Spaces Invariant under Unitary Representations and the Bracket.

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For $f, g \in L^2(\mathbb{R}^d)$,

$$[f, g](x) = \sum_{k \in \mathbb{Z}^d} f(x + k)g(x + k), \quad x \in \mathbb{R}^d.$$ 

- $[f, f](x) \geq 0$ a. e. $x \in \mathbb{R}^d$ and $[f, f] \equiv 0 \iff f \equiv 0$.
- $[\cdot, \cdot]$ is a sesquilinear hermitian symmetric map.
- $[f, g]$ is $\mathbb{Z}^d$-periodic and

$$\int_{[0,1)^d} |[f, g](x)|\,dx \leq \|f\|_2\|g\|_2.$$ 

The bracket defines and $L^1([0, 1)^d)$-valued inner product in the Hilbert space $L^2(\mathbb{R}^d)$.

- Jia, Michelli (1991); de Boor, DeVore, Ron (1994).
With \( T_kg(x) = g(x + k) \)

\[
\langle f, T_kg \rangle_2 = \int_{\mathbb{R}^d} \hat{f}(\omega) \overline{\hat{g}(\omega)} e^{-2\pi i k \cdot \omega} d\omega = \int_{[0,1)^d} [\hat{f}, \hat{g}](\omega) e^{-2\pi i k \cdot \omega} d\omega.
\]

Denote by \( \langle f \rangle_{\mathbb{Z}^d} := \text{span}\{ T_k f : k \in \mathbb{Z}^d \} \) the shift-invariant space generated by \( f \in L^2(\mathbb{R}^d) \),

\[
\langle f \rangle_{\mathbb{Z}^d} \perp \langle g \rangle_{\mathbb{Z}^d} \iff [\hat{f}, \hat{g}](\omega) = 0 \text{ a.e. } \omega \in [0,1)^d.
\]

For \( f \in L^2(\mathbb{R}^d) \) denote by \( \mathcal{M}_f \) the space of all \( m : \mathbb{R}^d \rightarrow \mathbb{C} \) that are \( \mathbb{Z}^d \)-periodic and

\[
\| m \|_{\mathcal{M}_f} := \left( \int_{[0,1)^d} |m(\omega)|^2 [\hat{f}, \hat{f}](\omega) d\omega \right)^{1/2} < \infty.
\]
Let $f \in L^2(\mathbb{R}^d)$. The map $J_f$ defined by $J_f(m) = (mf)\hat{\cdot}$ is an isometric isomorphism from $\mathcal{M}_f$ onto $\langle f \rangle_{\mathbb{Z}^d}$.

**Corollary:** $g \in \langle f \rangle_{\mathbb{Z}^d}$ if and only if there exists $m \in \mathcal{M}_f$ such that $\hat{g} = m\hat{f}$.

Denote by $\mathbb{P}\langle f \rangle_{\mathbb{Z}^d}$ the orthogonal projection of $L^2(\mathbb{R}^d)$ onto $\langle f \rangle_{\mathbb{Z}^d}$. Then,

$$\left(\mathbb{P}\langle f \rangle_{\mathbb{Z}^d}(g)\right)\hat{\cdot} = \frac{\hat{g}, \hat{f}}{\hat{f}, \hat{f}} \mathbf{1}_{\{[\hat{f}, \hat{f}] > 0\}} \hat{f}.$$
Reproducing properties of $\mathcal{O}(f) = \{T_kf : k \in \mathbb{Z}^d\}$

- **(a)** $\mathcal{O}(f)$ is an orthonormal basis for $\langle f \rangle_{\mathbb{Z}^d} \iff [\hat{f}, \hat{f}] = 1$ a.e.

- **(b)** $\mathcal{O}(f)$ is a Riesz basis for $\langle f \rangle_{\mathbb{Z}^d}$ with bounds $0 < A \leq B < \infty$ if and only if $A \leq [\hat{f}, \hat{f}] \leq B$ a.e.

- **(c)** $\mathcal{O}(f)$ is a frame for $\langle f \rangle_{\mathbb{Z}^d}$ with bounds $0 < A \leq B < \infty$ if and only if $A \mathbf{1}_{\{[\hat{f}, \hat{f}] > 0\}} \leq [\hat{f}, \hat{f}] \leq B \mathbf{1}_{\{[\hat{f}, \hat{f}] > 0\}}$ a.e.

(a) appears in a paper of R. P. Gosselin (1963) dedicated to the study of cardinal series.

Earliest reference to (c) is due to J. Benedetto and S. Li (1993/1998). This result follows from the representation of the Frame operator of $\mathcal{O}(f)$,

$$\mathcal{F}_f(g) := \sum_{k \in \mathbb{Z}^d} \langle g, T_kf \rangle T_kf, \quad g \in \langle f \rangle_{\mathbb{Z}^d},$$

in terms of the bracket: $\mathcal{F}_f(g) = [\hat{f}, \hat{f}]\hat{g}$.
Other results for $\mathcal{O}(f) = \{ T_k f : k \in \mathbb{Z}^d \}$

- **(a)** $\mathcal{O}(f)$ has a biorthogonal system of the form $\mathcal{O}(\tilde{f})$ with $\tilde{f} \in \langle f \rangle_{\mathbb{Z}^d}$ if
  $$\frac{1}{[\hat{f}, \hat{f}]} \in L^1([0,1]^d).$$
  In this case $\tilde{f} = \frac{1}{[\hat{f}, \hat{f}]} \hat{f}$.

- **(b)** (d=1) $\mathcal{O}(f)$ is $\ell^2$-linearly independent in $L^2(\mathbb{R})$ if $[\hat{f}, \hat{f}] > 0$ a.e.

- **(c)** (d=1) $\mathcal{O}(f)$ is a Schauder basis for $\langle f \rangle_{\mathbb{Z}}$ if $[\hat{f}, \hat{f}]$ is a Muckenhoupt $A_2$ weight in $[0,1)$.

\[ (*) \] A sequence $(x_n)_{n=1}^\infty$ in a Hilbert space $\mathbb{H}$ is $\ell^2$-linearly independent if whenever $(c_n)_{n=1}^\infty \in \ell^2(\mathbb{N})$ and
\[ \lim_{n \to \infty} \| \sum_{k=1}^{n} c_n x_n \| = 0, \] then $c_n = 0$ for all $n \in \mathbb{N}$. 
Group von Neumann algebras

Let Γ be a discrete countable group.

- The right regular representation of Γ is \( \rho : \Gamma \to \mathcal{U}(\ell^2(\Gamma)) \) given by \((\rho(\gamma)a)(\gamma_1) = a(\gamma_1 \gamma)\) or equivalently \(\rho(\gamma)\delta_{\gamma_1} = \delta_{\gamma_1 \gamma^{-1}}\).

- The right von Neumann algebra of Γ is
  \[
  \mathcal{R}(\Gamma) := \overline{\text{span} \{\rho(\gamma) : \gamma \in \Gamma\}}^{\text{WOT}}.
  \]

- The trace of \( F \in \mathcal{R}(\Gamma) \) is given by \( \tau(F) = \langle F\delta_e, \delta_e \rangle_{\ell^2(\Gamma)} \).

- For \( 1 \leq p < \infty \), and \( F \in \mathcal{R}(\Gamma) \), let \( \|F\|_p := (\tau(|F|^p))^{1/p} \), where \( |F| = \sqrt{F^*F} \).

The left regular representation of Γ is \( \lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma)) \) given by \((\lambda(\gamma)a)(\gamma_1) = a(\gamma^{-1} \gamma_1)\) or equivalently \(\lambda(\gamma)\delta_{\gamma_1} = \delta_{\gamma \gamma_1} \).
Non commutative Lebesgue spaces

- Non commutative Lebesgue spaces over $\Gamma$: For $1 \leq p < \infty$,

$$L^p(\mathcal{R}(\Gamma)) := \text{span} \{ \rho(\gamma) : \gamma \in \Gamma \} \| \cdot \|^p.$$

and $L^\infty(\mathcal{R}(\Gamma)) := \mathcal{R}(\Gamma)$ with the operator norm.

- The trace can be defined for any element of $L^p(\mathcal{R}(\Gamma))$, $1 \leq p \leq \infty$ and

$$L^\infty(\mathcal{R}(\Gamma)) \subset L^p(\mathcal{R}(\Gamma)) \subset L^1(\mathcal{R}(\Gamma)).$$

- $L^2(\mathcal{R}(\Gamma))$ is a Hilbert space with

$$\langle F_1, F_2 \rangle_2 = \tau(F_2^*F_1)$$

and $\{ \rho(\gamma) : \gamma \in \Gamma \}$ is an orthonormal basis of $L^2(\mathcal{R}(\Gamma))$. 
Plancherel Theorem

- For $F \in L^1(\mathcal{R}(\Gamma))$ its Fourier coefficients are defined by
  \[
  \hat{F}(\gamma) = \tau(F \rho(\gamma)), \quad \gamma \in \Gamma.
  \]

- For $a \in \ell^2(\Gamma)$ its Fourier series is defined by
  \[
  \mathcal{F}_\Gamma(a) = \sum_{\gamma \in \Gamma} a(\gamma) \rho(\gamma)^*.
  \]

Plancherel Theorem:

(a) For $F \in L^2(\mathcal{R}(\Gamma))$, \( \hat{F} := (\hat{F}(\gamma))_{\gamma \in \Gamma} \in \ell^2(\Gamma) \) and \( \|F\|_2 = \|\hat{F}\|_{\ell^2(\Gamma)} \).

(b) For $a = (a(\gamma))_{\gamma \in \Gamma} \in \ell^2(\Gamma)$, the series \( \sum_{\gamma \in \Gamma} a(\gamma) \rho(\gamma)^* \) converges in the $L^2(\mathcal{R}(\Gamma))$ norm to an operator $F := \mathcal{F}_\Gamma(a) \in L^2(\mathcal{R}(\Gamma))$ such that $\hat{F}(\gamma) = a(\gamma)$ and \[
\|\mathcal{F}_\Gamma(a)\|_2 = \|a\|_{\ell^2(\Gamma)}.
\]
For $F \in L^1(\mathcal{R}(\Gamma))$ selfadjoint, the **support** of $F$ is the minimal orthogonal projection $s_F$ of $\ell^2(\Gamma)$ such that

$$ F = s_F F = Fs_F. $$

It holds that $s_F \in \mathcal{R}(\Gamma)$ and

$$ s_F = \mathbb{P}_{(\ker(F))_\perp} = \mathbb{P}_{\text{Ran}(F)}. $$
Dual integrable representations

Let $\Pi : \Gamma \longrightarrow U(\mathbb{H})$ be a unitary representation of countable discrete group $\Gamma$ on the Hilbert space $\mathbb{H}$.

**Definition.** The unitary representation $\Pi$ is said to be **dual integrable** if there exists a function, called **bracket**, $[\cdot, \cdot]_\Pi : \mathbb{H} \times \mathbb{H} \rightarrow L^1(\mathcal{R}(\Gamma))$ such that

$$
\langle f, \Pi(\gamma)g \rangle_\mathbb{H} = \tau([f, g]_\Pi \rho(\gamma)), \quad f, g \in \mathbb{H}, \gamma \in \Gamma. \quad (2)
$$

The bracket of a dual integrable representation is sesquilinear map that satisfies

- (I) $[f, g]_\Pi^* = [g, f]_\Pi$
- (II) $[f, \Pi(\gamma)g]_\Pi = \rho(\gamma)[f, g]_\Pi$ and $[\Pi(\gamma)f, g]_\Pi = [f, g]_\Pi \rho(\gamma)^*$
- (III) $[f, f]_\Pi$ is nonnegative, and $\|[f, f]_\Pi\|_1 = \|f\|_\mathbb{H}^2$. 
Equivalent conditions

The following conditions are equivalent for a unitary representation $\Pi$ of a discrete countable group $\Gamma$ on a Hilbert space $\mathbb{H}$:

- $\Pi$ is dual integrable
- $\Pi$ is unitary equivalent to a subrepresentation of a direct sum of countable many copies of the right regular representation.
- $\Pi$ is square integrable, that is, there exists a dense subspace $D \subset \mathbb{H}$ such that for each $f \in D$, $\sum_{\gamma \in \Gamma} |\langle g, \Pi(\gamma) f \rangle|^2 < \infty$ for all $g \in \mathbb{H}$.
- $\Pi$ admits a Helson map, that is, there exists a $\sigma$-finite measure space $(M, \nu)$ and a linear isometry $\mathcal{H} : \mathbb{H} \rightarrow L^2(M, L^2(\mathcal{R}(\Gamma)))$ such that

$$\mathcal{H}[\Pi(\gamma)f](x) = \mathcal{H}[f](x) \rho(\gamma)^*, \quad x \in M, \gamma \in \Gamma, \ f \in \mathbb{H}.$$
Example of Helson maps

- If $\mathcal{H}$ is a Helson map for a dual integrable representation $\Pi$, the bracket is given by

$$[f, g]_\Pi = \int_M \mathcal{H}[g](x)^* \mathcal{H}[f](x) d\nu(x), \quad f, g \in \mathbb{H}.$$ 

- **Example 1**: A Helson map for the **left regular representation**

$(\lambda(\gamma)a)(\gamma_1) = a(\gamma^{-1}\gamma_1)$ is the group Fourier series $\mathcal{F}_\Gamma : \ell^2(\Gamma) \to L^2(\mathcal{R}(\Gamma))$ since it is a linear (surjective) isometry that satisfies

$$\mathcal{F}_\Gamma(\lambda(\gamma)a) = \mathcal{F}_\Gamma(a) \rho(\gamma)^*.$$ 

Therefore,

$$[a, b]_\lambda = (\mathcal{F}_\Gamma(b))^* \mathcal{F}_\Gamma(a), \quad a, b \in \ell^2(\Gamma).$$
Example 2: The Gabor representation (abelian)

- The **Gabor representation** \( G : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathcal{U}(L^2(\mathbb{R}^d)) \) is given by

  \[ G(k, \ell)f(x) = T_k M_\ell f(x) = e^{2\pi i x \cdot \ell} f(x + k), \quad x \in \mathbb{R}^d. \]

- A Helson map for \( G \) is the **Zak transform**

  \[ Z : L^2(\mathbb{R}^d) \rightarrow L^2([0, 1)^d \times [0, 1)^d) \] given by

  \[ Zf(x, \omega) := \sum_{\ell \in \mathbb{Z}^d} f(x + \ell) e^{-2\pi i \ell \cdot \omega} \]

- Therefore \( G \) is dual integrable and

  \[ [f, g]_G = \overline{Zg} \cdot Zf, \quad \text{on } [0, 1)^d \times [0, 1)^d. \]
Group actions on $L^2(X, \mu)$

- $\sigma : \Gamma \times X \to X$ is an **action** if the map $x \to \sigma_\gamma(x) := \sigma(\gamma, x)$ is $\mu$-measurable, $\sigma(e, x) = x$ for all $x \in X$, and

  $$\sigma(\gamma_1, \rho(\gamma_2, x)) = \sigma(\gamma_1 \gamma_2, x), \quad \gamma_1, \gamma_2 \in \Gamma, \ x \in X.$$ 

- The action $\sigma$ is **regular** if for each $\gamma \in \Gamma$ the measure $\mu_\gamma(E) = \mu(\sigma_\gamma(E)), E \subset X$, is absolutely continuous with respect to $\mu$ with positive Radon-Nikodym derivative $J_\sigma : \Gamma \times X \to \mathbb{R}^+$ so that

  $$d\mu(\sigma_\gamma(x)) = J_\sigma(\gamma, x)d\mu(x).$$

- The action $\sigma$ has the **tiling property** if there exists a $\mu$-measurable set $C \subset X$ such that $\{\sigma_\gamma(C)\}_{\gamma \in \Gamma}$ is a $\mu$-almost disjoint covering of $X$. 
Example 3

- $\Pi_\sigma : \Gamma \rightarrow \mathcal{U}(L^2(X))$ given by
  \[(\Pi_\sigma(\gamma)f)(x) = J_\sigma(\gamma, x)^{-1/2}f(\sigma(\gamma^{-1}, x))\]

  is a unitary representation of $\Gamma$ in $L^2(X)$.

- A Helson map for the representation $\Pi_\sigma$ is the non commutative Zak transform $Z_\sigma : L^2(X, \mu) \rightarrow L^2(C, L^2(\mathcal{R}))$ (isometric isomorphism) given by
  \[Z_\sigma[f](x) = \sum_{\gamma \in \Gamma}(\Pi_\sigma(\gamma)f)(x)\rho(\gamma), \quad x \in C.\]

- Therefore, the representation $\Pi_\sigma$ is dual integrable and
  \[[f, g]_{\Pi_\sigma} = \int_C (Z_\sigma[g])(x)^*(Z_\sigma[f])(x) d\mu(x).\]
Let $\Pi : \Gamma \longrightarrow \mathcal{U}(\mathbb{H})$ be a unitary representation of a countable discrete group $\Gamma$ on the separable Hilbert space $\mathbb{H}$.

- A closed subspace $V$ of $\mathbb{H}$ is $\Pi$-invariant if $\Pi(\gamma)(V) \subseteq V$ for all $\gamma \in \Gamma$.
- If $\mathcal{A}$ is a subset of $\mathbb{H}$, the $\Pi$-invariant space generated by $\mathcal{A}$ is
  \[ \langle \mathcal{A} \rangle_\Pi := \text{span} \left\{ \Pi(\gamma)f : f \in \mathcal{A}, \gamma \in \Gamma \right\}. \]
- Every $\Pi$-invariant space $V \subset \mathbb{H}$ is of the form $V = \langle \mathcal{A} \rangle_\Pi$ for some countable set $\mathcal{A} \subset \mathbb{H}$.
- When $\mathcal{A} = \{f\}$ we write $\langle \mathcal{A} \rangle_\Pi = \langle f \rangle_\Pi$ and the space is called principal.

**Proposition 1.** For every $\Pi$-invariant spaces $V \subset \mathbb{H}$, there exist a countable set $\mathcal{A} = \{f_i\}_{i \in I}$ such that $\langle f_i \rangle_\Pi \perp \langle f_j \rangle_\Pi$ for $i \neq j$ and
  \[ V = \bigoplus_{i \in I} \langle f_i \rangle_\Pi. \]
Let $\Pi : \Gamma \to U(\mathbb{H})$ be a dual integrable representation of a countable discrete group $(\Gamma, +)$ on the separable Hilbert space $\mathbb{H}$ with Helson map $\mathcal{H}$.

$$\langle f \rangle_\Pi \perp \langle g \rangle_\Pi \iff [f, g] = 0$$

**Proposition 2.** Let $f \in \mathbb{H}$. The map

$$S_f\left(\sum_{\gamma \in \Gamma} a(\gamma) \Pi(\gamma)f\right) = s_{[f, f]} \sum_{\gamma \in \Gamma} a(\gamma)\rho(\gamma)^*$$

defined on span $\{\Pi(\gamma)f : \gamma \in \Gamma\}$ is well defined and can be extended to a linear surjective isometry $S_f : \langle f \rangle_\Pi \to L^2(\mathcal{R}(\Gamma), [f, f]_\Pi)$ satisfying

$$S_f(\Pi(\gamma)g) = S_f(g)\rho(\gamma)^*.$$

**Proposition 3.** Let $f \in \mathbb{H}, g \in \langle f \rangle_\Pi$ if and only if there exists $G \in L^2(\mathcal{R}(\Gamma), [f, f]_\Pi)$ such that $\mathcal{H}[g] = \mathcal{H}[f]G$. In this case

$$[f, g]_\Pi = [f, f]_\Pi G.$$
Reproducing properties of orbits

The orbit generated by $\mathcal{A} = \{\phi_i\}_{i \in I} \subset \mathbb{H}$ is

$$\mathcal{O}_\Pi(\mathcal{A}) = \{\Pi(\gamma)\phi_i : i \in I, \gamma \in \Gamma\}.$$

- $\mathcal{O}_\Pi(\mathcal{A})$ is an orthonormal basis for $\langle \mathcal{A} \rangle_\Pi \iff [\phi_i, \phi_j] = \delta_{i,j}I_2(\Gamma)$.

**Theorem 4.** TFAE:

(a) $\mathcal{O}_\Pi(\mathcal{A})$ is a frame for $\langle \mathcal{A} \rangle_\Pi$ with frame bounds $0 < A \leq B < \infty$.

(b) $A[f, f]_\Pi \leq \sum_{i \in I} |[f, \phi_i]_\Pi|^2 \leq B[f, f]_\Pi$ for all $f \in \langle \mathcal{A} \rangle_\Pi$.

**Proposition 5.** Let $\phi \in \mathbb{H}$. TFAE:

(a) $\mathcal{O}_\Pi(\phi)$ is a frame for $\langle \phi \rangle_\Pi$ with frame bounds $0 < A \leq B < \infty$.

(b) $A_s[\phi, \phi]_\Pi \leq [\phi, \phi]_\Pi \leq B_s[\phi, \phi]_\Pi$.

Recall: $s[\phi, \phi]_\Pi = \mathbb{P}(\ker([\phi, \phi]_\Pi)^\perp)$. 
Theorem 6. Let $V \subset \mathbb{H}$ be a $\Pi$-invariant space. There exists a countable set $\mathcal{A}$ such that $\mathcal{O}_\Pi(\mathcal{A})$ is a Parseval frame for $V$.

**Proof**

1. By Proposition 1 there exist a countable set $\mathcal{A} = \{f_i\}_{i \in I}$ such that $V = \bigoplus_{i \in I} \langle f_i \rangle_\Pi$. (orthogonal)

2. For each $i \in I$, let $F_i := s_{[f_i, f_i]} [f_i, f_i]^{-1/2} \in L^2(\mathcal{R}(\Gamma), [f_i, f_i]_\Pi)$. By Proposition 2 there exists $\phi_i \in \mathbb{H}$ such that $\mathcal{H}[\phi_i](x) = \mathcal{H}[f_i](x) F_i$.

3. Proposition 3 proves that $\langle \phi_i \rangle_\Pi = \langle f_i \rangle_\Pi$.

4. By Proposition 5 and (2), $\mathcal{O}_\Pi(\phi_i)$ is a Parseval frame for $\langle f_i \rangle_\Pi$:

$$[\phi_i, \phi_i]_\Pi = \int_M \mathcal{H}[\phi_i](x)^* \mathcal{H}[\phi_i](x) dx = F_i [f_i, f_i]_\Pi F_i = s_{[f_i, f_i]}.$$

The result follows from (1) and (4).
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