Perfect Sequence Covering Arrays

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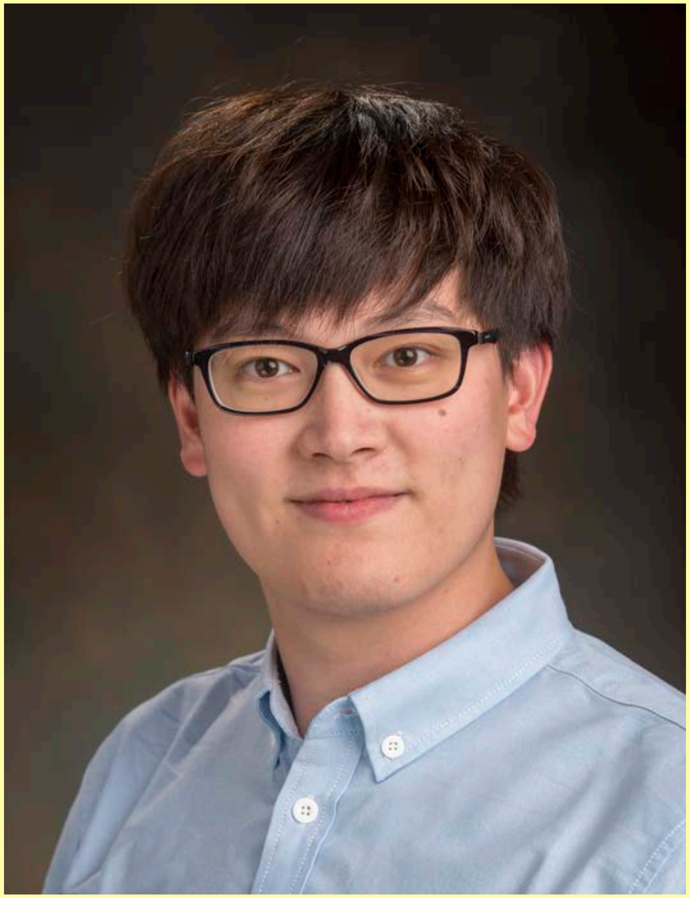
Joint work with Jingzhou Na and Shuxing Li

Joint work with Jingzhou Na and Shuxing Li

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Outline

- Perfect sequence covering arrays
- Central questions and motivation
- Combinatorial nonexistence
- Combinatorial constructions and bounds
- Yuster construction
- Reinterpretation
- New results
- Open problems

Perfect Sequence Covering Arrays

12345	43215	35214	14523	25413	53412
13254	52314	24315	15432	34512	42513

multiset of 5-sequences that covers
each ordered 3-subsequence exactly 2 times

PSCA(5, 3, 2)

The number of sequences in a PSCA(n, k, λ) is $k!\lambda$

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Central Questions

- What is the smallest value of λ for which a PSCA(n, k, λ) exists? Write this as $g(n,k)$

- How can examples attaining $\lambda = g(n,k)$ be constructed? How can we search for them efficiently?

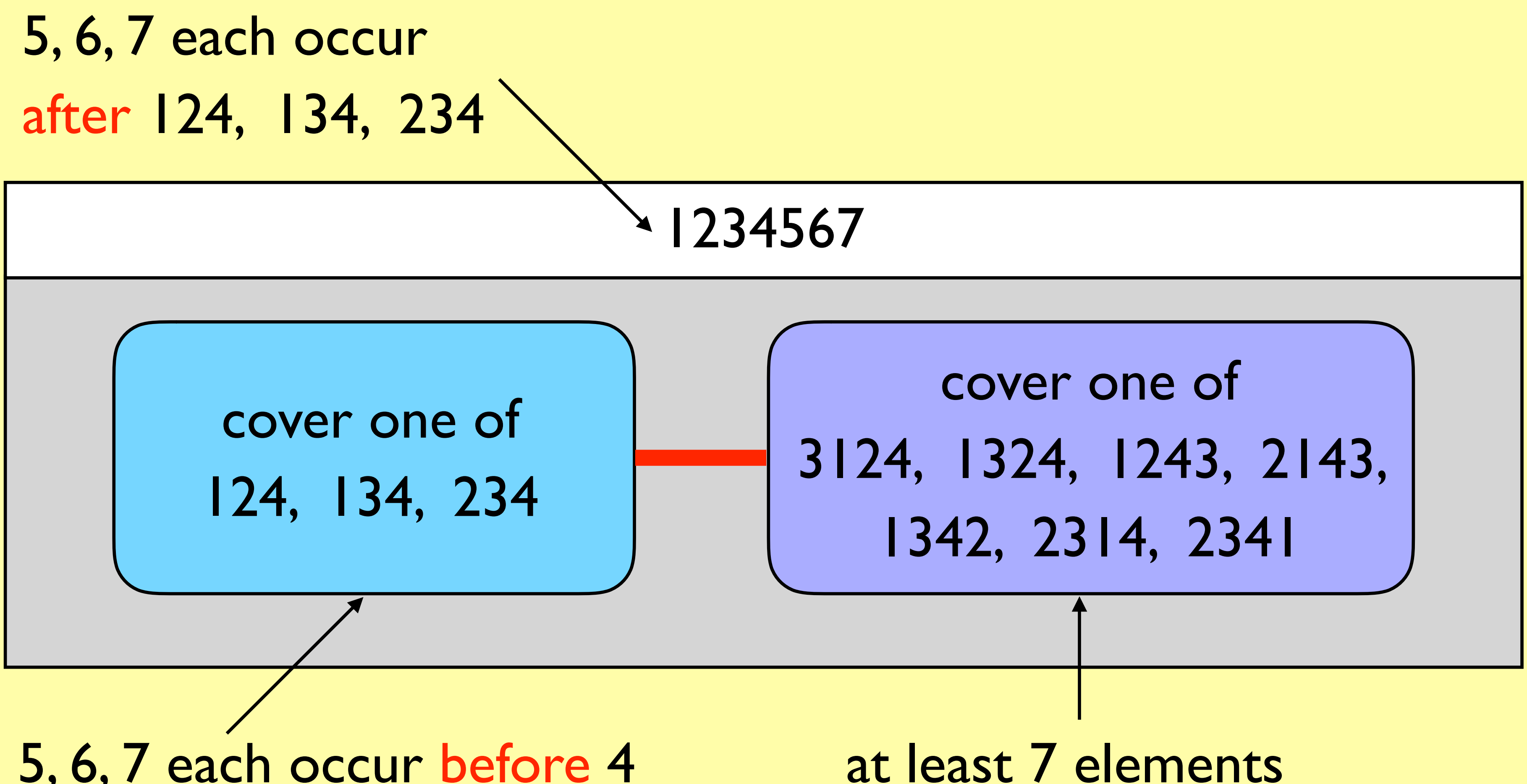
- How can we prove nonexistence of a PSCA(n, k, λ) ?

Motivation

- PSCAs are of practical interest
  - ⋆ use a PSCA(n, k, 1) to efficiently test whether faults arise under all possible orderings of at most k out of n events
- PSCAs are of theoretical interest: connections to
  - ⋆ directed packings
  - ⋆ directed designs
  - ⋆ completely scrambling sets of permutations
  - ⋆ group theory

Combinatorial Nonexistence

Theorem (Klein 2004). There is no PSCA(7, 4, 1)



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Combinatorial Constructions

- $g(n, n) = 1$: all length n sequences

- $g(n, 2) = 1$:

$12\cdots n$	$n\cdots 21$

- (Levenshtein 1991) $g(n, n-1) = 1$
  - ⋆ partition the length n sequences into n disjoint sets, each a PSCA $(n, n-1, 1)$, using perfect single-deletion-correcting codes

Known values of $g(n, k)$

$g(n, k)$ is smallest value of λ for which a PSCA (n, k, λ) exists

n \ k	2	3	4	5	6	7	8	9
2	1							
3	1	1						
4	1	1	1					
5	1	>1?	1	1				
6	1	>1?	>1?	1	1			
7	1	>1?	>1?	>1?	1	1		
8	1	>1?	>1?	>1?	>1?	1	1	
9	1	>1?	>1?	>1?	>1?	>1?	1	1
10	1	>1?	>1?	>1?	>1?	>1?	>1?	1

Levenshtein 1991 conjecture: >1

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Combinatorial Bounds

- $g(n, k) \geq g(n-1, k)$
  - ⋆ delete each occurrence of symbol *n*

12345	43215	35214	14523	25413	53412
13254	52314	24315	15432	34512	42513

PSCA(5, 3, 2)

1234■	4321■	3■214	14■23	2■413	■3412
132■4	■2314	2431■	1■432	34■12	42■13

PSCA(4, 3, 2)

Known values of $g(n, k)$

$g(n, k)$ is smallest value of λ for which a PSCA(n, k, λ) exists

n \ k	2	3	4	5	6	7	8	9
2	1							
3	1	1						
4	1	1	1					
5	1	>1?	1	1				
6	1		>1?	1	1			
7	1			>1?	1	1		
8	1				>1?	1	1	
9	1					>1?	1	1
10	1						>1?	1

Levenshtein 1991 conjecture: >1

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2	1							
3	1	1						
4	1	1	1					
5	1	>1	1	1				
6	1		1	1	1			
7	1			>1	1	1		
8	1				>1	1	1	
9	1					>1?	1	1
10	1						>1?	1

Mathon & van Trung 1999

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Known values of $g(n, k)$

$g(n, k)$ is smallest value of λ for which a PSCA (n, k, λ) exists

n \ k	2	3	4	5	6	7	8	9
2	1							
3	1	1						
4	1	1	1					
5	1	>1	1	1				
6	1		1	1	1			
7	1			>1	1	1		
8	1				>1	1	1	
9	1					>1?	1	1
10	1						>1?	1

Levenshtein 1991 conjecture (modified): >1

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Known values of $g(n, k)$

$g(n, k)$ is smallest value of λ for which a PSCA (n, k, λ) exists

n \ k	2	3	4	5	6	7	8	9
2	1							
3	1	1						
4	1	1	1					
5	1	>1	1	1				
6	1		1	1	1			
7	1		>1	>1	1	1		
8	1				>1	1	1	
9	1					>1?	1	1
10	1						>1?	1

Klein 2004

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2	1							
3	1	1						
4	1	1	1					
5	1	>1	1	1				
6	1	>1	1	1	1			
7	1	>1	>1	>1	1	1		
8	1	>1	>1	>1	>1	1	1	
9	1	>1	>1	>1	>1	>1?	1	1
10	1	>1	>1	>1	>1	>1?	>1?	1

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Combinatorial Bounds

- $g(n, k) \geq g(n-1, k)$
  - ⋆ delete each occurrence of symbol n

- (Chee Colbourn Horsley Zhou 2013) $g(2n, n) > 1$ for $n > 2$
  - ⋆ matrix rank arguments, results on covering arrays

Known values of $g(n, k)$

$g(n,k)$ is smallest value of λ for which a PSCA(n, k, λ) exists

n \ k	2	3	4	5	6	7	8	9
2	1							
3	1	1						
4	1	1	1					
5	1	>1	1	1				
6	1	>1	1	1	1			
7	1	>1	>1	>1	1	1		
8	1	>1	>1	>1	>1	1	1	
9	1	>1	>1	>1	>1	>1?	1	1
10	1	>1	>1	>1	>1	>1?	>1?	1

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Yuster Construction

X	12345	43215	35214	14523	25413	53412

Covered 2 times: 123 321 145 541

Covered 0 times: 132 231 154 451

Covered 1 time: all other 52 ordered 3-subsequences

$X\sigma$	13254	52314	24315	15432	34512	42513

$\sigma = 13254$ as element of symmetric group S_5

$X\sigma = \{x\sigma : x \in X\}$ (convention: $x\sigma$ means x then σ)

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Covered 2 times: 123 321 145 541

Covered 0 times: 132 231 154 451

Covered 1 time: all other 52 ordered 3-subsequences

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$g(n, k)$ is smallest value of λ for which a PSCA (n, k, λ) exists

n \ k	2	3	4	5	6	7	8	9
2	1							
3	1	1						
4	1	1	1					
5	1	2	1	1				
6	1	>1	1	1	1			
7	1	>1	>1	>1	1	1		
8	1	>1	>1	>1	>1	1	1	
9	1	>1	>1	>1	>1	>1?	1	1
10	1	>1	>1	>1	>1	>1?	>1?	1

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Yuster Construction

X	12345	43215	35214	14523	25413	53412
$X\sigma$	13254	52314	24315	15432	34512	42513

PSCA(5, 3, 2)

Theorem (Yuster 2020). $g(5, 3) = 2$

"Proving additional exact values of $g(n, k)$ which are not of unit multiplicity in addition to $g(5, 3)$ also seems challenging."

Yuster 2020

Yuster Construction

X	12345	43215	35214	14523	25413	53412
$X\sigma$	13254	52314	24315	15432	34512	42513

Order 2 subgroup $G = \langle \sigma \rangle = \langle 13254 \rangle$ of S_5

G	43215 G	35214 G	14523 G	25413 G	53412 G

Reinterpret PSCA as union of left cosets of subgroup G

(convention: $x\sigma$ means x then σ)

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Prescribed Subgroup

- Seek PSCA (n, k, λ) as union of left cosets of prescribed subgroup G (not necessarily cyclic nor order 2) of S_n
  - ⋆ reduces size of search space to its $|G|^{\text{th}}$ root (approx)
  - ⋆ sufficient to search over one representative of each conjugacy class of subgroups

- Adapt search algorithm of Mathon & van Trung 1999 for finding spreads of an incidence structure
  - ⋆ recursively track shrinking set of rows and columns of incidence matrix

Prescribed Subgroup

Subsequence

Coset

	1234	1235	1236	1237	...
C_1	0	0	1	0	...
C_2	0	0	1	2	...
C_3	1	1	0	1	...
C_4	1	0	0	1	...
C_5	0	4	1	2	...
C_6	1	0	0	1	...
C_7	1	2	1	1	...
C_8	0	0	3	1	...
C_9	1	0	1	0	...

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Prescribed Subgroup

Subsequence

Coset

	1234	1235	1236	1237	...
C_1	0	0	1	0	...
C_2	0	0	1	2	...
C_3	1	1	0	1	...
C_4	1	0	0	1	...
C_5	0	4	1	2	...
C_6	1	0	0	1	...
C_7	1	2	1	1	...
C_8	0	0	3	1	...
C_9	1	0	1	0	...
Target	3	3	3	3	...

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Prescribed Subgroup

Subsequence

Coset

	1234	1235	1236	1237	...
C_1	0	0	1	0	...
C_2	0	0	1	2	...
C_3	1	1	0	1	...
C_4	1	0	0	1	...
C_6	1	0	0	1	...
C_7	1	2	1	1	...
C_8	0	0	3	1	...
C_9	1	0	1	0	...
Target	3	3	3	3	...

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Prescribed Subgroup

Subsequence

Coset

	1234	1235	1236	1237	...
C_1	0	0	1	0	...
C_2	0	0	1	2	...
C_3	1	1	0	1	...
C_4	1	0	0	1	...
C_6	1	0	0	1	...
C_7	1	2	1	1	...
C_8	0	0	3	1	...
C_9	1	0	1	0	...
Target	3	3	3	3	...

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Prescribed Subgroup

Subsequence

Coset

	1234	1235	1236	1237	...
C_1	0	0	1	0	...
C_2	0	0	1	2	...
C_3	1	1	0	1	...
C_4	1	0	0	1	...
C_6	1	0	0	1	...
C_7	1	2	1	1	...
C_8	0	0	3	1	...
C_9	1	0	1	0	...
Total	2	3	1	2	...
Target	3	3	3	3	...

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Prescribed Subgroup

Subsequence

Coset

	1234	1235	1236	1237	...
C_1	0	0	1	0	...
C_2	0	0	1	2	...
C_3	1	1	0	1	...
C_4	1	0	0	1	...
C_6	1	0	0	1	...
C_7	1	2	1	1	...
C_9	1	0	1	0	...
Total	2	3	1	2	...
Target	3	3	3	3	...

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Prescribed Subgroup

Subsequence

Coset

	1234	1235	1236	1237	...
C_1	0	0	1	0	...
C_3	1	1	0	1	...
C_4	1	0	0	1	...
C_6	1	0	0	1	...
C_7	1	2	1	1	...
C_9	1	0	1	0	...
Total	2	3	1	2	...
Target	3	3	3	3	...

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Prescribed Subgroup

Subsequence

Coset

	1234	1235	1236	1237	...
C_1	0		1	0	...
C_3	1		0	1	...
C_4	1		0	1	...
C_6	1		0	1	...
C_7	1		1	1	...
C_9	1		1	0	...
Total	2	3	1	2	...
Target	3	3	3	3	...

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Prescribed Subgroup

Subsequence

Coset

	1234	1235	1236	1237	…
C_1	0		1	0	…
C_3	1		0	1	…
C_4	1		0	1	…
C_6	1		0	1	…
C_7	1		1	1	…
C_9	1		1	0	…
Total	2	3	1	2	…
Target	3	3	3	3	…

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Known values of $g(n, k)$

$g(n, k)$ is smallest value of λ for which a PSCA(n, k, λ) exists

n \ k	2	3	4	5	6	7	8	9
2	1							
3	1	1						
4	1	1	1					
5	1	2	1	1				
6	1	**2**	1	1	1			
7	1	**2**	**2**	**2 / 3 / 4**	1	1		
8	1	**2 / 3**	>1	>1	>1	1	1	
9	1	**2 / 3 / 4**	>1	>1	>1	>1?	1	1
10	1	>1	>1	>1	>1	>1?	>1?	1

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New Results

Theorem (Na Jedwab Li 2022+).

$g(6, 3) = g(7, 3) = g(7, 4) = 2$ and $g(7, 5) = 2$ or 3 or 4

and $g(8, 3) = 2$ or 3 and $g(9, 3) = 2$ or 3 or 4

For which isomorphism classes of subgroups G of S_n does there exist PSCA(n, k, λ) that is a union of left cosets of G ?

Isomorphism Classes of Subgroups

(n, k, λ)	left cosets of $G \cong$	right cosets of $G \cong$
(4, 3, 1)	C_2	none
(5, 4, 1)	C_4 and $C_2 \times C_2$	C_2
(6, 4, 1)	S_4	S_4
(6, 5, 1)	D_{10} and S_4	C_2
(7, 6, 1)	S_4	C_2
(5, 3, 2)	C_4 and $C_2 \times C_2$ and S_3	C_2
(6, 3, 2)	C_4 and D_{12} and A_4	C_4 and D_{12} and A_4
(7, 3, 2)	C_6 and S_3	C_2
(7, 4, 2)	C_6	C_2
(8, 3, 3)	C_2	

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Open Problems

- Determine further exact values of $g(n, k)$
- Why do left cosets seem to give richer existence pattern than right cosets?
- Construct a new infinite family of PSCAs with small λ
- Find more combinatorial nonexistence results

Known values of $g(n, k)$

$g(n, k)$ is smallest value of λ for which a PSCA (n, k, λ) exists

n \ k	2	3	4	5	6	7	8	9
2	1							
3	1	1						
4	1	1	1					
5	1	2	1	1				
6	1	**2**	1	1	1			
7	1	**2**	**2**	**2 / 3 / 4**	1	1		
8	1	**2 / 3**	>1	>1	>1	1	1	
9	1	**2 / 3 / 4**	>1	>1	>1	>1?	1	1
10	1	>1	>1	>1	>1	>1?	>1?	1

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Known values of $g(n, k)$

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n \ k	2	3	4	5	6	7	8	9
2	1							
3	1	1						
4	1	1	1					
5	1	2	1	1				
6	1	2	1	1	1			
7	1	2	2	2 / 3 / 4	1	1		
8	1	3	>2	>1	>1	1	1	
9	1	2 / 3 / 4	>1	>1	>1	>1?	1	1
10	1	>1	>1	>1	>1	>1?	>1?	1

Independent research by

Dan Gentle and Ian Wanless

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