: $2^r \times 2^r$ Sylvester Hadamard [r-fold Kronecker product of $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$]

Character tables are scaled unitary matrices, i.e., have rows which are equal norm and orthogonal.

ETFs from Difference Sets

Theorem

Form Φ by removing the rows indexed by $D \subset G$ of size k from the character table of finite abelian G and scaling each entry by $1/\sqrt{k}$. Then the columns of Φ form an ETF if and only if D is a difference set.

$\zeta = e^{2\pi i/3}$

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix}$$

simplex ETF

$\zeta = e^{2\pi i/7}$

$$\Phi = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 & \zeta^5 & \zeta^6 \\ 1 & \zeta^3 & \zeta^6 & \zeta^2 & \zeta^5 & \zeta & \zeta^4 \end{pmatrix}$$

Ding, Feng 2007; Strohmer, Heath 2003; Xia, Zhou, Giannakis 2005; Godsil, Roy 2009

Gabor-Weyl-Heisenberg Frames

For $d \in \mathbb{N}$, $d > 1$, and $\zeta \in \mathbb{C}$ a primitive dth root of unity, define $T_d, M_d \in \mathcal{U}(d)$ by

$$T_d = \operatorname{circ}(0, 1, 0, \ldots, 0), \quad M_d = \operatorname{diag}(1, \zeta, \ldots, \zeta^{d-1}).$$

Let $\psi \in \mathbb{C}^d$. If the orbit

$$\{M_d^\kappa T_d^k \psi : k, \kappa \in \{0, \ldots, d-1\}\}$$

is an ETF, then it is a Gabor-Weyl-Heisenberg ETF.

Strong Zauner's Conjecture: For every $d > 1$, there exists a Gabor-Weyl-Heisenberg ETF in $\mathbb{C}^d$.

Zauner 1999

$\mathcal{G}(3)$ AS GABOR-WEYL-HEISENBERG ETF

Let ζ be a primitive 3rd root of unity and $\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^\top$.

$$\mathcal{G}(3) = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \zeta & \zeta^2 & 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta & 0 & 0 & 0 & 1 & \zeta^2 & \zeta \end{array} \right)$$
$$= \left(M_3^0 T_3^0 \psi \ M_3^1 T_3^0 \psi \ M_3^2 T_3^0 \psi \middle| M_3^0 T_3^1 \psi \ M_3^1 T_3^1 \psi \ M_3^2 T_3^1 \psi \middle| M_3^0 T_3^2 \psi \ M_3^1 T_3^2 \psi \ M_3^2 T_3^2 \psi \right)$$

Bojarovska, Paternostro 2015; Haas, Cahill, Tremain, Casazza 2017

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$$= \left(M_3^0T_3^0\psi \;\; M_3^1T_3^0\psi \;\; M_3^2T_3^0\psi \,\middle|\, M_3^0T_3^1\psi \;\; M_3^1T_3^1\psi \;\; M_3^2T_3^1\psi \,\middle|\, M_3^0T_3^2\psi \;\; M_3^1T_3^2\psi \;\; M_3^2T_3^2\psi \right)$$

If D is a (d, k, λ)-difference set with $k < d-1$ and $\mathbb{1}_D$ the normalized indicator vector of D.

$\{M_d^\kappa T_d^k \mathbb{1}_D k, \kappa \in \{0, \ldots, d-1\}\}$ is a tight frame for $\mathbb{C}^d$ which is biangular not equiangular.

Bojarovska, Paternostro 2015; Haas, Cahill, Tremain, Casazza 2017

Gabor-Weyl-Heisenberg ETFs Over Finite Fields

For appropriate q and d, let D be the so-called Singer difference set in $\mathbb{Z}/d\mathbb{Z}$.

For primitive element $\alpha \in \mathbb{F}_{q^2}^{\times}$, set $\omega = \alpha^{(q^2-1)/d}$, and define

$$T_d = \operatorname{circ}(0, 1, 0, \ldots, 0), \quad M_d = \operatorname{diag}(1, \omega, \ldots, \omega^{d-1}).$$

Then $\{M_d^{\kappa} T_d^{k} \mathbb{1}_D : k, \kappa \in \{0, \ldots, d-1\}\}$ is an equiangular projective 2-design.

Iverson, K, Mixon 2020

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FURTHER QUESTIONS

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- Does the mapping $u : \mathbb{H}^d \to \mathfrak{u}(d, \mathbb{H})$ yield particularly nice subspace configurations when applied to maximal MUBs/Kerdock codes (Kantor 1995)?

Thanks for your attention!

I look forward to the discussion.

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`https://www.math.colostate.edu/~king/`