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Spaces Invariant under Unitary Representations and the Bracket.

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The Bracket

For $f, g \in L^2(\mathbb{R}^d)$,

$$[f,g](x) = \sum_{k \in \mathbb{Z}^d} f(x+k)\overline{g(x+k)}, \quad x \in \mathbb{R}^d.$$

- $[f,f](x) \geqslant 0$ a. e. $x \in \mathbb{R}^d$ and $[f,f] \equiv 0 \iff f \equiv 0$.
- $[\cdot,\cdot]$ is a sesquilinear hermitian symmetric map.
- $[f,g]$ is $\mathbb{Z}^d$-periodic and

$$\int_{[0,1)^d} |[f,g](x)| dx \leqslant \|f\|_2 \|g\|_2 \, .$$

The bracket defines and $L^1([0,1)^d)$-valued inner product in the Hilbert space $L^2(\mathbb{R}^d)$.

- Jia, Michelli (1991); de Boor, DeVore, Ron (1994).

Results with the bracket

- With $T_k g(x) = g(x+k)$

$$\langle f, T_k g \rangle_2 = \int_{\mathbb{R}^d} \widehat{f}(\omega)\overline{\widehat{g}(\omega)} e^{-2\pi i k\cdot\omega} d\omega = \int_{[0,1)^d} [\widehat{f}, \widehat{g}](\omega) e^{-2\pi i k\cdot\omega} d\omega \,. \tag{1}$$

Denote by $\langle f \rangle_{\mathbb{Z}^d} := \overline{\text{span}\{T_k f : k \in \mathbb{Z}^d\}}$ the shift-invariant space generated by $f \in L^2(\mathbb{R}^d)$,

- $\langle f \rangle_{\mathbb{Z}^d} \perp \langle g \rangle_{\mathbb{Z}^d} \iff [\widehat{f}, \widehat{g}](\omega) = 0$ a.e. $\omega \in [0,1)^d$.

For $f \in L^2(\mathbb{R}^d)$ denote by $\mathcal{M}_f$ the space of all $m : \mathbb{R}^d \to \mathbb{C}$ that are $\mathbb{Z}^d$-periodic and

$$\|m\|_{\mathcal{M}_f} := \left(\int_{[0,1)^d} |m(\omega)|^2 [\widehat{f}, \widehat{f}](\omega) d\omega \right)^{1/2} < \infty \,.$$

Results with the bracket

- Let $f \in L^2(\mathbb{R}^d)$. The map J_f defined by $J_f(m) = (m\widehat{f})^{\vee}$ is an isometric isomorphism from $\mathcal{M}_f$ onto $\langle f \rangle_{\mathbb{Z}^d}$.

- **Corollary:** $g \in \langle f \rangle_{\mathbb{Z}^d}$ if and only if there exists $m \in \mathcal{M}_f$ such that $\widehat{g} = m\widehat{f}$.

- Denote by $\mathbb{P}_{\langle f \rangle_{\mathbb{Z}^d}}$ the orthogonal projection of $L^2(\mathbb{R}^d)$ onto $\langle f \rangle_{\mathbb{Z}^d}$. Then,

$$(\mathbb{P}_{\langle f \rangle_{\mathbb{Z}^d}}(g))\widehat{\ } = \frac{[\widehat{g}, \widehat{f}]}{[\widehat{f}, \widehat{f}]} \mathbf{1}_{\{[\widehat{f},\widehat{f}]>0\}} \widehat{f}.$$

Reproducing properties of $\mathcal{O}(f) = \{T_k f : k \in \mathbb{Z}^d\}$

- **(a)** $\mathcal{O}(f)$ is an orthonormal basis for $\langle f \rangle_{\mathbb{Z}^d} \iff [\widehat{f}, \widehat{f}] = 1$ a.e.
- **(b)** $\mathcal{O}(f)$ is a Riesz basis for $\langle f \rangle_{\mathbb{Z}^d}$ with bounds $0 < A \leqslant B < \infty$ $\iff A \leqslant [\widehat{f}, \widehat{f}] \leqslant B$ a.e.
- **(c)** $\mathcal{O}(f)$ is a frame for $\langle f \rangle_{\mathbb{Z}^d}$ with bounds $0 < A \leqslant B < \infty$ $\iff A\mathbf{1}_{\{[\widehat{f},\widehat{f}]>0\}} \leqslant [\widehat{f}, \widehat{f}] \leqslant B\mathbf{1}_{\{[\widehat{f},\widehat{f}]>0\}}$ a.e.

(a) appears in a paper of R. P. Gosselin (1963) dedicated to the study of cardinal series.
Earliest reference to **(c)** is due to J. Benedetto and S. Li (1993/1998).
This result follows from the representation of the Frame operator of $\mathcal{O}(f)$,

$$\mathcal{F}_f(g) := \sum_{k \in \mathbb{Z}^d} \langle g, T_k f \rangle T_k f, \quad g \in \langle f \rangle_{\mathbb{Z}^d},$$

in terms of the bracket: $\mathcal{F}_f(g)\widehat{\ } = [\widehat{f}, \widehat{f}]\widehat{g}$.

Other results for $\mathcal{O}(f) = \{ T_k f : k \in \mathbb{Z}^d \}$

- **(a)** $\mathcal{O}(f)$ has a biorthogonal system of the form $\mathcal{O}(\widetilde{f})$ with $\widetilde{f} \in \langle f \rangle_{\mathbb{Z}^d}$ $\Longleftrightarrow \dfrac{1}{[\widehat{f}, \widehat{f}]} \in L^1([0,1)^d)$. In this case $\widehat{\widetilde{f}} = \dfrac{1}{[\widehat{f}, \widehat{f}]} \widehat{f}$.
- **(b)** (d=1) $\mathcal{O}(f)$ is ℓ^2-linearly independent in $L^2(\mathbb{R})$(*) $\Longleftrightarrow [\widehat{f}, \widehat{f}] > 0$ a.e.
- **(c)** (d=1) $\mathcal{O}(f)$ is a Schauder basis for $\langle f \rangle_{\mathbb{Z}} \Longleftrightarrow [\widehat{f}, \widehat{f}]$ is a Muckenhoupt A_2 weight in $[0,1)$.

(*) A sequence $(x_n)_{n=1}^{\infty}$ in a Hilbert space $\mathbb{H}$ is ℓ^2-linearly independent if whenever $(c_n)_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ and $\lim_{n\to\infty} \| \sum_{k=1}^{n} c_n x_n \| = 0$, then $c_n = 0$ for all $n \in \mathbb{N}$.

Group von Neumann algebras

Let Γ be a discrete countable group.

- The right regular representation of Γ is $\rho : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ given by $(\rho(\gamma)a)(\gamma_1) = a(\gamma_1\gamma)$ or equivalently $\rho(\gamma)\delta_{\gamma_1} = \delta_{\gamma_1\gamma^{-1}}$.
- The right von Neumann algebra of Γ is

$$\mathcal{R}(\Gamma) := \overline{\text{span}\ \{\rho(\gamma) : \gamma \in \Gamma\}}^{WOT}.$$

- The trace of $F \in \mathcal{R}(\Gamma)$ is given by $\tau(F) = \langle F\delta_e, \delta_e\rangle_{\ell^2(\Gamma)}$.
- For $1 \leqslant p < \infty$, and $F \in \mathcal{R}(\Gamma)$, let $\|F\|_p := (\tau(|F|^p))^{1/p}$, where $|F| = \sqrt{F^*F}$.

The left regular representation of Γ is $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ given by $(\lambda(\gamma)a)(\gamma_1) = a(\gamma^{-1}\gamma_1)$ or equivalently $\lambda(\gamma)\delta_{\gamma_1} = \delta_{\gamma\gamma_1}$.

Non commutative Lebesgue spaces

- Non commutative Lebesgue spaces over Γ: For $1 \leqslant p < \infty$,

$$L^p(\mathcal{R}(\Gamma)) := \overline{\text{span}\ \{\rho(\gamma) : \gamma \in \Gamma\}}^{\|\ \|_p}.$$

  and $L^\infty(\mathcal{R}(\Gamma)) := \mathcal{R}(\Gamma)$ with the operator norm.

- The trace can be defined for any element of $L^p(\mathcal{R}(\Gamma)),\ 1 \leqslant p \leqslant \infty$ and

$$L^\infty(\mathcal{R}(\Gamma)) \subset L^p(\mathcal{R}(\Gamma)) \subset L^1(\mathcal{R}(\Gamma))\,.$$

- $L^2(\mathcal{R}(\Gamma))$ is a Hilbert space with

$$\langle F_1, F_2 \rangle_2 = \tau(F_2^* F_1)$$

  and $\{\rho(\gamma) : \gamma \in \Gamma\}$ is an orthonormal basis of $L^2(\mathcal{R}(\Gamma))$.

Plancherel Theorem

- For $F \in L^1(\mathcal{R}(\Gamma))$ its Fourier coefficients are defined by

$$\widehat{F}(\gamma) = \tau(F\rho(\gamma)), \ \gamma \in \Gamma.$$

- For $a \in \ell^2(\Gamma)$ its Fourier series is defined by

$$\mathcal{F}_\Gamma(a) = \sum_{\gamma \in \Gamma} a(\gamma)\rho(\gamma)^* .$$

Plancherel Theorem:

(a) For $F \in L^2(\mathcal{R}(\Gamma))$, $\widehat{F} := (\widehat{F}(\gamma))_{\gamma\in\Gamma} \in \ell^2(\Gamma)$ and $\|F\|_2 = \|\widehat{F}\|_{\ell^2(\Gamma)}$.

(b) For $a = (a(\gamma))_{\gamma\in\Gamma} \in \ell^2(\Gamma)$, the series $\sum_{\gamma\in\Gamma} a(\gamma)\rho(\gamma)^*$ converges in the $L^2(\mathcal{R}(\Gamma))$ norm to an operator $F := \mathcal{F}_\Gamma(a) \in L^2(\mathcal{R}(\Gamma))$ such that $\widehat{F}(\gamma) = a(\gamma)$ and

$$\|\mathcal{F}_\Gamma(a)\|_2 = \|a\|_{\ell^2(\Gamma)}.$$

The support of a selfadjoint operator

- For $F \in L^1(\mathcal{R}(\Gamma))$ selfadjoint, the **support** of F is the minimal orthogonal projection s_F of $\ell^2(\Gamma)$ such that

$$F = s_F F = F s_F \, .$$

- It holds that $s_F \in \mathcal{R}(\Gamma)$ and

$$s_F = \mathbb{P}_{(\ker(F))^{\perp}} = \mathbb{P}_{\overline{\mathrm{Ran}(F)}} \, .$$

Dual integrable representations

Let $\Pi : \Gamma \longrightarrow \mathcal{U}(\mathbb{H})$ be a unitary representation of countable discrete group Γ on the Hilbert space $\mathbb{H}$.

- **Definition.** The unitary representation Π is said to be **dual integrable** if there exists a function, called **bracket**, $[\cdot,\cdot]_\Pi : \mathbb{H} \times \mathbb{H} \mapsto L^1(\mathcal{R}(\Gamma))$ such that

$$\langle f, \Pi(\gamma)g\rangle_{\mathbb{H}} = \tau([f,g]_\Pi\, \rho(\gamma)), \quad f,g \in \mathbb{H}, \gamma \in \Gamma. \tag{2}$$

- The bracket of a dual integrable representation is sesquilinear map that satisfies
- (I) $[f,g]_\Pi^* = [g,f]_\Pi$
- (II) $[f,\Pi(\gamma)g]_\Pi = \rho(\gamma)[f,g]_\Pi$ and $[\Pi(\gamma)f,g]_\Pi = [f,g]_\Pi\rho(\gamma)^*$
- (III) $[f,f]_\Pi$ is nonnegative, and $\|[f,f]_\Pi\|_1 = \|f\|_{\mathbb{H}}^2$.

Equivalent conditions

The following conditions are equivalent for a unitary representation Π of a discrete countable group Γ on a Hilbert space $\mathbb{H}$:

- Π is dual integrable
- Π is unitary equivalent to a subrepresentation of a direct sum of countable many copies of the right regular representation.
- Π is square integrable, that is, there exists a dense subspace $D \subset \mathbb{H}$ such that for each $f \in D$, $\sum_{\gamma \in \Gamma} |\langle g, \Pi(\gamma) f \rangle|^2 < \infty$ for all $g \in \mathbb{H}$.
- Π admits a Helson map, that is, there exists a σ-finite measure space (M, ν) and a linear isometry $\mathcal{H} : \mathbb{H} \longrightarrow L^2(M, L^2(\mathcal{R}(\Gamma)))$ such that

$$\mathcal{H}[\Pi(\gamma) f](x) = \mathcal{H}[f](x) \rho(\gamma)^*, \qquad x \in M, \gamma \in \Gamma, \ f \in \mathbb{H}.$$

Example of Helson maps

- If $\mathcal{H}$ is a Helson map for a dual integrable representation Π, the bracket is given by

$$[f,g]_\Pi = \int_M \mathcal{H}[g](x)^* \mathcal{H}[f](x) d\nu(x), \quad f,g \in \mathbb{H}.$$

- **Example 1**: A Helson map for the **left regular representation** $(\lambda(\gamma)a)(\gamma_1) = a(\gamma^{-1}\gamma_1)$ is the group Fourier series $\mathcal{F}_\Gamma : \ell^2(\Gamma) \to L^2(\mathcal{R}(\Gamma))$ since it is a linear (surjective) isometry that satisfies

$$\mathcal{F}_\Gamma(\lambda(\gamma)a) = \mathcal{F}_\Gamma(a)\rho(\gamma)^* \,.$$

Therefore.

$$[a,b]_\lambda = (\mathcal{F}_\Gamma(b))^* \mathcal{F}_\Gamma(a), \quad a,b \in \ell^2(\Gamma)\,.$$

Example 2: The Gabor representation (abelian)

- The **Gabor representation** $\mathcal{G} : \mathbb{Z}^d \times \mathbb{Z}^d \mapsto \mathcal{U}(L^2(\mathbb{R}^d))$ is given by

$$\mathcal{G}(k,\ell)f(x) = T_k M_\ell f(x)) = e^{2\pi i x\cdot\ell} f(x+k)\,, \quad x \in \mathbb{R}^d.$$

- A Helson map for $\mathcal{G}$ is the **Zak transform** $Z : L^2(\mathbb{R}^d) \longrightarrow L^2([0,1)^d \times [0,1)^d)$ given by

$$Zf(x,\omega) := \sum_{\ell\in\mathbb{Z}^d} f(x+\ell)e^{-2\pi i \ell\cdot\omega}$$

- Therefore $\mathcal{G}$ is dual integrable and

$$[f,g]_{\mathcal{G}} = \overline{Zg}\cdot Zf, \quad \text{on } [0,1)^d \times [0,1)^d\,.$$

Group actions on $L^2(X,\mu)$

- $\sigma : \Gamma \times X \to X$ is an **action** if the map $x \to \sigma_\gamma(x) := \sigma(\gamma, x)$ is μ-measurable, $\sigma(e,x) = x$ for all $x \in X$, and

$$\sigma(\gamma_1, \rho(\gamma_2, x)) = \sigma(\gamma_1\gamma_2, x)\,, \quad \gamma_1, \gamma_2 \in \Gamma,\ x \in X.$$

- The action σ is regular if for each $\gamma \in \Gamma$ the measure $\mu_\gamma(E) = \mu(\sigma_\gamma(E)), E \subset X$, is absolutely continuous with respect to μ with positive Radon-Nikodym derivative $J_\sigma : \Gamma \times X \to \mathbb{R}^+$ so that

$$d\mu(\sigma_\gamma(x)) = J_\sigma(\gamma, x) d\mu(x)\,.$$

- The action σ has the tiling property if there exists a μ-measurable set $C \subset X$ such that $\{\sigma_\gamma(C)\}_{\gamma\in\Gamma}$ is a μ-almost disjoint covering of X.

Example 3

- $\Pi_\sigma : \Gamma \to \mathcal{U}(L^2(X))$ given by
$$(\Pi_\sigma(\gamma)f)(x) = J_\sigma(\gamma, x)^{-1/2} f(\sigma(\gamma^{-1}, x))$$
is a unitary representation of Γ in $L^2(X)$.

- A Helson map for the representation Π_σ is the non commutative Zak transform $Z_\sigma : L^2(X, \mu) \to L^2(C, L^2(\mathcal{R}))$ (isometric isomorphism) given by
$$Z_\sigma[f](x) = \sum_{\gamma \in \Gamma} (\Pi_\sigma(\gamma)f)(x)\rho(\gamma), \quad x \in C.$$

- Therefore, the representation Π_σ is dual integrable and
$$[f, g]_{\Pi_\sigma} = \int_C (Z_\sigma[g])(x)^* (Z_\sigma[f])(x) d\mu(x).$$

Π-invariant spaces

Let $\Pi : \Gamma \longrightarrow \mathcal{U}(\mathbb{H})$ be a unitary representation of a countable discrete group Γ on the separable Hilbert space $\mathbb{H}$.

- A closed subspace V of $\mathbb{H}$ is Π-invariant if $\Pi(\gamma)(V) \subset V$ for all $\gamma \in \Gamma$.
- If $\mathcal{A}$ is a subset of $\mathbb{H}$, the Π-invariant space generated by $\mathcal{A}$ is

$$\langle \mathcal{A} \rangle_\Pi := \overline{\text{span } \{\Pi(\gamma)f : f \in \mathcal{A}, \gamma \in \Gamma\}}^{\mathbb{H}}.$$

- Every Π-invariant space $V \subset \mathbb{H}$ is of the form $V = \langle \mathcal{A} \rangle_\Pi$ for some countable set $\mathcal{A} \subset \mathbb{H}$.
- When $\mathcal{A} = \{f\}$ we write $\langle \mathcal{A} \rangle_\Pi = \langle f \rangle_\Pi$ and the space is called principal.

Proposition 1. For every Π-invariant spaces $V \subset \mathbb{H}$, there exist a countable set $\mathcal{A} = \{f_i\}_{i \in I}$ such that $\langle f_i \rangle_\Pi \perp \langle f_j \rangle_\Pi$ for $i \neq j$ and

$$V = \bigoplus_{i \in I} \langle f_i \rangle_\Pi.$$

Results with the bracket

Let $\Pi : \Gamma \longrightarrow \mathcal{U}(\mathbb{H})$ be a dual integrable representation of a countable discrete group $(\Gamma, +)$ on the separable Hilbert space $\mathbb{H}$ with Helson map $\mathcal{H}$.

- $\langle f \rangle_\Pi \perp \langle g \rangle_\Pi \Leftrightarrow [f, g] = 0$

Proposition 2. Let $f \in \mathbb{H}$. The map

$$S_f(\sum_{\gamma \in \Gamma} a(\gamma)\Pi(\gamma)f) = s_{[f,f]} \sum_{\gamma \in \Gamma} a(\gamma)\rho(\gamma)^*$$

defined on span $\{\Pi(\gamma)f : \gamma \in \Gamma\}$ is well defined and can be extended to a linear surjective isometry $S_f : \langle f \rangle_\Pi \to L^2(\mathcal{R}(\Gamma), [f, f]_\Pi)$ satifying

$$S_f(\Pi(\gamma)g) = S_f(g)\rho(\gamma)^* \,.$$

Proposition 3. Let $f \in \mathbb{H}$. $g \in \langle f \rangle_\Pi$ if and only if there exists $G \in L^2(\mathcal{R}(\Gamma), [f, f]_\Pi)$ such that $\mathcal{H}[g] = \mathcal{H}[f]G$. In this case

$$[f, g]_\Pi = [f, f]_\Pi G.$$

Reproducing properties of orbits

The **orbit** generated by $\mathcal{A} = \{\phi_i\}_{i\in I} \subset \mathbb{H}$ is

$$\mathcal{O}_\Pi(\mathcal{A}) = \{\Pi(\gamma)\phi_i : i \in I, \gamma \in \Gamma\}.$$

- $\mathcal{O}_\Pi(\mathcal{A})$ is an orthonormal basis for $\langle \mathcal{A} \rangle_\Pi \iff [\phi_i, \phi_j] = \delta_{i,j}\mathbb{I}_{\ell^2(\Gamma)}$.

Theorem 4. TFAE:

(a) $\mathcal{O}_\Pi(\mathcal{A})$ is a frame for $\langle \mathcal{A} \rangle_\Pi$ with frame bounds $0 < A \leqslant B < \infty$.

(b) $A[f,f]_\Pi \leqslant \sum_{i\in I} |[f,\phi_i]_\Pi|^2 \leqslant B[f,f]_\Pi$ for all $f \in \langle \mathcal{A} \rangle_\Pi$.

Proposition 5. Let $\phi \in \mathbb{H}$. TFAE:

(a) $\mathcal{O}_\Pi(\phi)$ is a frame for $\langle \phi \rangle_\Pi$ with frame bounds $0 < A \leqslant B < \infty$.

(b) $As_{[\phi,\phi]_\Pi} \leqslant [\phi,\phi]_\Pi \leqslant Bs_{[\phi,\phi]_\Pi}$.

Recall: $s_{[\phi,\phi]_\Pi} = \mathbb{P}_{(\ker([\phi,\phi]_\Pi)^\perp}$.

Parseval frame of orbits

Theorem 6. Let $V \subset \mathbb{H}$ be a Π-invariant space. There exists a countable set $\mathcal{A}$ such that $\mathcal{O}_\Pi(\mathcal{A})$ is a Parseval frame for V.

Proof

- (1) By Proposition 1 there exist a countable set $\mathcal{A} = \{f_i\}_{i \in I}$ such that $V = \bigoplus_{i \in I} \langle f_i \rangle_\Pi$. (orthogonal)
- (2) For each $i \in I$, let $F_i := s_{[f_i,f_i]_\Pi} [f_i, f_i]_\Pi^{-1/2} \in L^2(\mathcal{R}(\Gamma), [f_i, f_i]_\Pi)$. By Proposition 2 there exists $\phi_i \in \mathbb{H}$ such that $\mathcal{H}[\phi_i](x) = \mathcal{H}[f_i](x) F_i$.
- (3) Proposition 3 proves that $\langle \phi_i \rangle_\Pi = \langle f_i \rangle_\Pi$.
- (4) By Proposition 5 and (2), $\mathcal{O}_\Pi(\phi_i)$ is a Parseval frame for $\langle f_i \rangle_\Pi$:

$$[\phi_i, \phi_i]_\Pi = \int_M \mathcal{H}[\phi_i](x)^* \mathcal{H}[\phi_i](x) dx = F_i [f_i, f_i]_\Pi F_i = s_{[f_i,f_i]_\Pi}.$$

- The resut follows from (1) and (4).

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