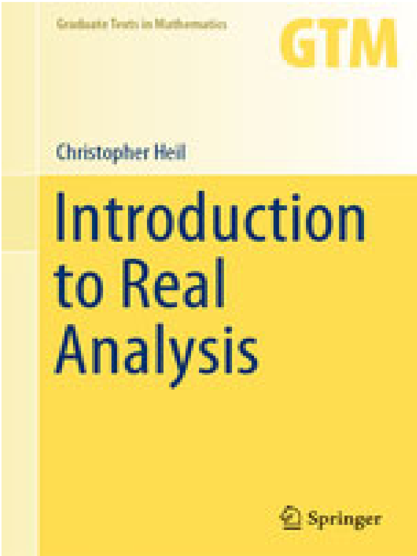
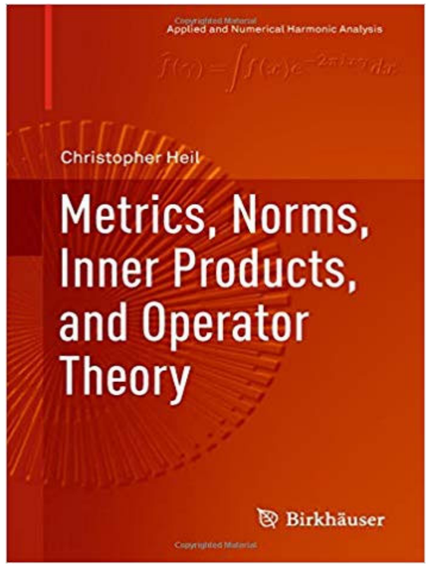
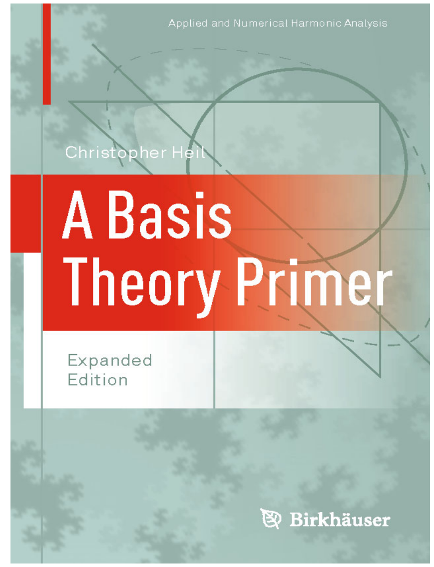
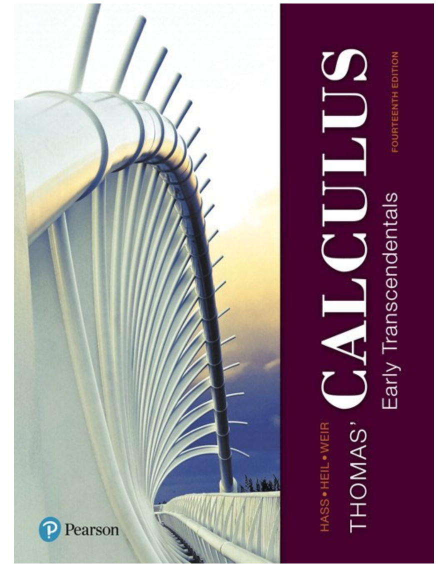

Graduate Texts in Mathematics
GTM
Christopher Heil
Introduction to Real Analysis
Springer


Applied and Numerical Harmonic Analysis
Christopher Heil
Metrics, Norms, Inner Products, and Operator Theory
Birkhäuser


Applied and Numerical Harmonic Analysis
Christopher Heil
A Basis Theory Primer
Expanded Edition
Birkhäuser


HASS • HEIL • WEIR
THOMAS' CALCULUS
Early Transcendentals
FOURTEENTH EDITION
Pearson

Frames, Erasures, and Excess in Infinite Dimensions

Christopher Heil

Georgia Tech

Codex Seminar, December 8, 2020

Frames (Duffin and Schaeffer, 1952)

A sequence $\{x_n\}_{n\in\mathbb{N}}$ **in a Hilbert space** H **is a frame if** $\exists\, A, B > 0$ **such that**

$$\forall\, x \in H, \quad A\,\|x\|^2 \leq \sum_n |\langle x, x_n\rangle|^2 \leq B\,\|x\|^2$$

In this case (DS, 1952):

- **The frame operator** $Sx = \sum_n \langle x, x_n\rangle\, x_n$ **is a topological isormorphism** $S\colon H \to H$

- **The canonical dual frame is** $\{\widetilde{x}_n\}_{n\in\mathbb{N}}$ **where** $\widetilde{x}_n = S^{-1}x_n$

- $x = \displaystyle\sum_n \langle x, \widetilde{x}_n\rangle\, x_n = \sum_n \langle x, x_n\rangle\, \widetilde{x}_n$ **for every** $x \in H$

- **If** $x = \sum c_n x_n$, **then** $\sum_n |c_n|^2 \;=\; \sum_n \left|\langle x, \widetilde{x}_n\rangle\right|^2 \;+\; \sum_n \left|c_n - \langle x, \widetilde{x}_n\rangle\right|^2$

- * $\displaystyle\sum_{n\neq m} |\langle x_m, \widetilde{x}_n\rangle|^2 \;=\; \frac{1 \;-\; |\langle x_m, \widetilde{x}_m\rangle|^2 \;-\; |1 - \langle x_m, \widetilde{x}_m\rangle|^2}{2}$

- **If** $\langle x_m, \widetilde{x}_m\rangle = 1$, **then** $\langle x_m, \widetilde{x}_n\rangle = 0$ **for** $n \neq m$.

- **The removal of a vector from a frame leaves either a frame or an incomplete set:**

$$\langle x_m, \widetilde{x}_m\rangle \neq 1 \quad \Longrightarrow \quad \{x_n\}_{n\neq m} \textbf{ is a frame}$$

$$\langle x_m, \widetilde{x}_m\rangle = 1 \quad \Longrightarrow \quad \{x_n\}_{n\neq m} \textbf{ is incomplete}$$

Multiplicative Completion (Boas and Pollard, 1948)

Problem

If $\{f_k\}_{k\in\mathbb{Z}}$ **is an ONB for** $L^2[0,1]$ **and** $F \subseteq \mathbb{Z}$ **is finite then** $\{f_k\}_{k\notin F}$ **is incomplete.**

Does $\exists\, m \in L^\infty[0,1]$ **such that** $\{f_k \cdot m\}_{k\notin F}$ **is complete?**

Multiplicative Completion (Boas and Pollard, 1948)

Problem

If $\{f_k\}_{k\in\mathbb{Z}}$ is an ONB for $L^2[0,1]$ and $F \subseteq \mathbb{Z}$ be finite then $\{f_k\}_{k\notin F}$ is incomplete.

Does $\exists\, m \in L^\infty[0,1]$ such that $\{f_k \cdot m\}_{k\notin F}$ is complete?

Yes: Proof for $F = \{0\}$

Chose m so $f_0/m \notin L^2$. If $\langle f, f_k \cdot m\rangle = 0$ for $k \neq 0$ then $\langle f \cdot \overline{m}, f_k\rangle = 0$ for $k \neq 0$, so

$$f \cdot \overline{m} \;=\; c f_0$$

Hence $f = cf_0/m$. Contradiction if $c \neq 0$. Therefore $\{f_k \cdot m\}_{k\neq 0}$ is complete.

A (not entirely trivial) induction extends to finite F.

Constructing m

Choose $E \subseteq [0,1]$ **with** $|E| > 0$ **and** $\inf_{x \in E} |f_0(x)| = \varepsilon > 0$.

Choose $\psi \in L^\infty \backslash L^2$ **with** $\mathbf{supp}(\psi) \subseteq E$. **Set**

$$m(x) = \begin{cases} 1, & x \notin \mathbf{supp}(\psi), \\ 1/\psi(x), & x \in \mathbf{supp}(\psi). \end{cases}$$

Then

$$f_0(x)/m(x) \geq \varepsilon\, \psi(x), \quad x \in \mathbf{supp}(\psi),$$

so $f_0/m \notin L^2$.

Generalizes to frames and less structured systems in function spaces.

Lemma. Let $\mathcal{A}$ be a solid Banach space of measurable functions on a measure space (X, Σ, μ). Assume for each measurable $E \subseteq X$ there is a measurable function ψ on X such that $\{\psi \neq 0\} \subseteq E$ and $\psi \notin \mathcal{A}$. If $f_1, \ldots, f_N \in \mathcal{A}$, then $\exists\, g \in L^\infty(\mu)$ such that

(a) $g(x) \neq 0$ for $x \in X$, and

(b) $f/g \notin \mathcal{A}$ for all $f \in \mathrm{span}\{f_1, \ldots, f_N\} \setminus \{0\}$.

Need "nonatomicness"

Multiplicative completion fails in $\ell^2(\mathbb{Z})$.

Surprising Equivalences (Talalyan, and Price and Zink, 1957–1975)

If $\{f_n\}_{n\in\mathbb{N}} \subseteq L^2(\mu)$, **where** (X,μ) **is a separable measure space with** $\mu(\mu) = 1$, **then TFAE:**

(a) If $\varepsilon > 0$ **then** $\exists\, S \subseteq X$ **such that** $\mu(S) > 1-\varepsilon$ **and** $\{f_n\, \chi_S\}$ **is complete in** $L^2(S)$.

(b) If f **is finite a.e. and** $\varepsilon > 0$ **then** $\exists\, S \subseteq X$ **and** $g \in \mathrm{span}\{f_n\}$ **such that** $\mu(S) > 1-\varepsilon$ **and** $|f-g| < \varepsilon$ **on** S.

(c) There exists a bounded, nonnegative function m **such that** $\{mf_n\}$ **is complete in** $L^2(\mu)$.

There are a wide variety of related results by Kazarian.

<u>Weighted Exponentials</u>: Fix $g \in L^2[0,1]$, **set** $\mathcal{E}_g = \{e^{2\pi inx} g(x)\}_{n\in\mathbb{Z}}$

(a) $\mathcal{E}_g$ **is complete** $\iff$ $g \neq 0$ **a.e.**

(b) $\mathcal{E}_g$ **is minimal** $\iff$ $1/g \in L^2[0,1]$

(c) $\mathcal{E}_g$ **is a Schauder basis** $\iff$ $|g|^2 \in \mathcal{A}_2[0,1]$

(d) $\mathcal{E}_g$ **is a Bessel sequence** $\iff$ $g \in L^\infty[0,1]$

(e) $\mathcal{E}_g$ **is a frame sequence** $\iff$ $\exists\, A, B > 0$ **such that** $A \leq |g(x)|^2 \leq B$ **for a.e.** x **with** $g(x) \neq 0$

(f) $\mathcal{E}_g$ **is a Riesz basis** $\iff$ $\exists\, A, B > 0$ **such that** $A \leq |g(x)|^2 \leq B$ **a.e.**

(g) $\mathcal{E}_g$ **is an orthonormal basis** $\iff$ $|g(x)| = 1$ **a.e.**

<u>Attribution</u>

(c) Hunt, Muckenhaupt, Wheeden (1973)

(e) Benedetto and Li (1998)

Multiplicative Completion for $\{e^{2\pi inx}\}_{n\neq 0}$. **Let** $e_n(x) = e^{2\pi inx}$.

- $1/x \notin L^2[0,1]$ **so** $\{xe_n\}_{n\neq 0}$ **is complete**
- **Biorthogonal system is** $\widetilde{e}_n(x) = \dfrac{e^{2\pi inx} - 1}{x}$ **for** $n \neq 0$
- $\{xe_n\}_{n\neq 0}$ **is minimal (and complete, so exact).**
- **If** $\{xe_n\}_{n\neq 0}$ **were a Schauder basis then** $1 \leq \|xe_n\|_2 \, \|\widetilde{e}_n\|_2 \leq 2C$
- $\|xe_n\|_2 = 3^{-1/2}$ **and** $\|\widetilde{e}_n\|_2 \;=\; 4\pi n \int_0^{\pi n} \frac{\sin^2 u}{u^2}\, du \;\to\; \infty$ **as** $n \to \infty$
- $\{xe_n\}_{n\neq 0}$ **is not a Schauder basis**
- **(Other results by Yoon/H., 2012)**

The original Gabor system, generated by the Gaussian at the critical density, has similar properties:

$$\mathcal{G}(\phi, 1, 1) \;=\; \{M_n T_k \phi\}_{k,n\in\mathbb{Z}} \;=\; \{e^{2\pi inx}\, e^{-(x-k)^2}\}_{k,n\in\mathbb{Z}}$$

Both systems have density 1. **Why?**

Beurling Density for $\Lambda \subseteq \mathbb{R}^2$

Let $Q_r(z)$ be the square centered at z with side lengths r.

Lower Beurling density: $D^-(\Lambda) \;=\; \liminf_{r\to\infty} \inf_{z\in\mathbb{R}^2} \dfrac{\#(\Lambda \cap Q_r(z))}{r^2}$

Upper Beurling density: $D^+(\Lambda) \;=\; \limsup_{r\to\infty} \sup_{z\in\mathbb{R}^2} \dfrac{\#(\Lambda \cap Q_r(z))}{r^2}$

Examples: $D^{\pm}(\alpha\mathbb{Z}\times\beta\mathbb{Z}) = \dfrac{1}{\alpha\beta}$, $\quad D^-(\alpha\mathbb{Z}\times\beta\mathbb{Z}^+) = 0$, $\quad D^+(\alpha\mathbb{Z}\times\beta\mathbb{Z}^+) = \dfrac{1}{\alpha\beta}$

<u>Conjecture</u>

A Gabor system $\{M_bT_ag\}_{(a,b)\in\Lambda}$ is complete $\implies D^-(\Lambda) \geq 1$

Beurling Density for $\Lambda \subseteq \mathbb{R}^2$

Let $Q_r(z)$ be the square centered at z with side lengths r.

Lower Beurling density: $D^-(\Lambda) \;=\; \liminf_{r\to\infty} \inf_{z\in\mathbb{R}^2} \dfrac{\#(\Lambda \cap Q_r(z))}{r^2}$

Upper Beurling density: $D^+(\Lambda) \;=\; \limsup_{r\to\infty} \sup_{z\in\mathbb{R}^2} \dfrac{\#(\Lambda \cap Q_r(z))}{r^2}$

Examples: $D^{\pm}(\alpha\mathbb{Z}\times\beta\mathbb{Z}) = \dfrac{1}{\alpha\beta}, \quad D^-(\alpha\mathbb{Z}\times\beta\mathbb{Z}^+) = 0, \quad D^+(\alpha\mathbb{Z}\times\beta\mathbb{Z}^+) = \dfrac{1}{\alpha\beta}$

Conjecture

A Gabor system $\{M_bT_ag\}_{(a,b)\in\Lambda}$ is complete $\implies D^-(\Lambda) \geq 1$

H./Walnut, 1995 (builds on H. Landau, 1964)

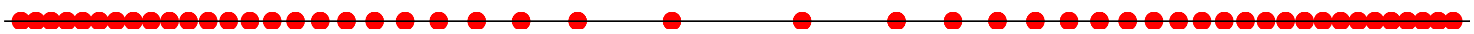
False: $\exists$ complete Gabor system with $D^+(\Lambda) < \varepsilon$

Translates of Gaussians (Zalik, 1978) $\phi(x) = e^{-x^2}$

$\exists\, \Gamma \subseteq \mathbb{R}$ **such that** $\{T_a\phi\}_{a\in\Gamma}$ **is complete**

Complete in $L^2(\mathbb{R})$, **not just a subspace like PW!**

For this example, $D^+(\Gamma) = \infty$ **as a subset of** $\mathbb{R}$

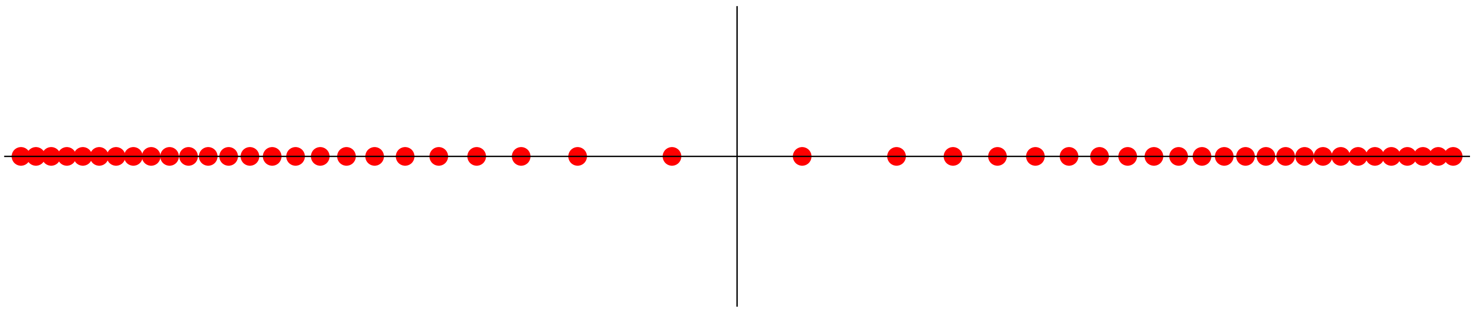


$\mathbb{R}$

Complete Gabor system with no modulations

$\{T_a\phi\}_{a\in\Gamma}$ **is a Gabor system** $\{M_bT_a\phi\}_{(a,b)\in\Lambda}$ **with** $\Lambda = \Gamma \times \{0\}$

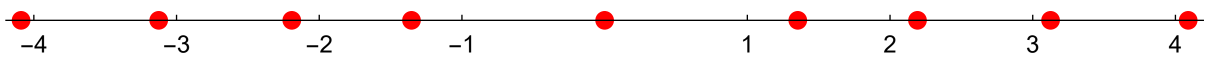
Complete, yet $D^-(\Lambda) = 0$ **and** $D^+(\Lambda) = \infty$ **as a subset of** $\mathbb{R}^2$



$\mathbb{R}^2$

__Olevskii (1997) and Olevskii/Ulanovskii (2004)__

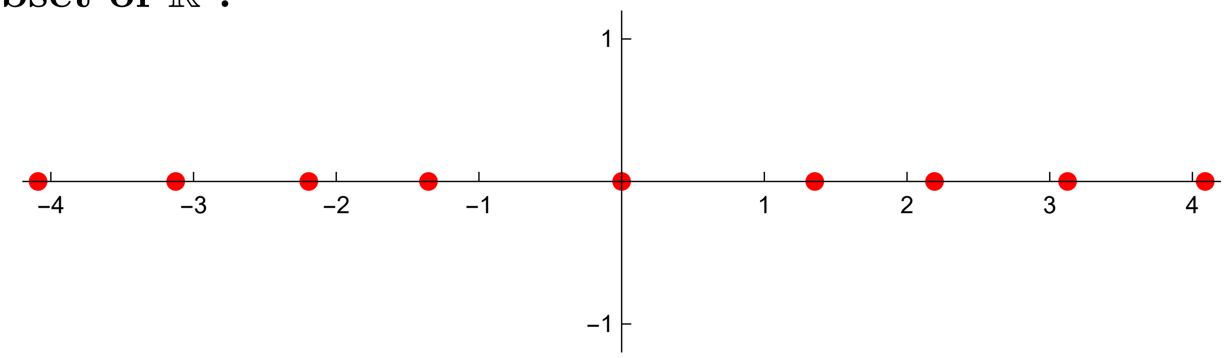
$\exists$ **(nice!)** g **and** Γ **a (small!) perturbation of** $\mathbb{Z}$ **such that** $\{T_a\phi\}_{a\in\Gamma}$ **is complete in** $L^2(\mathbb{R})$. **Density:** $D^{\pm}(\Gamma) = 1$ **as a subset of** $\mathbb{R}$**:**



$\mathbb{R}$

__As a Gabor System__

$\{T_a\phi\}_{a\in\Gamma}$ **is a Gabor system with** $\Lambda = \Gamma \times \{0\}$. **Complete, but** $D^{\pm}(\Lambda) = 0$ **as a subset of** $\mathbb{R}^2$**:**



$\mathbb{R}^2$

But how nice can these systems be? Frames? Bases?

Olson–Zalik Conjecture (1992)

No set of translates $\{T_a g\}_{a\in\Gamma}$ can be a Schauder basis for $L^2(\mathbb{R})$

Results in this direction:

- **Olson/Zalik: $\{T_a g\}_{a\in\Gamma}$ cannot be a Riesz basis for $L^2(\mathbb{R})$**

- **Christensen/Deng/H. (1999):**

  $\{M_b T_a g\}_{(a,b)\in\Lambda}$ **is a frame for** $L^2(\mathbb{R}) \implies D^-(\Lambda) \geq 1$

  Corollary: $\{T_a g\}_{a\in\Gamma}$ cannot be a frame for $L^2(\mathbb{R})$

Definition: $\{T_a g\}_{a\in\Gamma}$ is a Schauder basis if there is some enumeration $\Gamma = \{a_k\}_{k\in\mathbb{N}}$ such that

$$f \;=\; \sum_{k=1}^{\infty} c_k(f)\, T_{a_k} g \ \textbf{uniquely}, \qquad \textbf{for all } f \in L^2(\mathbb{R})$$

Olson–Zalik Conjecture (1992)

No set of translates $\{T_a g\}_{a\in\Gamma}$ **can be a Schauder basis for** $L^2(\mathbb{R})$

<u>**Results in this direction:**</u>

- **Olson/Zalik:** $\{T_a g\}_{a\in\Gamma}$ **cannot be a Riesz basis for** $L^2(\mathbb{R})$

- **Christensen/Deng/H. (1999):**

  $\{M_b T_a g\}_{(a,b)\in\Lambda}$ **is a frame for** $L^2(\mathbb{R}) \implies D^-(\Lambda) \geq 1$

  Corollary: $\{T_a g\}_{a\in\Gamma}$ **cannot be a frame for** $L^2(\mathbb{R})$

- **Deng/H. (2000):**

  $\{M_b T_a g\}_{(a,b)\in\Lambda}$ **is a Schauder basis for** $L^2(\mathbb{R}) \implies D^+(\Lambda) \leq 1$

- **Deng/H. (2000):**

  $g \in L^2 \backslash L^1 \implies \{T_a g\}_{(a,b)\in\Gamma}$ **not a Schauder basis**

Olson–Zalik Conjecture (1992)

No set of translates $\{T_a g\}_{a\in\Gamma}$ can be a Schauder basis for $L^2(\mathbb{R})$

The full Olson–Zalik conjecture is still open!

Conjecture: Density of Gabor Schauder bases (Deng/H.)

If $\{M_b T_a g\}_{(a,b)\in\Lambda}$ is a Schauder basis for $L^2(\mathbb{R})$, then $D^{\pm}(\Lambda) = 1$

Olson–Zalik is a Corollary if true:

$\{T_a g\}_{a\in\Gamma}$ can never be a Schauder basis for $L^2(\mathbb{R})$, because $\Lambda = \Gamma \times \{0\} \subseteq \mathbb{R}^2$ has density $D^{-}(\Lambda) = 0$

Gabor Schauder bases do exist:

If $g(x) = |x|^{-1/4}\,\chi_{[-\frac{1}{2},\frac{1}{2}]}$ **then** $\{M_n T_k g\}_{k,n\in\mathbb{Z}}$ **is a Gabor Schauder basis that is not a Riesz basis**

H./Powell (2006)

$$\{M_n T_k g\}_{k,n\in\mathbb{Z}} \text{ is Schauder basis for } L^2(\mathbb{R}) \iff |Zg|^2 \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$$

Product $\mathcal{A}_2$ weights

$w \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$ **if** $\forall$ **intervals** $I,\ J \subseteq \mathbb{R}$,

$$\left(\frac{1}{|I|\,|J|}\int_J\int_I w(x,y)\,dx\,dy\right)\left(\frac{1}{|I|\,|J|}\int_I\int_J \frac{1}{w(x,y)}\,dx\,dy\right) \le C.$$

Conditionality

The Schauder basis condition is with respect to a particular class of enumerations of $\mathbb{Z}^2$.

What about exact Gabor systems? Exact = minimal and complete

Exact is "slightly less" than a Schauder basis

<u>**Example from before**</u> **with** $\phi(x) = e^{-x^2}$

$\{M_n T_k \phi\}_{(k,n)\neq(0,0)}$ **is exact;** $\Lambda = \mathbb{Z}^2 \setminus \{(0,0)\}$ **has density** $D^{\pm}(\Lambda) = 1$

<u>**Ascensi/Lyubarskii/Seip (2008)**</u> **with** $\phi(x) = e^{-x^2}$

If $\Lambda = \Big\{(-1,0),\ (1,0),\ (0,\pm\sqrt{2n}),\ (\pm\sqrt{2n},0)\Big\}$ **then** $\{M_b T_a \phi\}_{(a,b)\in\Lambda}$ **is exact**

Density: $D^-(\Lambda) = 0$**,** $D^+(\Lambda) = \infty$**.**

<u>**Used in the proof**</u>**:**

Λ **is the zero set of the entire function** $s(z) = \dfrac{z^2-1}{z^2}\sin\dfrac{\pi z^2}{2}.$

<u>**Question**</u>

Does there exist a set of translations $\{T_a g\}_{a\in\Gamma}$ **that is exact in** $L^2(\mathbb{R})$**?**

Translations Again

Translations $\{T_a g\}_{a\in\Gamma}$ are finitely linearly independent in $L^2(\mathbb{R})$

Same is true in $L^p(\mathbb{R})$ if $1 \le p \le 2$ via the Fourier transform:

$$\sum_{k=1}^{N} c_k g(x-a_k) = 0 \implies \left(\sum_{k=1}^{N} c_k e^{2\pi i\xi a_k}\right)\widehat{g}(\xi) = 0 \implies \widehat{g} = 0.$$

But if $p > 2$ then $\widehat{g}$ is a distribution:

$$\sum_{k=1}^{N} c_k g(x-a_k) = 0 \implies \varphi\,\widehat{g} = 0 \not\Longrightarrow \widehat{g} = 0$$

Distributions can be supported at a single point, functions cannot.

$$\textbf{Q.}\ \exists\, g \in L^p(\mathbb{R}) \quad \textbf{s.t.} \quad \left(\sum_{k=1}^{N} c_k e^{2\pi i \xi a_k}\right) \widehat{g}(\xi) = 0?$$

Theorem (Edgar/Rosenblatt, 1979)

Assume $S \subseteq \mathbb{R}^n$ **is closed and has Hausdorff dimension** $\leq n-1$. **If** $g \in L^p(\mathbb{R}^n)$ **and supp**$(\widehat{g}) \subseteq S$, **then** $p \geq 2n/(n-1)$.

Corollary

If $g \in L^p(\mathbb{R}^n)$ **and** $p < 2n/(n-1)$, **then translates of** g **are linearly independent.**

Proof sketch

The zero set of a trigonometric polynomial φ **on** $\mathbb{R}^n$ **is the intersection of an analytic variety in** $\mathbb{C}^n$ **with** $\mathbb{R}^n$. **The Hausdorff dimension of this zero set is** $\leq n-1$.

Corollary

If $g \in L^p(\mathbb{R}^n)$ **and** $p < 2n/(n-1)$, **then translates of** g **are linearly independent. (Rosenblatt, 1995, added** $p = 2n/(n-1)$**)**

Theorem (Edgar/Rosenblatt, 1979)

If $2n/(n-1) < p < \infty$, **then there exists a function** $g \in L^p(\mathbb{R}^n)$ **that has linearly dependent translates.**

Example

Translates are independent in $L^1(\mathbb{R})$ **for** $1 \leq p < \infty$,

but translates are independent in $L^2(\mathbb{R}^2)$ **for** $1 \leq p \leq 4$.

The HRT Conjecture (1996)

Time-frequency translates are finitely linearly independent. That is, if:

(a) $0 < \int_{-\infty}^{\infty} |g(x)|^2 \, dx < \infty$**,**

(b) $\{(a_k, b_k)\}_{k=1}^{N}$ **are finitely many distinct points in** $\mathbb{R}^2$**,**

then:

$$\sum_{k=1}^{N} c_k \, e^{2\pi i b_k x} \, g(x - a_k) \;=\; 0 \qquad \Longrightarrow \qquad c_1 = \cdots = c_N = 0.$$

This conjecture is open in the generality stated, even if we require g **to be continuous (or smoother).**

HRT Subconjecture

If $g\colon \mathbb{R} \to \mathbb{R}$ **is continuous and** $0 \; < \; \int_{-\infty}^{\infty} |g(x)|^2 \, dx \; < \; \infty$**, then**

$$c_1 \, g(x) \; + \; c_2 \, g(x-1) \; + \; c_3 \, g(x) e^{2\pi i x} \; + \; c_4 \, g(x-\sqrt{2}) e^{2\pi i \sqrt{3} x} \; = \; 0$$

implies

$$c_1 = c_2 = c_3 = c_4 = 0.$$

That is,

$$\Big\{ g(x), \; g(x-1), \; g(x) e^{2\pi i x}, \; g(x-\sqrt{2}) \, e^{2\pi i \sqrt{3} x} \Big\}$$

is linearly independent.

This is open even if we assume that g **is Schwartz-class.**

Frames (Duffin and Schaeffer, 1952)

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space H is a frame if $\exists\, A, B > 0$ such that

$$\forall\, x \in H, \quad A\,\|x\|^2 \leq \sum_n |\langle x, x_n\rangle|^2 \leq B\,\|x\|^2$$

Quote from H., "What is a Frame?"

Sadly, Duffin and Schaeffer both passed away before anyone thought to ask why they called such a system a "frame". Is it because $A\,\|f\|^2$ and $B\,\|f\|^2$ "frame" the sum between them? We will never know.

Frames (Duffin and Schaeffer, 1952)

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space H is a frame if $\exists\, A, B > 0$ such that

$$\forall\, x \in H, \quad A\,\|x\|^2 \leq \sum_n |\langle x, x_n\rangle|^2 \leq B\,\|x\|^2$$

<u>Quote from H., "What is a Frame?"</u>

Sadly, Duffin and Schaeffer both passed away before anyone thought to ask why they called such a system a "frame". Is it because $A\,\|f\|^2$ and $B\,\|f\|^2$ "frame" the sum between them? We will never know.

But there's more to the story . . .

Mathematics Genealogy Project

Richard James Duffin

MathSciNet

Ph.D. University of Illinois at Urbana-Champaign 1935

Dissertation: *Galvanomagnetic and Thermomagnetic Phenomena*

Advisor 1: Harold Meade Mott-Smith

Advisor 2: David Gordon Bourgin

Students:

Click here to see the students ordered by family name.

Name	School	Year	Descendants
Bott, Raoul	Carnegie Mellon University	1949	1241
Weinberger, Hans	Carnegie Mellon University	1950	27
Boyce, Elsa	Carnegie Mellon University	1954	
Rogers, Edwin	Carnegie Mellon University	1962	14
Duris, Charles	Carnegie Mellon University	1963	1
Karlovitz, Les	Carnegie Mellon University	1964	1
Mangrad, Moshe	Carnegie Mellon University	1964	
Patrick, Merrell	Carnegie Mellon University	1964	2
Peterson, Elmor	Carnegie Mellon University	1964	43
Porsching, Thomas	Carnegie Mellon University	1964	14
Winter, Mary	Carnegie Mellon University	1966	2
Bhargava, Srinivasamurthy	Carnegie Mellon University	1972	14
Morley, Thomas	Carnegie Mellon University	1976	5
Buckwalter, Jeff	Carnegie Mellon University	1978	

According to our current on-line database, Richard Duffin has 14 students and 1378 descendants.

THANK YOU.