Real equiangular line systems in low dimensions

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April 2021

A set of lines through the origin of $\mathbb{R}^d$ is called **equiangular** if the angle between any pair of lines is constant

Example: $\mathbb{R}^2$

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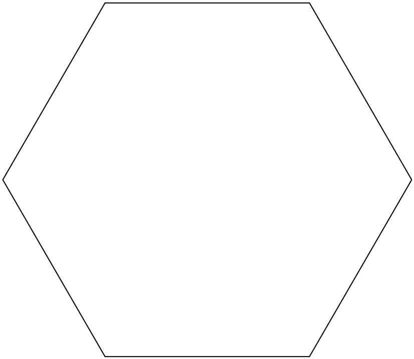

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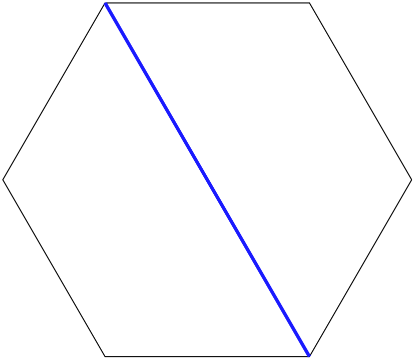

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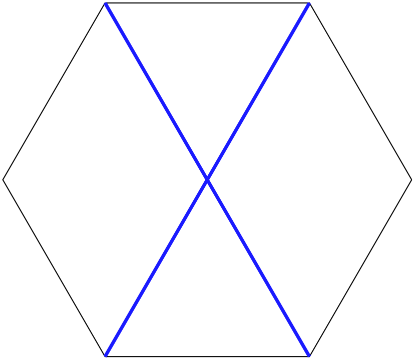

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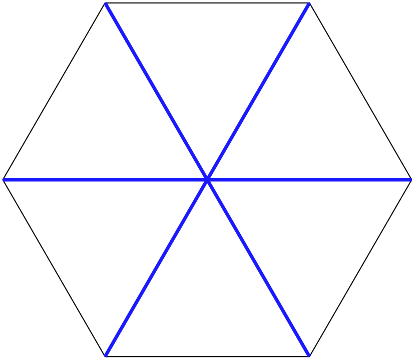

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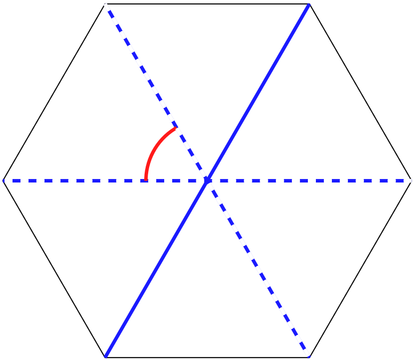

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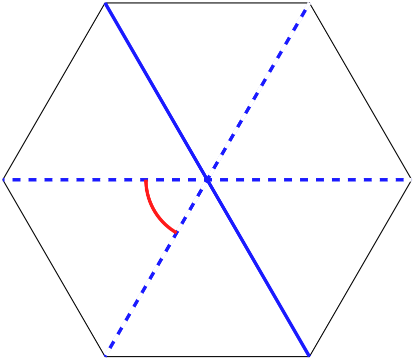

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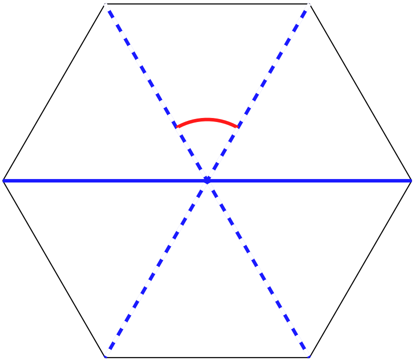

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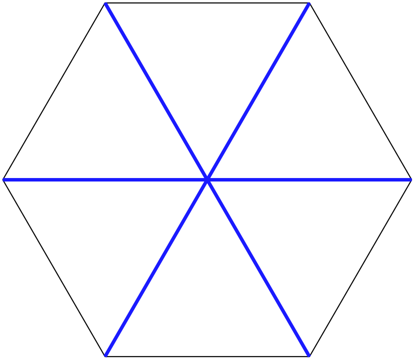

These three lines are equiangular

Non-example: $\mathbb{R}^2$

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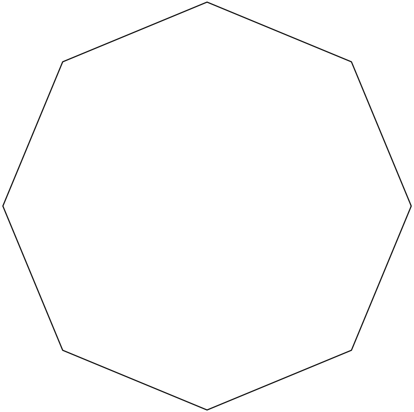

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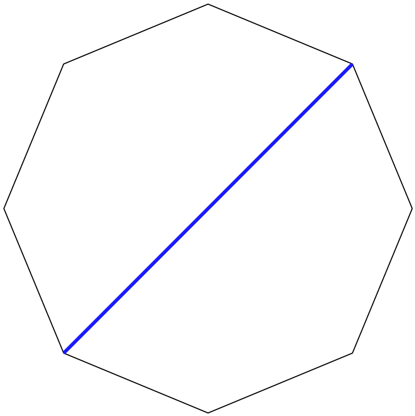

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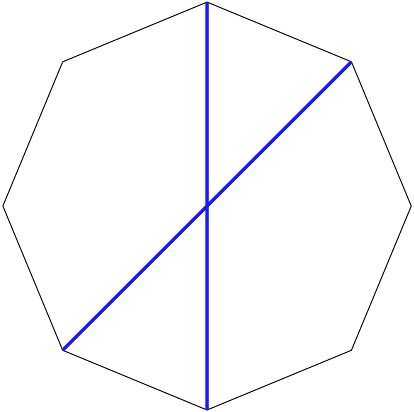

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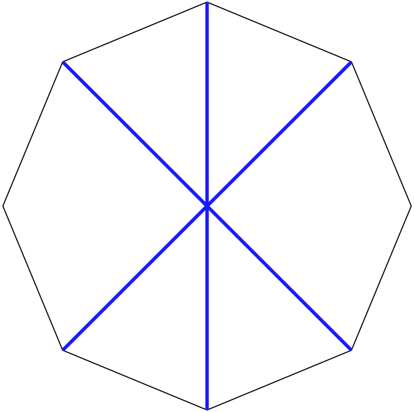

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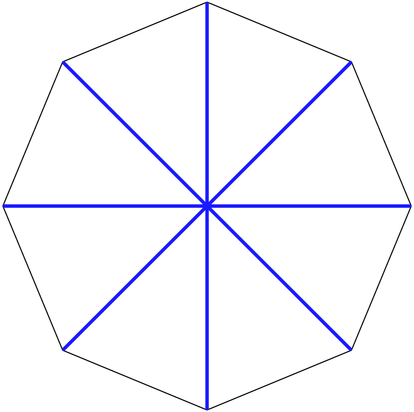

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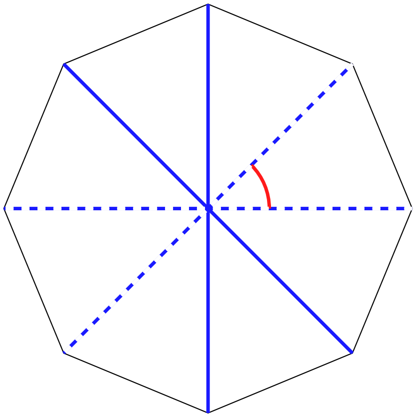

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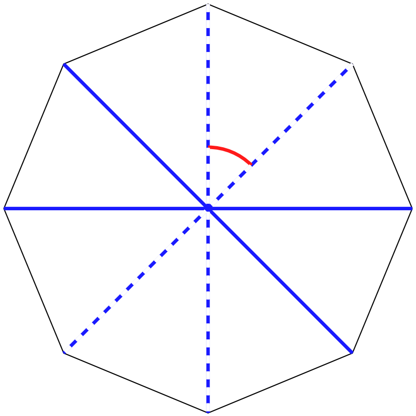

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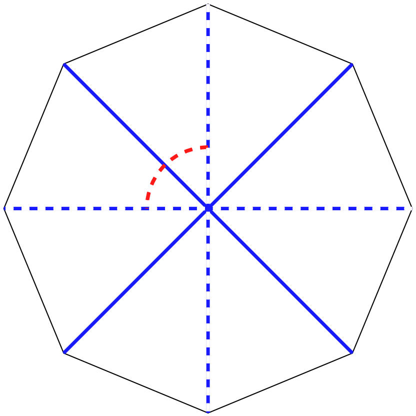

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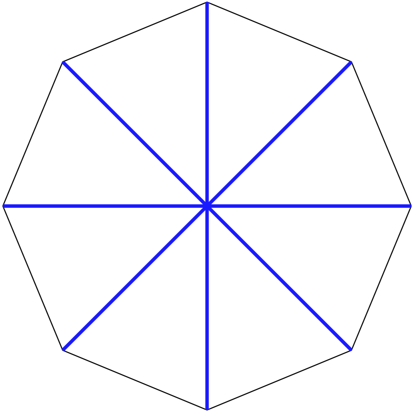
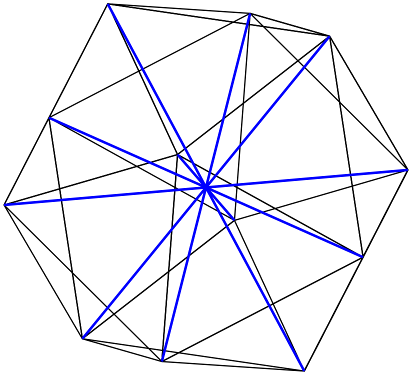


These four lines are **not** equiangular

Example: $\mathbb{R}^3$

A set of lines through the origin of $\mathbb{R}^d$ is called **equiangular** if the angle between any pair of lines is constant



These six lines are equiangular

Main problem and history

Problem: given d, what is the largest possible size $N(d)$ of an equiangular line system in $\mathbb{R}^d$?

- Haantjes (1948): $N(2) = 3$ and $N(3) = N(4) = 6$
- van Lint and Seidel (1966): $N(5) = 10$, $N(6) = 16$, and $N(7) \geqslant 28$
- Lemmens and Seidel (1973): $N(7) = \cdots = N(13) = 28$, $N(15) = 36$, $N(21) = 126$, $N(22) = 176$, $N(23) = 276$

Recent work

Okuda-Yu (2016), King-Tang (2019), Glazyrin-Yu (2018), de Laat-Machado-de Oliveira Filho-Vallentin: semidefinite programming upper bounds;

Bukh (2016), Jiang-Polyanski (2017), Balla-Dräxler-Keevash-Sudakov (2018), Jiang-Tidor-Yao-Zhang-Zhao: asymptotics for fixed angle;

Szöllősi (2017), Lin-Yu (2020), GG-Syatriadi-Yatsyna: lower bounds for $N(18)$;

GG-Koolen-Munemasa-Szöllősi (2016), GG (2018), Azarija-Marc (2018), GG-Syatriadi-Yatsyna: upper bounds for $N(14)$, $N(16)$, $N(17)$, $N(18)$, $N(19)$, and $N(20)$;

Szöllősi-Östergård (2018), Lin-Yu (2020), Gillespie, Cao-Koolen-Lin-Yu: miscellaneous.

State of the art

$N(d)$: largest possible size of an equiangular line system in $\mathbb{R}^d$.

d	2	3	4	5	6	7 – 14	15	16	17	18	19	20	21	22	23 – 41
$N(d)$	3	6	6	10	16	28	36	40	48	57 60	72 74	90 94	126	176	276

Plan

- **Part I:** Constructions of equiangular lines and $N(18) \geqslant 57$

- **Part II:** Tools to show that $N(14) = 28$, $N(16) = 40$, and $N(17) = 48$

GG, J. Syatriadi, P. Yatsyna, Combinatorica (to appear)
GG, J. Syatriadi, P. Yatsyna, arXiv:2104.04330

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From lines to matrices

Equiangular lines $l_1, \ldots, l_n$ — angle: $\alpha > 0$

$\Updownarrow$

Unit spanning vectors $\mathbf{v}_i \; : \; l_i = \langle \mathbf{v}_i \rangle$ — $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \pm\alpha$

$\Updownarrow$

Gram matrix $M = (\langle \mathbf{v}_i, \mathbf{v}_j \rangle)_{ij}$

$$\begin{pmatrix} 1 & \pm\alpha & \pm\alpha \\ \pm\alpha & 1 & \pm\alpha \\ \pm\alpha & \pm\alpha & 1 \end{pmatrix}$$

$\Updownarrow$

Seidel matrix $S = \dfrac{(M - I)}{\alpha}$

$$\begin{pmatrix} 0 & \pm1 & \pm1 \\ \pm1 & 0 & \pm1 \\ \pm1 & \pm1 & 0 \end{pmatrix}$$

Multiplicity of the smallest eigenvalue

Unit vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathbb{R}^d$

n vectors

$$\Updownarrow$$

$$B = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

rank $= d$

$$\Updownarrow$$

Gram matrix $M = B^\top B$

smallest eigenvalue $[0]^{n-d}$

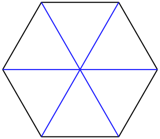
$$\Updownarrow$$

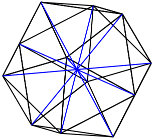
Seidel matrix $S = \dfrac{(M - I)}{\alpha}$

smallest eigenvalue $\left[\dfrac{-1}{\alpha}\right]^{n-d}$

Absolute bound $\mathbb{R}^d$: $$N(d) \leqslant \frac{d(d+1)}{2}.$$

- Equality achieved when $d = 2, 3, 7, 23$
- Widely believed to be complete list

$\mathbb{R}^2$: 

$\mathbb{R}^3$: 

$\mathbb{R}^7$: 28 equiangular lines spanned by all permutations of $\langle 1,1,1,1,1,1,-3,-3 \rangle$.

$\mathbb{R}^{23}$: 276 equiangular lines obtained from the 253 blocks and 23 points of the Witt design: 4-$(23,7,1)$ design.

Inside the Witt design

Many of the largest known constructions of equiangular lines are found inside the 23-dimensional 276 equiangular lines:

$\mathbb{R}^{23}$: 276 equiangular lines. Lemmens-Seidel (1973)
$\mathbb{R}^{22}$: 176 equiangular lines. Lemmens-Seidel (1973)
$\mathbb{R}^{21}$: 126 equiangular lines. Lemmens-Seidel (1973)
$\mathbb{R}^{20}$: 90 equiangular lines. Taylor (1973)
$\mathbb{R}^{19}$: 72 equiangular lines. Asche (1973)
$\mathbb{R}^{18}$: 56 equiangular lines. Lin-Yu (2020)
$\mathbb{R}^{17}$: 48 equiangular lines. Lemmens-Seidel (1973)

d	2	3	4	5	6	7 – 14	15	16	17	18	19	20	21	22	23–41
$N(d)$	3	6	6	10	16	28	36	40	48	57 60	72 74	90 94	126	176	276

A new construction

Consider subsets $\mathcal{S} \subset \mathbb{Z}^{18}$ such that

- $\langle \mathbf{v}, \mathbf{v} \rangle = 10$ for all $\mathbf{v} \in \mathcal{S}$;
- $\langle \mathbf{v}, \mathbf{w} \rangle = \pm 2$ for all $\mathbf{v} \neq \mathbf{w} \in \mathcal{S}$.

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\end{array}\right]$$

Seidel matrix: $S = (B^\top B - 10I)/2$. B is the matrix above

Characteristic polynomial $\chi_S(x)$:
$(x+5)^{39}(x-4)(x-7)(x-9)^2(x-11)^9(x-13)^4(x-15)$

Asche's lines

Let $\mathcal{A}$ be the Seidel matrix of Asche's construction of 72 equiangular lines in $\mathbb{R}^{19}$.

The Seidel matrices of the constructions of 54 Szöllősi (2017) and 56 Lin-Yu (2020) equiangular lines in $\mathbb{R}^{18}$ can both be obtained as principal submatrices of $\mathcal{A}$.

Char poly: $\chi_{\mathcal{A}}(x) = (x+5)^{53}(x-13)^{16}(x-19)^3$

Interlacing: any principal submatrix of $\mathcal{A}$ of order 57 must have 13 as an eigenvalue.

New construction II (not inside Asche)

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+&-&-&0&0&0&0&0&0&+&0&0&0&0&0&0&0&-&-&-&0&+&+&+&0&-&-&+&+&0&-&-&-&-&0&-&+&-&0&-&0&0&-&+&0&-&-&0&-&0&+&0&+&-&0&+&-\\
+&-&0&-&0&+&-&+&-&0&-&+&0&0&0&+&0&+&0&0&0&0&+&-&0&0&0&-&0&0&0&+&-&0&+&-&-&0&+&0&0&-&0&+&-&0&-&0&+&+&-&-&-&+&-&0&-
\end{array}\right]
$$

Seidel matrix: $T = (B^\top B - 10I)/2$. B is the matrix above

Characteristic polynomial $\chi_T(x)$:
$(x+5)^{39}(x-11)^{11}(x-15)(x^2-19x+76)(x^2-20x+87)^2$

Part II: Tools to show that $N(14) = 28$, $N(16) = 40$, and $N(17) = 48$


GG, J. Syatriadi, P. Yatsyna, Combinatorica (to appear)
GG, J. Syatriadi, P. Yatsyna, arXiv:2104.04330

Candidate characteristic polynomials

S: Seidel matrix; $\phi_d(x)$: polynomial in $\mathbb{Z}[x]$ with degree d.

If S corresponds to 29 equiangular lines in $\mathbb{R}^{14}$ then

$$\chi_S(x) = (x+5)^{15}(x-5)^4\phi_{10}(x).$$

If S corresponds to 41 equiangular lines in $\mathbb{R}^{16}$ then

$$\chi_S(x) = (x+5)^{25}(x-7)^3\phi_{13}(x).$$

If S corresponds to 49 equiangular lines in $\mathbb{R}^{17}$ then

$$\chi_S(x) = (x+5)^{32}(x-9)^4\phi_{13}(x).$$

Modular restrictions on coefficients of $\chi_S(x)$

S: Seidel matrix of order n.

(**weakly**) (2^{i-1})

$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ is **type 2** if 2^i divides a_i for each $i \geqslant 1$.

$\chi_S(x-1)$ is weakly type 2 and type 2 if n is even.

Top coefficients: $\chi_S(x) = x^n + 0x^{n-1} - \binom{n}{2}x^{n-2} + \ldots$

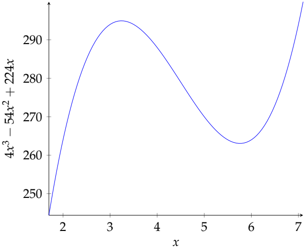
McKee-Smyth-Robinson algorithm: efficiently computes all totally-real polynomials in $\mathbb{Z}[x]$ with fixed top coefficients.

candidate char polys $\subset$ McKee-Smyth-Robinson + weakly type 2

Example: Robinson algorithm + type 2

Find all totally-real, type-2 $f(x) = x^4 - 18x^3 + 112x^2 + a_3x + a_4$

Derivative $f'(x) = 4x^3 - 54x^2 + 224x + a_3$ must also be totally real.



$$\implies a_3 \in \{-294, \dots, -264\}.$$

Since 8 divides a_3, we must have $a_3 \in \{-288, -280, -272, -264\}$.

Apply the same idea to the resulting four polynomials to obtain

$$\begin{gathered} x^4 - 18x^3 + 112x^2 - 288x + 256 \\ x^4 - 18x^3 + 112x^2 - 280x + 224 \\ x^4 - 18x^3 + 112x^2 - 280x + 240 \\ x^4 - 18x^3 + 112x^2 - 272x + 192 \\ x^4 - 18x^3 + 112x^2 - 264x + 144 \end{gathered}$$

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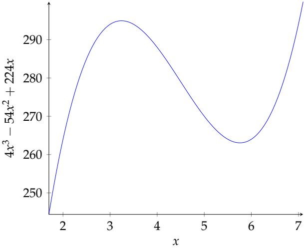

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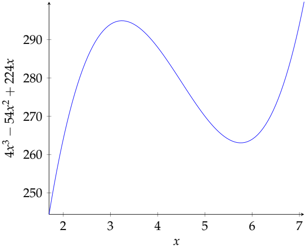

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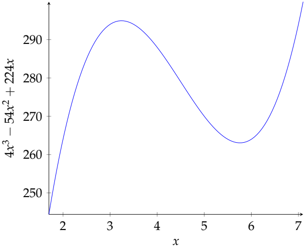

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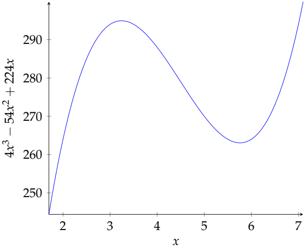

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$$

Candidate characteristic polynomials

S: Seidel matrix.

If S corresponds to 29 equiangular lines in $\mathbb{R}^{14}$ then S is one of **31** candidate char polys. *comp. time: 1 minute*

If S corresponds to 41 equiangular lines in $\mathbb{R}^{16}$ then S is one of **22** candidate char polys. *comp. time: 5 minutes*

If S corresponds to 49 equiangular lines in $\mathbb{R}^{17}$ then S is one of **194** candidate char polys. *comp. time: 2.5 hours*

S: Seidel matrix of order n odd;
$S[i]$: matrix obtained by deleting ith row and column of S.

$$\textbf{Key identity:} \quad \sum_{i=1}^{n} \chi_{S[i]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\chi_S(x) \tag{1}$$

Strategy: For each candidate char poly $p(x)$ find all potential solutions to (1):

$\mathrm{Deck}(p)$: set of $\mathfrak{f}(x) \in \mathbb{Z}[x]$ s.t.

- $\mathfrak{f}(x) = x^{n-1} - \binom{n-1}{2}x^{n-3} + \dots$ interlaces $p(x)$
- $\mathfrak{f}(x-1)$ is type 2

Search for solutions (**interlacing configurations**) to

$$\sum_{\mathfrak{f}(x) \in \mathrm{Deck}(p)} n_{\mathfrak{f}}\, \mathfrak{f}(x) = \frac{\mathrm{d}}{\mathrm{d}x}p(x); \qquad n_{\mathfrak{f}} \in \mathbb{N} \cup \{0\}. \tag{2}$$

Example: finding interlacing configurations

For $p(x) = x^7 - 21x^5 + 26x^4 + 71x^3 - 132x^2 + 45x + 10$, **find all** interlacing configurations.

$$\mathrm{Deck}(p) = \left\{ \begin{array}{l} \mathfrak{f}_1(x) = x^6 - 15x^4 + 8x^3 + 51x^2 - 40x - 5 \\ \mathfrak{f}_2(x) = x^6 - 15x^4 + 16x^3 + 27x^2 - 48x + 19 \\ \mathfrak{f}_3(x) = x^6 - 15x^4 + 16x^3 + 43x^2 - 80x + 35 \\ \mathfrak{f}_4(x) = x^6 - 15x^4 + 24x^3 + 3x^2 - 24x + 11 \end{array} \right\}$$

interlacing configurations: $3\mathfrak{f}_1(x) + 2\mathfrak{f}_2(x) + 2\mathfrak{f}_4(x) = \frac{\mathrm{d}}{\mathrm{d}x}p(x)$

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$$\chi_S(x) = p(x)$$

Candidate char polys interlacing configurations

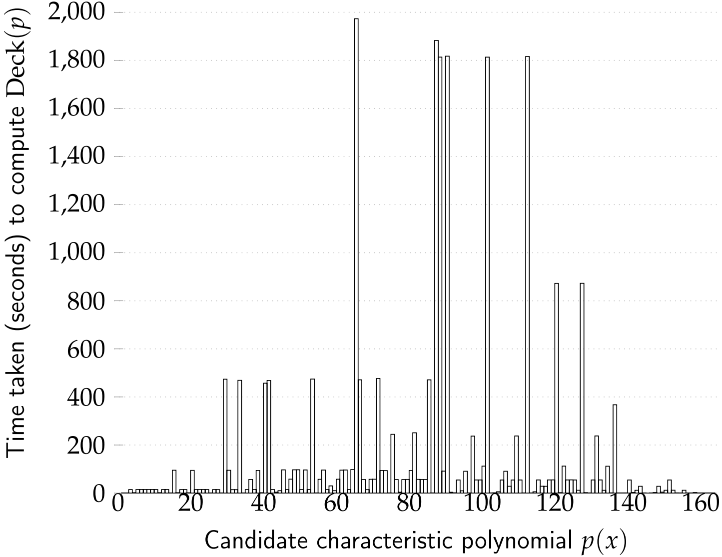
Using (2) we can prune our lists of candidate char polys:

For 29 lines in $\mathbb{R}^{14}$ only **6** out of **31** candidate char polys have interlacing configurations. *comp. time: 1 minute*

For 41 lines in $\mathbb{R}^{16}$ only **2** out of **22** candidate char polys have interlacing configurations. *comp. time: 4 minutes*

For 49 lines in $\mathbb{R}^{17}$ only **28** out of **194** candidate char polys have interlacing configurations. *comp. time: 6.5 hours*

For 166 candidate char polys for 49 lines in $\mathbb{R}^{17}$:



Warranted polynomials

$p(x)$: candidate char poly of degree n;
$\mathrm{Deck}(p)$: set of $\mathfrak{f}(x) \in \mathbb{Z}[x]$ s.t.

- $\mathfrak{f}(x) = x^{n-1} - \binom{n-1}{2}x^{n-3} + \dots$ interlaces $p(x)$
- $\mathfrak{f}(x-1)$ is type 2

$\mathfrak{g}(x) \in \mathrm{Deck}(p)$ is called **warranted** if $n_{\mathfrak{g}} > 0$ for every solution to

$$\sum_{\mathfrak{f}(x) \in \mathrm{Deck}(p)} n_{\mathfrak{f}}\, \mathfrak{f}(x) = \frac{\mathrm{d}}{\mathrm{d}x} p(x); \qquad n_{\mathfrak{f}} \in \mathbb{N} \cup \{0\}.$$

If there is a warranted polynomial in $\mathfrak{g}(x) \in \mathrm{Deck}(p)$ we can reduce to polynomials that are *compatible* with $\mathfrak{g}(x)$.

Applying warranty and compatibility I

Using warranted polynomials and compatibility, we can prune our lists further.

29 lines in $\mathbb{R}^{14}$: **3** out of the **6** remaining candidate char polys have two incompatible warranted polynomials.

41 lines in $\mathbb{R}^{16}$: **2** out of the **2** remaining candidate char polys have two incompatible warranted polynomials.

49 lines in $\mathbb{R}^{17}$: **11** out of the **28** remaining candidate char polys have two incompatible warranted polynomials.

Applying warranty and compatibility II

Using warranted polynomials and compatibility, we can prune our lists even further.

29 lines in $\mathbb{R}^{14}$: **1** out of the **3** remaining candidate char polys has a warranted polynomial $g(x)$ for which there is no interlacing configuration on the subset of $\mathrm{Deck}(p)$ that is compatible with $g(x)$.

49 lines in $\mathbb{R}^{17}$: **8** out of the **17** remaining candidate char polys have a warranted polynomial $g(x)$ for which there is no interlacing configuration on the subset of $\mathrm{Deck}(p)$ that is compatible with $g(x)$.

Compatibility: motivation

M: integer symmetric matrix of order n; $\mathbf{u}_\lambda$: unit λ-eigenvector;
$\Sigma(M)$: set of simple eigenvalues of M; $\Lambda_M(x)$: min poly of M;

$$\Sigma_M(x) := \prod_{\lambda \in \Sigma(M)} (x - \lambda); \qquad q(x) = \frac{\Lambda_M(x)}{\Sigma_M(x)}.$$

Spectral Decomposition Theorem:

$$q(M) = \sum_{\lambda \in \Sigma(M)} q(\lambda) \mathbf{u}_\lambda \mathbf{u}_\lambda^\top.$$

- The above matrix is an integer matrix.
- For each $\lambda \in \Sigma(M)$, up to sign, the ith entry of $\mathbf{u}_\lambda$ is determined by $\chi_{M[i]}(x)$.
- Gives us a **compatibility** condition for polys in $\mathrm{Deck}(p)$.

Compatibility: definition

M: integer symmetric matrix of order n; $\mathbf{u}_\lambda$: unit λ-eigenvector;
$\Sigma(M)$: set of simple eigenvalues of M; $\Lambda_M(x)$: min poly of M;

$$\Sigma_M(x) := \prod_{\lambda \in \Sigma(M)} (x - \lambda); \qquad q(x) = \frac{\Lambda_M(x)}{\Sigma_M(x)}.$$

Let $\mathfrak{f}(x) = \dfrac{\chi_M(x)}{\Lambda_M(x)} f(x) \in \mathrm{Deck}(\chi_M)$; **Angle**: $\alpha_\lambda(\mathfrak{f}) := \sqrt{\dfrac{f(\lambda)}{\Lambda'_M(\lambda)}}$

For $\lambda \in \Sigma(M)$ we have $\mathbf{u}_\lambda(i) = \pm\alpha_\lambda(\chi_{M[i]})$.
Thus,

$$(i,j)\text{-entry of } q(M) = \sum_{\lambda \in \Sigma(M)} \pm q(\lambda)\alpha_\lambda(\chi_{M[i]})\alpha_\lambda(\chi_{M[j]}).$$

Compatibility: definition

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Thus,

$$(i,j)\text{-entry of } q(M) = \sum_{\lambda\in\Sigma(M)} \pm q(\lambda)\alpha_\lambda(\chi_{M[i]})\alpha_\lambda(\chi_{M[j]}).$$

$\mathfrak{f}(x), \mathfrak{g}(x) \in \mathrm{Deck}(\chi_M)$ are **compatible** if $\exists\delta \in \{\pm 1\}^{\Sigma(M)}$ s.t.

$$\sum_{\lambda\in\Sigma(M)} q(\lambda)\delta(\lambda)\alpha_\lambda(\mathfrak{f})\alpha_\lambda(\mathfrak{g}) \in \mathbb{Z}.$$

Compatibility: computational considerations

S: Seidel matrix of order n odd;

$\Sigma(S)$: set of simple eigenvalues of S;

$$\Sigma_S(x) := \prod_{\lambda \in \Sigma(S)} (x - \lambda).$$

$\Lambda_S(x)$: min poly of S;

$$q(x) = \frac{\Lambda_S(x)}{\Sigma_S(x)};$$

- Checking compatibility can be computationally expensive
- The following result speeds up compatibility checks

Proposition.
Assume $\Sigma_S(x)$ is irreducible.
Let $\mathfrak{f}(x) = \frac{\chi_S(x)}{\Lambda_S(x)} f(x) \neq \mathfrak{g}(x) = \frac{\chi_S(x)}{\Lambda_S(x)} g(x) \in \mathrm{Deck}(\chi_S)$.
Let $\rho(x)$ be the min poly of $fg(\lambda)$ over $\mathbb{Q}$ for some $\lambda \in \Sigma(S)$.
Suppose that $\deg \rho = |\Sigma(S)|$ and $\rho(x^2)$ is irreducible.
Suppose $|\mathrm{Gal}(\rho(x))| < |\mathrm{Gal}(\rho(x^2))|$.
If $q(1) + q(0)$ is odd then $\mathfrak{f}(x)$ and $\mathfrak{g}(x)$ are not compatible.

Stubborn candidate char polys

29 lines in $\mathbb{R}^{14}$, surviving candidate char polys:

$$(x+5)^{15}(x-3)^2(x-5)^8(x-7)^2(x^2-15x+52)$$
$$(x+5)^{15}(x-3)(x-4)(x-5)^{10}(x-9)^2$$

49 lines in $\mathbb{R}^{17}$, surviving candidate char polys:

$$(x+5)^{32}(x-9)^{16}(x-16)$$
$$(x+5)^{32}(x-7)(x-9)^{14}(x-12)(x-15),$$
$$(x+5)^{32}(x-7)(x-8)(x-9)^{12}(x-11)^2(x-15),$$
$$(x+5)^{32}(x-9)^{13}(x-11)^2(x^2-21x+92),$$
$$(x+5)^{32}(x-7)^2(x-8)(x-9)^{10}(x-11)^2(x-13)^2,$$
$$(x+5)^{32}(x-9)^{13}(x-13)^2(x^2-17x+64),$$
$$(x+5)^{32}(x-9)^{12}(x-11)^3(x^2-19x+72),$$
$$(x+5)^{32}(x-7)(x-9)^{10}(x-11)^4(x^2-19x+76),$$
$$(x+5)^{32}(x-4)(x-9)^{10}(x-11)^6$$

Summary and further reading

$N(d)$: largest possible size of an equiangular line system in $\mathbb{R}^d$.

d	2	3	4	5	6	7 – 14	15	16	17	18	19	20	21	22	23–41
$N(d)$	3	6	6	10	16	28	36	40	48	57 60	72 74	90 94	126	176	276

Equiangular lines in low dimensional Euclidean spaces
G.R.W Greaves, J. Syatriadi, and P. Yatsyna
Combinatorica (to appear)

Equiangular lines in Euclidean spaces: dimensions 17 and 18
G.R.W Greaves, J. Syatriadi, and P. Yatsyna
arXiv:2104.04330