Upgraded free independence phenomena for independent unitaries

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Motivation: Asymptotic freeness

Voiculescu [2] showed that if the random matrices $X_1^{(n)}, X_2^{(n)}, \dots$ are independent, then the non-commutative moments $\mathrm{tr}(X_{i_1}^{(n)} \dots X_{i_k}^{(n)})$ can be described in the large-n limit by **free independence / freeness**.

To give an example without wading deeply into definitions

Example

Consider $U_1^{(n)}, U_2^{(N)}, \dots$ independent random unitary matrices, chosen uniformly according to the Haar measure on $\mathbb{U}_n$. Then

$$\lim_{n \to \infty} \mathrm{tr}((U_{i_1}^{(n)})^{\pm 1} \dots (U_{i_k}^{(n)})^{\pm 1})$$

is 1 if the word $g_{i_1}^{\pm 1} \dots g_{i_k}^{\pm 1}$ is the identity in the free group $F_\infty = \langle g_1, g_2, \dots \rangle$, and is almost surely 0 otherwise.

Motivation: Haar unitaries

What is a Haar unitary?

The unitary group $\mathbb{U}_n = \{U \in M_n(\mathbb{C}) : U^* U = I\}$ has a unique shift-invariant probability measure, called the *Haar measure*.

A *Haar random unitary* is a random variable U with values in $\mathbb{U}_n$, such that for any Borel set S, $\mathbb{P}(U \in S) = \text{Haar}(S)$.

Motivation: Haar unitaries

The distribution of eigenvalues of $U^{(n)}$ is asymptotically uniform on the circle:

Let $U^{(n)}$ be a Haar random unitary with (random) eigenvalues $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$. For $k \in \mathbb{Z}$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} (\lambda_j^{(n)})^k = \delta_{k=0} = \int_0^1 (e^{2\pi i t})^k \, dt.$$

The same holds when x^k is replaced by $f(x)$ for a continuous function on the circle.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} f(\lambda_j^{(n)}) = \lim_{n \to \infty} \operatorname{tr}_n(f(U^{(n)})) = \int_0^1 f(e^{2\pi i t}) \, dt.$$

Motivation II

Random matrix theory, and especially Voiculescu's theory of asymptotic freeness, has several applications:

1. Random initialization of neural networks.
2. Generating quantum expanders (Hastings).
3. Random inputs to Sherrington–Kirkpatrick model in spin glass theory.
4. Numerical computation of ranks of matrices over the free skew field. (Mai, Speicher, ...)
5. Analytical properties of von Neumann algebras.

First main result

Our first main result is motivated by a theorem of Houdayer and Ioana for free products of von Neumann algebras [3].

Theorem (Asymptotic freeness of approximate commutants)

Let $U_1^{(n)}$, $U_2^{(n)}$, ... be independent $n \times n$ Haar random unitary matrices. Let $B_1^{(n)}$, $B_2^{(n)}$, ... be random matrices on the same probability space such that $\|B_j^{(n)}\| \leq 1$ and $\lim_{n\to\infty} \|[U_j^{(n)}, B_j^{(n)}]\|_2 = 0$ almost surely. Then $B_1^{(n)}$, $B_2^{(n)}$, ... are almost surely asymptotically free (see below for definition).

Here $[A, B] = AB - BA$ is the commutator, and $\|A\|_2 = \mathrm{tr}_n(A^* A)^{1/2}$ is the normalized Hilbert-Schmidt norm.

Interpretation

Note we don't assume any sort of probabilistic independence for the $B_j^{(n)}$'s.

This result says that if $B_j^{(n)}$ approximately commutes with $U_j^{(n)}$, then the $B_j^{(n)}$'s must be approximately free. This is an extension or upgrade of the free independence of the $U_j^{(n)}$'s themselves.

From a logical viewpoint, the theorem upgrades asymptotic freeness to include some statements with quantifiers:

For all $B_j^{(n)}$ such that $B_j^{(n)}$ commutes with $U_j^{(n)}$ up to a δ error, the $B_j^{(n)}$'s are approximately freely independent up to degree k and error ε.

Extension to commutation chains

We can also generalize the result as follows:

Let $U_j^{(n)}$ be independent Haar random unitaries. Suppose that $V_{j,k}^{(n)}$ for $k \in \mathbb{N}$ are unitary random matrices. Suppose that

$$\lim_{n \to \infty} \|[U_j^{(n)}, V_{j,1}^{(n)}]\|_2 = 0$$

and

$$\lim_{n \to \infty} \|[V_{j,k}^{(n)}, V_{j,k_1}^{(n)}]\|_2 = 0,$$

and

$$\lim_{n \to \infty} \operatorname{tr}_n(f(V_{j,k}^{(n)})) = \int_0^1 f(e^{2\pi i t})\, dt \text{ for } f \in C(S^1).$$

Then $(V_{j,K}^{(n)})_{j \in \mathbb{N}}$ are asymptotically freely independent.

Non-commutative probability spaces

Definition

A **non-commutative probability space** is a $*$-algebra A ($+$, $\times$, and $*$ / adjoint) together with a map $\tau : A \to \mathbb{C}$ satisfying certain properties:

- $\tau(1) = 1$
- $\tau(ab) = \tau(ba)$ for all $a, b \in A$.
- $\tau(a^* a) \geq 0$.
- $\tau(a^* a) = 0$ implies $a = 0$.

Motivating examples:

- $A = M_n(\mathbb{C})$, $\tau = \mathrm{tr}_n = (1/n)\,\mathrm{Tr}_n$.
- $A = L^\infty[0,1]$, $\tau = \int \cdot \, dx$.

The elements of A represent "bounded random variables" that don't commute under multiplication.

Non-commutative probability spaces

GNS construction: Just like classical probability spaces, one can define an inner product

$$\langle a, b \rangle_\tau = \tau(a^* b)$$

and complete (A, τ) to a Hilbert space $L^2(A, \tau)$.

The algebra A acts on $L^2(A, \tau)$ by left and right multiplication, just like $L^\infty(\Omega, P)$ acts on $L^2(\Omega, P)$ by multiplication, so we can view $A \subseteq B(L^2(A, \tau))$.

Non-commutative probability spaces

(A, τ) is a *tracial von Neumann algebra* if A is closed in weak operator topology inside $B(L^2(A, \tau))$.

This is analogous to the fact that L^∞ of a measure space has a weak-$*$ topology as the dual of the L^1.

Group von Neumann algebras:

- G discrete group.
- $\ell^2(G)$ the square-summable functions on G.
- u_g the left shift operator on $\ell^2(G)$, which is unitary.
- $L(G)$ the von Neumann algebra generated by $(u_g)_{g \in G}$. This contains the group algebra $\mathbb{C}G$ as a WOT dense subset.

Example: $L(\mathbb{Z}) \cong L^\infty(S^1)$ using Fourier transform.

Free independence

Definition

Consider a non-commutative probability space (A, τ) and $*$-subalgebras $(A_i)_{i \in I}$. We say $(A_i)_{i \in I}$ are freely independent if whenever $i_1 \neq i_2 \neq \ldots \neq i_k$ and $a_j \in A_{i_j}$, then

$$\tau\left[(a_1 - \tau(a_1)) \ldots (a_k - \tau(a_k))\right] = 0.$$

Similarly, random variables x_1, x_2, ... are said to be freely independent if the $*$-algebras generated by them are freely independent.

Example: $A = L(F_\infty)$, $\tau(\sum_g a_g g) = a_e$, and A_i is the algebra generated by the g_i, the ith generator of F_∞.

Asymptotic free independence

Definition

Let $X_1^{(n)}$, $X_2^{(n)}$, ... be random matrices and assume almost surely $\limsup_{n\to\infty} \|X_i^{(n)}\| < \infty$ (operator norm). We say the $X_i^{(n)}$ are almost surely asymptotically ($*$-) freely independent if whenever $i_1 \neq i_2 \neq \dots \neq i_k$ and p_j is a polynomial in (x, x^*), then

$$\lim_{n\to\infty} \operatorname{tr}_n \left[(p_1(X_{i_1}^{(n)}) - \operatorname{tr}_n[p_1(X_{i_1}^{(n)})]) \dots (p_k(X_{i_k}^{(n)}) - \operatorname{tr}_n[p_k(X_{i_k}^{(n)})]) \right] = 0.$$

There are of course versions with "almost surely" replaced by "in probability" or "in expectation."

Example: Voiculescu showed that independent Haar unitaries $U_1^{(n)}$, $U_2^{(n)}$, ... are almost surely asymptotically freely independent. Same for GUE and GOE matrices.

Tools

Voiculescu's result, as well as our result, uses the following ingredients:

- Compute the expectation of the trace polynomial and show it goes to zero.
- Use high-dimensional concentration of measure to show that the sample value is close to the expectation with high probability.

For the concentration argument, we want the test functions to be Lipschitz, which is why the operator norm cut-off is a convenient assumption.

In addition our result will use an ϵ-net argument to cover all possible choices of $B_j^{(n)}$. The number of elements in the ϵ-net will still be small enough to be eaten by the concentration of measure bounds.

Diagonalization

To simplify the analysis, we swap out the Haar unitary $U_j^{(n)}$ for

$$V_j^{(n)} A^{(n)} (V_j^{(n)})^*,$$

where $V_j^{(n)}$ is a Haar unitary and

$$A^{(n)} = \operatorname{diag}(1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}), \qquad \omega_n = e^{2\pi i/n}.$$

It is an exercise to show that there is a coupling of $U_j^{(n)}$ and $V_j^{(n)}$ such that $\|U_j^{(n)} - V_j^{(n)} A^{(n)} (V_j^{(n)})^*\| \to 0$ almost surely.

Hence, it suffices to prove the result with $V_j^{(n)} A^{(n)} (V_j^{(n)})^*$ instead of $U_j^{(n)}$.

Understanding the approximate commutant

We assumed that $B_j^{(n)}$ asymptotically commutes with $U_j^{(n)}$. Write

$$B_j^{(n)} = V_j^{(n)} C_j^{(n)} (V_j^{(n)})^*,$$

so that

$$\|[A^{(n)}, C_j^{(n)}]\|_2 \to 0.$$

By entrywise computation

$$\begin{aligned} \|[A^{(n)}, C_j^{(n)}]\|_2^2 &= \frac{1}{n} \sum_{k,\ell=1}^{n} |(C_j^{(n)})_{k,\ell}|^2 |\omega_n^k - \omega_n^\ell|^2 \\ &\approx \frac{1}{n} \sum_{k,\ell=1}^{n} |(C_j^{(n)})_{k,\ell}|^2 \left| \frac{k}{n} - \frac{\ell}{n} \right|^2. \end{aligned}$$

Understanding the approximate commutant

Thus, $C_j^{(n)}$ commuting with $A^{(n)}$ means that $C_j^{(n)}$ is asymptotically supported on ϵ-bands around the diagonal. Let

$$\mathcal{D}_\epsilon^{(n)} = \{C \in M_n(\mathbb{C}) : C_{k,\ell} = 0 \text{ when } d_n(k,\ell) > n\epsilon\},$$

where distance is considered mod n.

Lemma

Let $A^{(n)}$ be as above and $\sup_n \|C^{(n)}\| < \infty$. We have $\|[A^{(n)}, C^{(n)}]\|_2 \to 0$ if and only if $d(C^{(n)}, \mathcal{D}_\epsilon^{(n)}) \to 0$ for every ϵ.

Technical note: We can also find a point in $\mathcal{D}_\epsilon^{(n)}$ close to $C^{(n)}$ with the operator norm bounded by $3\|C^{(n)}\|$.

Asymptotic free independence in expectation

Based on Voiculescu's work, we have the following:

Theorem

Let $V_j^{(n)}$ be independent Haar unitaries. Let $i_1 \neq i_2 \neq \ldots \neq i_k$. Then

$$\sup_{C_1^{(n)},\dots,C_k^{(n)} \in B_1^{M_n(\mathbb{C})}} \Big| \mathbb{E} \operatorname{tr}_n \Big[V_{i_1}^{(n)} (C_1^{(n)} - \operatorname{tr}_n(C_1^{(n)})) (V_{i_1}^{(n)})^* \dots$$

$$V_{i_k}^{(n)} (C_1^{(n)} - \operatorname{tr}_n(C_k^{(n)})) (V_{i_k}^{(n)})^* \Big] \Big| \to 0,$$

where $C_j^{(n)}$ ranges over the unit ball $B_j^{M_n(\mathbb{C})}$ in operator norm.

The free independence in expectation is *uniform* over the choice of $C_j^{(n)}$'s.

Concentration of measure

Lemma (Herbst-type concentration for unitaries)

Let $U_1^{(n)}, \dots, U_k^{(n)}$ be independent Haar random unitaries. Let $f : \mathbb{U}_n^k \to \mathbb{R}$ be Lipschitz with respect to $\|\cdot\|_2$ in the domain. Then

$$P(|f(X) - \mathbb{E}f(X)| \geq \delta) \leq 2\exp(-n^2\delta^2/12\|f\|_{\mathrm{Lip}}^2)$$

See E. Meckes' book [4]. Cf. isoperimetric inequality (Gromov, Milman), log-Sobolev inequality (Gross), Bakry-Émery criterion, Ledoux, Talagrand.

Covering numbers

For a set S in a metric space, the covering number $K_\epsilon(S)$ is the smallest cardinality of a set Ω such that the ϵ-balls centered at points of Ω cover S.

Lemma

Let $S_j^{(n)} \subseteq B_1^{M_n(\mathbb{C})}$. Suppose that

$$\limsup_{n \to \infty} \frac{1}{n^2} \log K_\epsilon(S_j^{(n)}) < \delta.$$

Let $i_1 \neq i_2 \neq \ldots \neq i_k$. Then almost surely

$$\limsup_{n \to \infty} \sup_{X_1 \in S_1^{(n)}} \ldots \sup_{X_k \in S_k^{(n)}} \left| \mathrm{tr}_n \left[V_{i_1}^{(n)} (X_1 - \mathrm{tr}_n(X_1))(V_{i_1}^{(n)})^* \ldots V_{i_k}^{(n)} (X_k - \mathrm{tr}_n(X_k))(V_{i_k}^{(n)})^* \right] \right| \leq 4k\epsilon + 2k\sqrt{12k\delta}.$$

Covering numbers

Sketch of proof: Note the trace that we want to estimate is a const $\cdot k$-Lipschitz function of the unitaries and of the X_j's. And for each choice of X_j, its expectation vanishes as $n \to \infty$.

For sufficiently large n, pick ϵ-dense subsets of $S_j^{(n)}$ with $\exp(n^2\delta)$ many elements. So the supremum over all choices of X_j can bounded by const $k\epsilon$ plus the maximum over X_j's in the ϵ-dense subsets.

For each such choice of X_j, estimate the probability (w.r.t. the $V_j^{(n)}$'s) of the trace deviating from its expectation using the concentration inequality with error parameter const $\cdot k\sqrt{k\delta}$. Then take a union bound over the choices of X_j's from the ϵ-dense subset.

Covering numbers

Summary:

- Number of choices of X_j in ϵ-net: $\exp(C_1 k n^2 \delta)$.
- Probability of error $> k\sqrt{k\delta}$ for each choice: $\exp(-Cn^2 k^3 \delta / k^2) = \exp(-C_2 n^2 k \delta)$.
- Total probability of error: $\exp(kn^2(C_1\delta - C_2\delta))$.

We choose the parameters so $C_2 > C_1$, and so the total probability of error will then be summable in n, and we can apply the Borel–Cantelli lemma.

Finally, we estimate the error for an arbitrary element from $S_j^{(n)}$ by the error from one in the ϵ-net plus $O(k\epsilon)$.

Covering numbers of diagonal bands

Lemma

Let $0 < \epsilon < R$. Then

$$\frac{1}{n^2} \log K_\epsilon(\{X \in \mathcal{D}_\epsilon^{(n)} : \|X\|_2 \leq R\}) \leq \text{const}\, \epsilon \log(\text{const}\, R/\epsilon).$$

This follows from standard estimates for the covering numbers of balls in Euclidean space and the fact that $\dim_{\mathbb{R}} \mathcal{D}_\epsilon^{(n)} \leq 2\epsilon n^2$.

Conclusion of proof

Sketch: Fix $*$-polynomials p_j and $i_1 \neq \ldots \neq i_k$. First fix ϵ. Since $(V_j^{(n)})^* p_j(B_{i_j}^{(n)}) V_j^{(n)}$ asymptotically commutes with $A^{(n)}$, we can modify it to $C_j^{(n)} \in \mathcal{D}_\epsilon^{(n)}$ where the error goes to zero as $n \to \infty$. Hence, $p_j(B_{i_j}^{(n)})$ is approximated by $V_j^{(n)} C_j^{(n)} (V_j^{(n)})^*$.

Almost surely

$$\limsup_{n \to \infty} \Big| \operatorname{tr}_n \Big[V_{i_1}^{(n)} (C_1^{(n)} - \operatorname{tr}_n(C_1^{(n)})) (V_{i_1}^{(n)})^* \ldots \\ V_{i_k}^{(n)} (C_k^{(n)} - \operatorname{tr}_n(C_k^{(n)})) (V_{i_k}^{(n)})^* \Big] \Big| \leq \text{const}\, k\epsilon + \text{const}\, k \sqrt{k\delta}$$

with $\delta = \text{const}\, \epsilon \log(R/\epsilon)$. Since ϵ was arbitrary, and since $i_1 \neq \ldots \neq i_k$ were arbitrary, we are done.

More general results

The commutants of $U_j^{(n)}$ are just one example of sets of $n \times n$ matrices that have small dimension, but the proof above would apply to anything where the covering numbers are subexponential in n^2.

The theory of 1-bounded entropy (Jung [5], Hayes [6]) gives several situations where the structural properties of the von Neumann algebra imply subexponential covering numbers for certain matrix approximations. (e.g. property Gamma, Cartan subalgebras, compare Voiculescu [7], Ge [8]).

1-bounded entropy $h(\mathcal{N} : \mathcal{M})$ is defined for any tracial von Neumann algebras $\mathcal{N} \subseteq \mathcal{M}$, and it is computed by looking at covering numbers of the space of matrix appproximations for a generating set of $\mathcal{N}$ that extend to matrix approximations for generators of $\mathcal{M}$.

More general results

If $h(\mathcal{N} : \mathcal{M}) = 0$, then there is a unique maximal $\mathcal{P}$ containing $\mathcal{N}$ with $h(\mathcal{P} : \mathcal{M}) = 0$, called the *Pinsker algebra* of $\mathcal{N}$ [9].

Analogy with dynamical systems:

concept	free version	dynamical version
finitary approximations	matrices with approximately correct moments	approximate orbits with correct behavior
entropy	1-bounded	dynamical/sofic
subobject	v.N. subalgebra	factor
free object	free product	Bernoulli action

Ultraproduct

Ultraproducts are a way to construct limiting objects for a sequence of tracial von Neumann algebra $\mathcal{M}_n$ (e.g. $M_n(\mathbb{C})$).

Take $\mathcal{U}$ an ultrafilter, equivalently $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$, so $\lim_{n\to\mathcal{U}} a_n$ exists for any bounded sequence. *Think of $\mathcal{U}$ as an object that allows you to take limits akin to passing to a subsequence, but maximally refined so that it works for any input sequence.*

Given $(\mathcal{M}_n, \tau_n)$, let

$$\prod_{n\to\mathcal{U}} \mathcal{M}_n = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathcal{M}_n, \sup_n \|x_n\| < \infty\} / \{(x_n)_{n\in\mathbb{N}} : \lim_{n\to\mathcal{U}} \|x_n\|_2 = 0\},$$

and

$$\tau_{\mathcal{U}}([x_n]_{n\in\mathbb{N}}) = \lim_{n\to\mathcal{U}} \tau_n(x_n).$$

Freeness in matrix ultraproduct

Theorem

Let $\mathcal{Q} = \prod_{n \to \mathcal{U}} M_n(\mathbb{C})$. Let $U_j^{(n)}$ be independent Haar random unitaries. Almost surely, the following statement holds: Let $u_j = [U_j^{(n)}]_{n \in \mathbb{N}} \in \mathcal{Q}$. Let $\mathcal{P}_j$ be the Pinsker algebra of u_j. Then the $\mathcal{P}_j$'s are freely independent.

Example: If $\mathcal{A}_j$ is any maximal amenable subalgebra containing u_j, then the $\mathcal{A}_j$'s are freely independent.

Remark: The proof is actually subtler than indicated up to this point of the talk, since the $\mathcal{P}_j$'s are not separable but in probability theory you can only intersect countably many events. This issue is solved by recasting the result in terms of *types* and using separability of the space of *formulas*.

Application to free products

Theorem

Let $(\mathcal{M}_i)_{i \in I}$ be separable von Neumann algebras that are Connes embeddable (i.e. they admit matrix approximations). Let $\mathcal{M} = *_{i \in I} \mathcal{M}_i$. Let $\mathcal{A}_i \subseteq \mathcal{M}^{\mathcal{V}}$ with $h(\mathcal{A}_i : \mathcal{M}^{\mathcal{V}}) = 0$ and $\mathcal{A}_i \cap \mathcal{M}_i$ diffuse. Then the $\mathcal{A}_i$'s are freely independent.

Application to free products

Proof sketch: We can reduce to the case that I is countable and $\mathcal{M}_i$ and $\mathcal{A}_i$ are contained in a separable $\mathcal{N} \subseteq \mathcal{M}^{\mathcal{V}}$ with $h(\mathcal{A}_i : \mathcal{N}) = 0$.

Construct an embedding $*_{i \in I} \mathcal{M}_i$ into $\mathcal{Q}$ through taking deterministic matrix approximations for $\mathcal{M}_i$ and conjugating them by the Haar random unitary $V_i^{(n)}$. This extends to an embedding of $\mathcal{N}$ into $\mathcal{Q}$ since $\mathcal{N}$ is separable and $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}^{\mathcal{V}}$.

Then $h(\mathcal{A}_i : \mathcal{Q}) \leq h(\mathcal{A}_i : \mathcal{N}) = 0$, and the construction of the embedding is arranged so that $\mathcal{A}_i \cap \mathcal{M}_i$ contains the element u_i from the earlier result. So the previous theorem implies free independence of the $\mathcal{A}_i$'s.

Application to free products

Theorem

Let $(\mathcal{M}_i)_{i \in I}$ be separable von Neumann algebras that are Connes embeddable (i.e. they admit some matrix approximations). Let $\mathcal{M} = *_{i \in I} \mathcal{M}_i$. Let $\mathcal{A}_i \subseteq \mathcal{M}^{\mathcal{V}}$ with $h(\mathcal{A}_i : \mathcal{M}^{\mathcal{V}}) = 0$ and $\mathcal{A}_i \cap \mathcal{M}_i$ diffuse. Then the $\mathcal{A}_i \vee \mathcal{M}_i$'s are freely independent.

Proof: Let $\mathcal{N}_i = \mathcal{M}_i \otimes \mathcal{R}$, so that $h(\mathcal{N}_i) = 0$. Let $\mathcal{N} = *_{i \in I} \mathcal{N}_i$. Since $\mathcal{A}_i \cap \mathcal{M}_i$ is diffuse, so is $\mathcal{A}_i \cap \mathcal{N}_i$. Thus, $\mathcal{B}_i = \mathcal{A}_i \vee \mathcal{N}_i$ has $h = 0$ as well. Now apply the previous theorem to the $\mathcal{N}_i$'s and $\mathcal{B}_i$'s.

Recovering cases of Houdayer–Ioana

Corollary

Let $\mathcal{M}_i$ be diffuse and Connes embeddable and let $\mathcal{M} = *_{i \in I} \mathcal{M}_i$. Let $\mathcal{A}_i = \mathcal{M}_i' \cap \mathcal{M}^{\mathcal{V}}$. Then the $\mathcal{A}_i$'s are freely independent.

Proof: Fix $\mathcal{C}_i \subseteq \mathcal{M}_i$ diffuse and commutative. Let $\mathcal{B}_i = \mathcal{C}_i' \cap \mathcal{M}^{\mathcal{V}} \supseteq \mathcal{A}_i \vee \mathcal{C}_i$. Then $h(\mathcal{B}_i) = 0$ because of general properties of h and also $\mathcal{B}_i \cap \mathcal{M}_i \supseteq \mathcal{C}_i$ is diffuse. Thus, by the previous result, the $\mathcal{B}_i \vee \mathcal{M}_i$'s are freely independent.

Recovering cases of Hayes–J.–Nelson–Sinclair

Corollary

Let $\mathcal{M} = \mathcal{M}_1 * \mathcal{M}_2$ with $\mathcal{M}_i$ Connes embeddable. Let $\mathcal{A} \subseteq \mathcal{M}$ with $h(\mathcal{A} : \mathcal{M}) = 0$ and $\mathcal{A} \cap \mathcal{M}_1$ diffuse. Then $\mathcal{A} \subseteq \mathcal{M}_1$.

Proof: Take some diffuse commutative $\mathcal{B} \subseteq \mathcal{M}_2$. By the previous result, $\mathcal{A} \vee \mathcal{M}_1$ and $\mathcal{M}_2 \vee \mathcal{B} = \mathcal{M}_2$ are freely independent. Since $\mathcal{M}_1 \subseteq \mathcal{A} \vee \mathcal{M}_1$ and $\mathcal{M} = \mathcal{M}_1 * \mathcal{M}_2$, this forces that $\mathcal{A} \vee \mathcal{M}_1 = \mathcal{M}_1$.

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