# Codes and Expansions (CodEx) Seminar

# Computing SIC-POVMs using Permutation Symmetries and Stark Units

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in collaboration with
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# Overview

- Zauner's conjecture
- numerical search
- symmetries & Fibonacci-Lucas SIC-POVMs
- exact solutions from numerical solutions
  - using overlaps
  - using permutations
- numerical and exact solutions from Stark units
- conclusions & outlook

# A Simple to State Problem

Are there $d^2$ vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d^2} \in \mathbb{C}^d$ in the complex vector space of dimension $d$ such that:

(i) $\langle \boldsymbol{v}_j | \boldsymbol{v}_j \rangle = 1$ for $j = 1, \ldots, d^2$

(ii) $|\langle \boldsymbol{v}_j | \boldsymbol{v}_k \rangle|^2 = \dfrac{1}{d+1}$ for $1 \le j < k \le d^2$

The vectors $\boldsymbol{v}_j$ form an equiangular tight frame/finite unit norm tight frame.

# A Simple to State Problem

Are there $d^2$ vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d^2} \in \mathbb{C}^d$ in the complex vector space of dimension $d$ such that:

$$\begin{array}{lll} \text{(i)} & \langle \boldsymbol{v}_j | \boldsymbol{v}_j \rangle = 1 & \text{for } j = 1, \ldots, d^2 \\ \text{(ii)} & |\langle \boldsymbol{v}_j | \boldsymbol{v}_k \rangle|^2 = \dfrac{1}{d+1} & \text{for } 1 \leq j < k \leq d^2 \end{array}$$

The vectors $\boldsymbol{v}_j$ form an equiangular tight frame/finite unit norm tight frame.

All solutions form a real algebraic variety, using $2d$ real variables per vector

$$\boldsymbol{v}_j = (a_{j,1} + ib_{j,1}, a_{j,2} + ib_{j,2}, \ldots, a_{j,d} + ib_{j,d})^T \qquad (i = \sqrt{-1})$$

$2d^3$ variables, $d^2$ equations (i) of degree $2$ and $\binom{d^2}{2}$ equations (ii) of degree $4$.

# Weyl-Heisenberg Group

- generators: $$H_d := \langle X, Z \rangle$$

  where $X := \sum_{j=0}^{d-1} |j+1\rangle\langle j|$ and $Z := \sum_{j=0}^{d-1} \omega_d^j |j\rangle\langle j|$

  $$(\omega_d := \exp(2\pi i/d))$$

- relations:

$$\left(\omega_d^c X^a Z^b\right)\left(\omega_d^{c'} X^{a'} Z^{b'}\right) = \omega_d^{a'b-b'a}\left(\omega_d^{c'} X^{a'} Z^{b'}\right)\left(\omega_d^c X^a Z^b\right)$$

- basis:

$$H_d/\zeta(H_d) = \left\{X^a Z^b \colon a, b \in \{0, \ldots, d-1\}\right\} \cong \mathbb{Z}_d \times \mathbb{Z}_d$$

  trace-orthogonal basis of all $d \times d$ matrices

# Constructing SIC-POVMs

**Ansatz:**

SIC-POVM that is the orbit under the Weyl-Heisenberg group $H_d$, i. e.,

$$\begin{aligned} |\boldsymbol{v}^{(a,b)}\rangle &:= X^a Z^b |\boldsymbol{v}^{(0,0)}\rangle \\ |\langle \boldsymbol{v}^{(a,b)}|\boldsymbol{v}^{(a',b')}\rangle|^2 &= \begin{cases} 1 & \text{for } (a,b) = (a',b'), \\ 1/(d+1) & \text{for } (a,b) \neq (a',b') \end{cases} \end{aligned}$$

$$|\boldsymbol{v}^{(0,0)}\rangle = \sum_{j=0}^{d-1} (x_{2j} + i x_{2j+1}) |j\rangle,$$

($x_0, \ldots, x_{2d-1}$ are real variables, $x_1 = 0$)

$\Longrightarrow$ we have to find only one *fiducial* vector $|\boldsymbol{v}^{(0,0)}\rangle$ instead of $d^2$ vectors

$\Longrightarrow$ polynomial equations with $2d-1$ variables, but already quite complicated for $d = 6$

# Jacobi Group (or Clifford Group)

- automorphism group of the Weyl-Heisenberg group $H_d$, i. e.

$$\forall T \in J_d : T^\dagger H_d T = H_d$$

- the action of $J_d$ on $H_d$ modulo phases corresponds to the symplectic group $\mathrm{SL}(2,\mathbb{Z}_d)$, i. e.

$$T^\dagger X^a Z^b T = \omega_d^c X^{a'} Z^{b'} \qquad \text{where } \begin{pmatrix} a' \\ b' \end{pmatrix} = \tilde{T} \begin{pmatrix} a \\ b \end{pmatrix},\ \tilde{T} \in \mathrm{SL}(2,\mathbb{Z}_d)$$

$\Longrightarrow$ homomorphism $J_d \to \mathrm{SL}(2,\mathbb{Z}_d)$

- additionally: complex conjugation (anti-unitary)

$$X^a Z^b \mapsto X^a Z^{-b} \qquad \text{corresponding to } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

# Zauner's Conjecture

[G. Zauner, Dissertation, Universität Wien, 1999]

**Conjecture:**
For every dimension $d \geq 2$ there exists a SIC-POVM whose elements are the orbit of a rank-one operator $E_0$ under the Weyl-Heisenberg group $H_d$.
What is more, $E_0$ commutes with an element $S$ of the Jacobi group $J_d$.
The action of $S$ on $H_d$ modulo the center has order three.

support for this conjecture (to date):

- numerical solutions for all dimensions $d \leq 193$, plus a few more
- exact algebraic solutions for some dimensions (see below)

one of the prize problems in
[Paweł Horodecki, Łukasz Rudnicki, Karol Życzkowski, Five open problems in quantum information, arXiv:2002.03233]

# Numerical Search for SIC-POVMs

- "second frame potential" for $2$-designs

$$\sum_{i,j=1}^{d^2} |\langle v^{(i)} \otimes v^{(i)} | v^{(j)} \otimes v^{(j)} \rangle|^2 = \sum_{i,j=1}^{d^2} |\langle v^{(i)} | v^{(j)} \rangle|^4 = d^2 \sum_{a,b=1}^{d} |\langle \psi | X^a Z^b | \psi \rangle|^4$$

- for any state $|\psi\rangle \in \mathbb{C}^d$

$$\begin{aligned} f(|\psi\rangle) &= \sum_{j,k=1}^{d} \Big| \sum_{\ell=1}^{d} \langle \psi | j+\ell \rangle \langle \ell | \psi \rangle \langle \psi | k+\ell \rangle \langle j+k+\ell | \psi \rangle \Big|^2 \\ &= \sum_{j,k=1}^{d} \Big| \underbrace{\sum_{\ell=1}^{d} \overline{\psi}_{j+\ell}\, \psi_\ell\, \overline{\psi}_{k+\ell}\, \psi_{j+k+\ell}}_{=:G(j,k)} \Big|^2 \qquad \geq \frac{2}{d+1} \end{aligned}$$

with equality iff $|\psi\rangle$ is a fiducial vector for a Weyl-Heisenberg SIC-POVM

- gradient descent to minimize $f(|\psi\rangle)$, subject to unit norm

# Numerical Search for SIC-POVMs

- for any state $|\psi\rangle \in \mathbb{C}^d$

$$f(|\psi\rangle) = \sum_{j,k=1}^{d} \left| \sum_{\ell=1}^{d} \overline{\psi}_{j+\ell}\, \psi_\ell\, \overline{\psi}_{k+\ell}\, \psi_{j+k+\ell} \right|^2 \geq \frac{2}{d+1}$$

  with equality iff $|\psi\rangle$ is a fiducial vector for a Weyl-Heisenberg SIC-POVM

- gradient descent to minimize $f(|\psi\rangle)$, subject to unit norm

- use $F(\vec{x}) = f\left(\frac{P\vec{x}}{\|P\vec{x}\|}\right)$ for an arbitrary vector $\vec{x} \in \mathbb{C}^d$, where $P$ is the projection onto a subspace (prescribed symmetry)

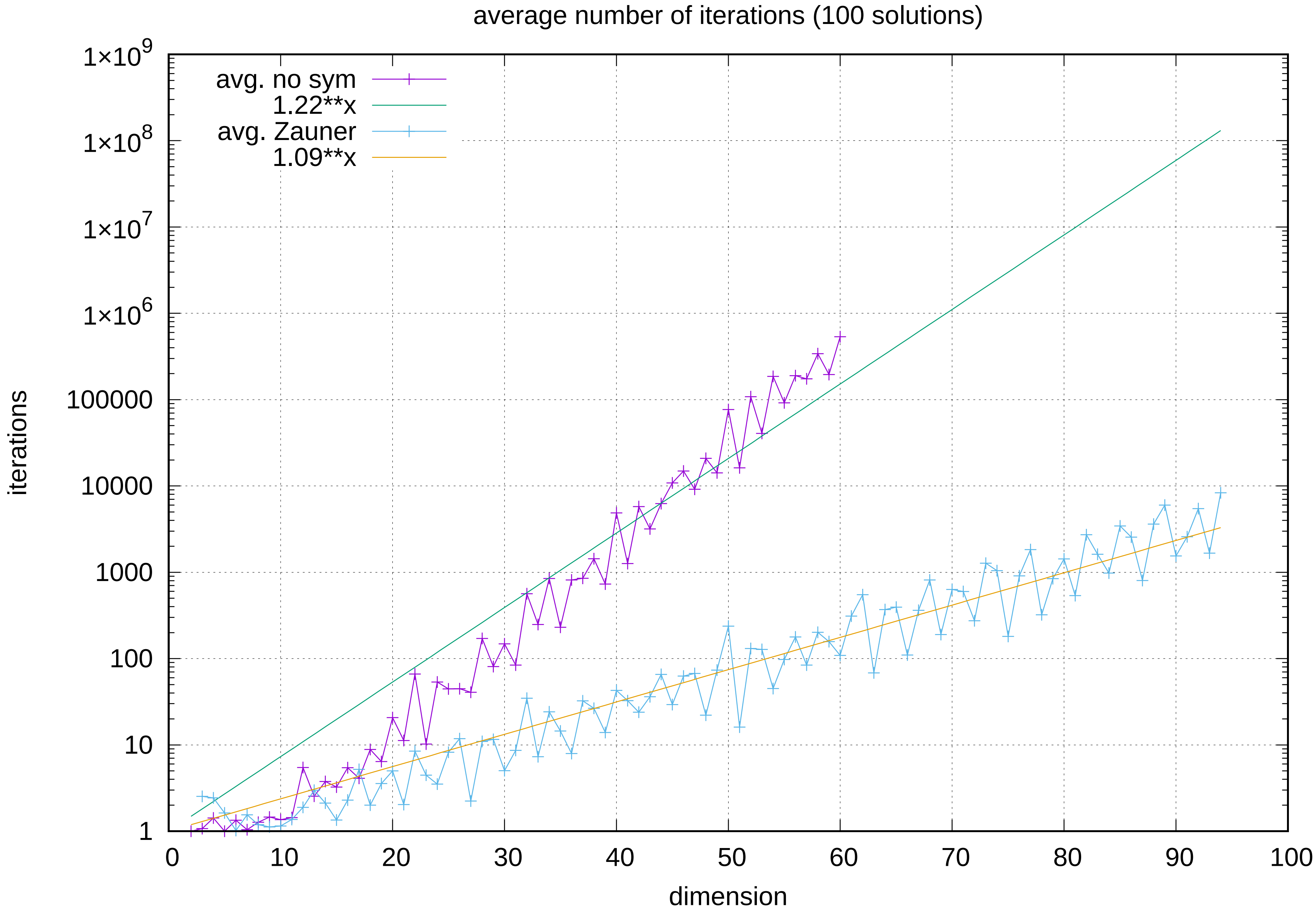
- chain rule yields a relatively simple formula for the gradient of $F(\vec{x})$ in terms of the gradient of $f$

- complexity $\mathcal{O}(d^3)$ for both the function and the gradient when storing $\mathcal{O}(d^2)$ intermediate values

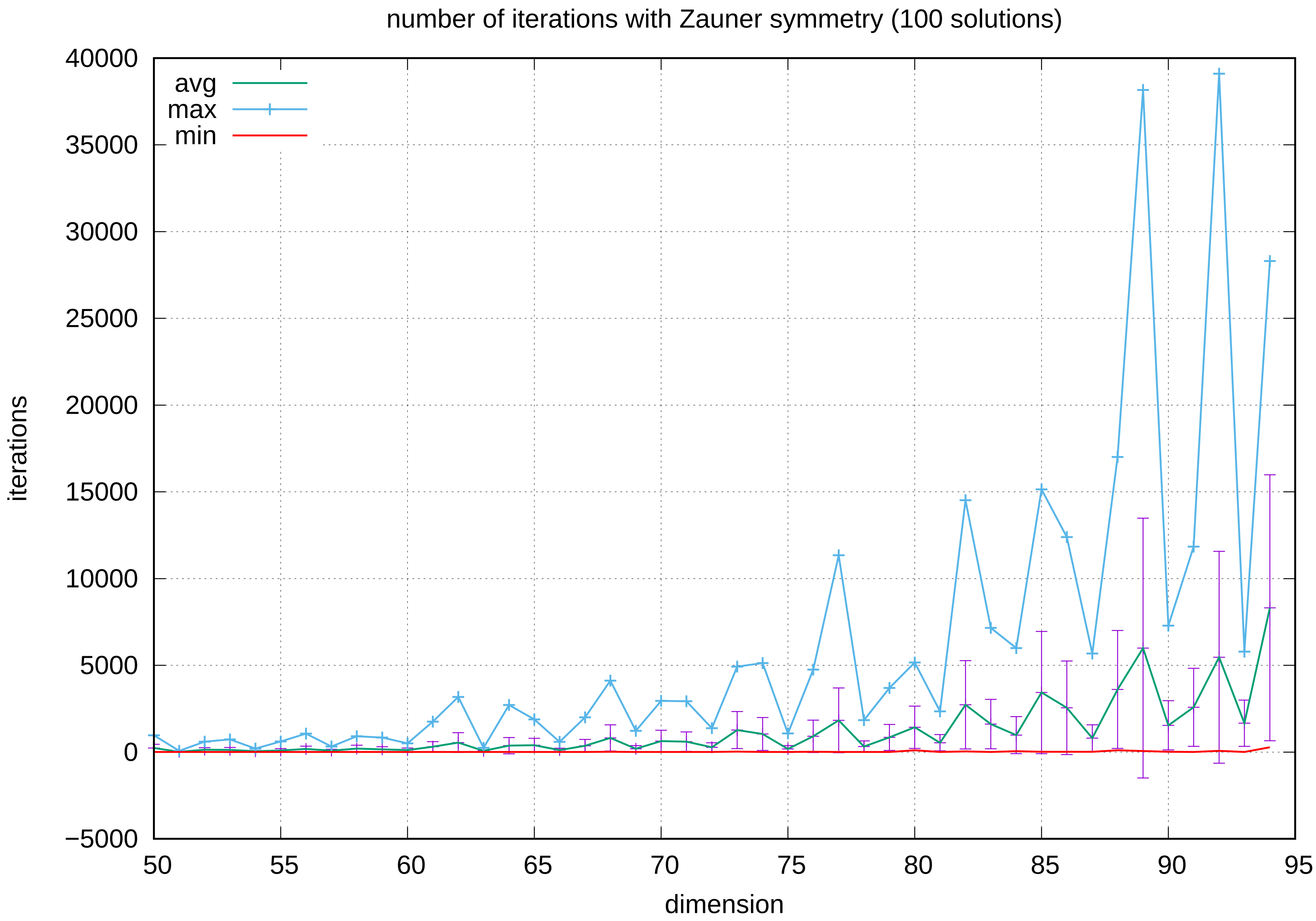
# Numerical Search for SIC-POVMs

- efficient implementation of $F(\vec{x})$ and its gradient in `C++` by Andrew Scott
- parallel computation of the function/gradient using OpenMP/CUDA
- minimization using limited-memory Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm
- search runs into local minima, we need many random initial points
- running many instances on HPC clusters by MPG and GWDG
- 
  - for $d = 189$: approx. $23.3 \times 10^6$ trials, $3.48$ CPU years
  - for $d = 190$: approx. $66.8 \times 10^6$ trials, $10.51$ CPU years
  - for $d = 193$: approx. $78.3 \times 10^6$ trials, $13.00$ CPU years
  - for $d = 5779$: $55065$ trials, $17.69$ GPU years, no success

# Average Number of Iterations



# Average Number of Iterations with Zauner Symmetry

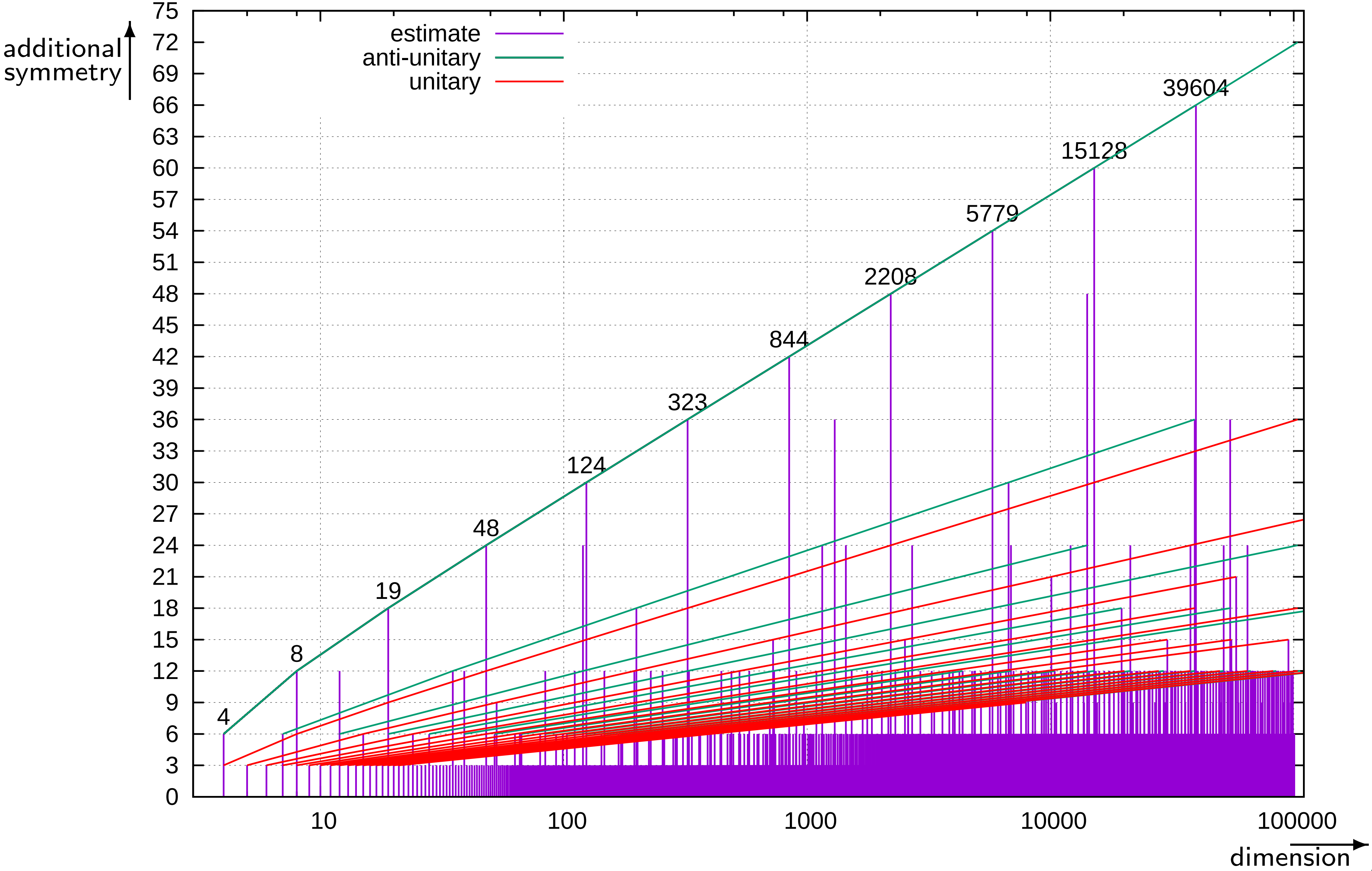


# Fibonacci-Lucas SIC-POVMs

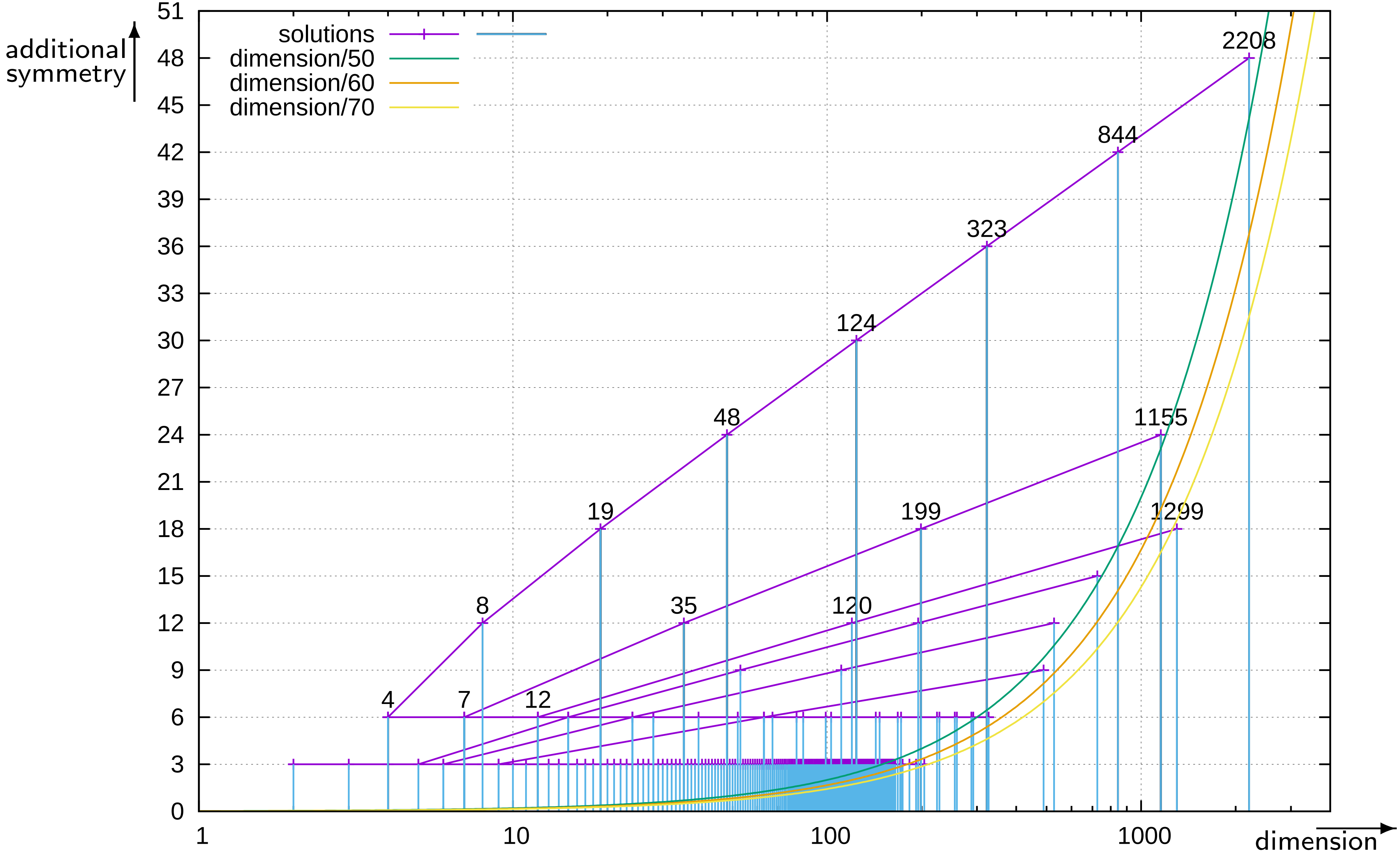
[Markus Grassl & Andrew J. Scott, JMP 58, December 2017, arXiv:1707.02944]

- (exact) symmetry analysis of a numerical solution for $d = 124$
  $\Longrightarrow$ symmetry group of order $30$ (prescribed order $6$)
- identified as part of a series of dimensions (related to Lucas numbers)
  $d = 4, 8, 19, 48, 124, 323, 844, 2208, 5779, 15128, 39604, \ldots$
- symmetry group of order $6k$ related to Fibonacci numbers, $F = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$
- new exact solutions for $d = 124$ and $d = 323$ (previously $d = 48$)
  (found using the symmetries and via Gröbner bases)
- new numerical solutions for $d = 844$ and $d = 2208$ (previously $d = 323$)
- generalisations related to generalised Fibonacci/Lucas numbers, using
  $A_m = \left(\begin{smallmatrix} 0 & 1 \\ 1 & m \end{smallmatrix}\right)$

# Symmetries of SIC-POVMs



# SIC-POVMs by Numerical Search



# Ray Class Field Conjecture

[Appleby, Flammia, McConnell & Yard, arXiv:1604.06098 & arXiv:1701.052000]
CodEx talks by M. Appleby, S. Flammia, G. Kopp

**Ray class field conjecture**

let $\mathbb{E}$ be the field containing all rank-one projection operators of a SIC-POVM

$$\mathbb{Q} \lhd \mathbb{K} = \mathbb{Q}(\sqrt{D}) \lhd \mathbb{E}_0 \lhd \mathbb{E}_1 \lhd \mathbb{E}$$

for the minimal field:

- $\mathbb{E}$ is the ray class field over $\mathbb{Q}(\sqrt{D})$ with conductor[a] $d'$ with ramification at both infinite places, $D$ is the squarefree part of $(d+1)(d-3)$
- $\mathbb{E}_1$ contains the overlap phases and equals the ray class field with ramification only allowed at the infinite place taking $\sqrt{D}$ to a positive real number
- $\mathbb{E}_0$ is the Hilbert class field $H_{\mathbb{K}}$, in particular $h = [\mathbb{E}_0 : \mathbb{K}]$ equals the class number of $\mathbb{K}$

[a] $d' = d$, or $d' = 2d$ for $d$ even

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**Ray class field conjecture**

let $\mathbb{E}$ be the field containing all rank-one projection operators of a SIC-POVM

$$\mathbb{Q} \lhd \mathbb{K} = \mathbb{Q}(\sqrt{D}) \lhd \mathbb{E}_0 \lhd \mathbb{E}_1 \lhd \mathbb{E}$$

- **"Fact 8"**: $\mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0)$ permutes the overlaps.
  For each $\sigma \in \mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0)$ there is a matrix $G_\sigma \in \mathrm{GL}(2, \mathbb{Z}/d'\mathbb{Z})$ such that[a]

  $$\sigma\big(\langle\psi|D_{\boldsymbol{p}}|\psi\rangle\big) = \langle\psi|D_{G_\sigma \boldsymbol{p}}|\psi\rangle.$$

  $G_\sigma$ commutes with matrices $F$ related to symmetries $U_F$ of the fiducial vector $|\psi\rangle$.

[a] $D_{\boldsymbol{p}} = D_{a,b} = (e^{\frac{i\pi}{d}})^{ab} X^a Z^b$

# Exact Solutions from Numerical Solutions

[Appleby, Chien, Flammia & Waldron, J. Phys. A. 51, 2018, arXiv:1703.05981]

- matrix group $\mathcal{M} = \{G_\sigma : \sigma \in \mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0)\}$, commutes with the symmetry
- projection operator $\Pi = |\psi\rangle\langle\psi|$

"Fact 8:" $$\sigma\big(\mathrm{Tr}(\Pi D_{\boldsymbol{p}})\big) = \mathrm{Tr}\big(\Pi D_{G_\sigma \boldsymbol{p}}\big)$$

- expansion coefficients $c_{\boldsymbol{p}} = \mathrm{Tr}(\Pi D_{\boldsymbol{p}})$ in the same orbit under $\mathcal{M}$ are related by Galois conjugation
- the coefficients of the polynomial $f_{\boldsymbol{p}_0}(z) = \prod_{\boldsymbol{p}\in\boldsymbol{p}_0^{\mathcal{M}}} (z - c_{\boldsymbol{p}})$

  lie in a number field of "small" degree
- find the exact minimal polynomials of those coefficients (requires high-precision numerical solution)
- find the roots of the exact polynomials $f_{\boldsymbol{p}_0}(z)$ in the ray class field
- compute $\Pi$ from the $d^2$ expansion coefficients $c_{\boldsymbol{p}}$
- exact solutions for some $d \leq 48$ ($d \leq 100$ work in progress)

# More Exact Solutions from Numerical Solutions

[Markus Grassl, Exact SIC-POVMs from permutation symmetries, in preparation]

- when $G_\sigma$ has determinant $1$, there exists a unitary $U_{G_\sigma} := T_\sigma$ with

$$\sigma\left(\mathrm{Tr}(\Pi D_{\boldsymbol{p}})\right) = \mathrm{Tr}(\Pi D_{G_\sigma \boldsymbol{p}}) = \mathrm{Tr}(\Pi T_\sigma D_{\boldsymbol{p}} T_\sigma^\dagger) = \mathrm{Tr}(T_\sigma^\dagger \Pi T_\sigma D_{\boldsymbol{p}})$$

$\Longrightarrow$ action of $T_\sigma^\dagger$ on the projection $\Pi$ and on the state $|\psi\rangle$

- when $G_\sigma = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right)$ is additionally diagonal, then $T_\sigma$ is a permutation matrix

- moreover, *assume* that $\sigma(D_{\boldsymbol{p}}) = D_{\boldsymbol{p}}$; then

$$\sigma(\Pi) = T_\sigma^\dagger \Pi T_\sigma$$

and hence

$$\sigma(\Pi_{j,k}) = \Pi_{\alpha j, \alpha k}$$

where the indices are computed modulo $d$

# More Exact Solutions from Numerical Solutions

- for the first column of $\Pi$ we have

$$\sigma(\Pi_{j,0}) = \Pi_{\alpha j,0} \qquad \text{for } j = 0, \ldots, d-1$$

- we can take the first column as (unnormalised) fiducial vector $\boldsymbol{v}$, unless it is zero (which was observed for $d = 26, 28, 62, 98, 228$)
  $\Longrightarrow$ $\sigma$ permutes the components of the fiducial vector, stabilising the first coordinate

- when the first column is zero, consider a non-zero column $k$:

$$\sigma(\Pi_{j,k}) = \Pi_{\alpha j,\alpha k} \stackrel{(*)}{=} \gamma \Pi_{\alpha j,k} \qquad \text{for } j = 0, \ldots, d-1$$

  $\Longrightarrow$ $\sigma$ gives rise to a projective permutation action
  $\Longrightarrow$ consider the action on ratios $v_j/v_{j'}$

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$^{(*)}$ $\Pi$ has rank one, so column $\alpha k$ is proportional to column $k$, i.e., $\Pi_{j,\alpha k} = \gamma \Pi_{j,k}$

# More Exact Solutions from Numerical Solutions

outline of the procedure:

- compute a numerical fiducial vector with prescribed symmetry $S$
- determine the diagonal matrices $G_\sigma \in \mathrm{SL}(2, \mathbb{Z}/d'\mathbb{Z})$ in the centraliser of $S$
- the diagonal matrices correspond to a subgroup $H \leq (Z/d'\mathbb{Z})^\times$
- consider the rescaled fiducial vector[a] $\boldsymbol{v}$ with $v_0 = 1$
- the coefficients of the polynomial $f_j(z) = \prod\limits_{\alpha \in H} (z - v_{\alpha j})$ lie in a number field of “small” degree, fixed by (a subgroup of) the Galois group
- similar as before, find the exact coefficients of $f_j(z)$ from a high-precision numerical solution, and then compute its exact roots
  $\Longrightarrow$ only $\mathcal{O}(d)$ numbers in a field of smaller degree

[a] assuming $v_0 \neq 0$ for simplicity here

# More Exact Solutions from Numerical Solutions

- the *assumption* that $\sigma(D_{\boldsymbol{p}}) = D_{\boldsymbol{p}}$ appears to be true
- new exact solutions for $57$ additional dimensions (so far)

$$d = 26, 38, 42, 49, 52, 57, 61, 62, 63, 65, 67, 73, 74, 78, 79, 84, 86, 91, 93, 95, \\ 97, 98, 103, 109, 111, 122, 127, 129, 133, 134, 139, 143, 146, 147, \\ 151, 155, 157, 163, 168, 169, 172, 181, 182, 183, 193, 199, \\ 201, 228, 259, 292, 327, 364, 399, 403, 489, 844, 1299$$

- fiducial vectors lie in a proper ("small") subfield of the ray class field from before, that intersects with the cyclotomic field $\mathbb{Q}(\zeta_{d'})$ trivially or in a smaller cyclotomic field
- "small ray class field conjecture":
  The minimal field containing a (suitably rescaled) fiducial vector is a ray class field whose conductor is a particular factor of the ideal $d\mathcal{O}_{\mathbb{K}}$ with ramification at one of the infinite places.

# Prime Dimensions $p \equiv 1 \bmod 3$

- for prime dimensions $d = p \equiv 1 \bmod 3$, the Zauner symmetry $F_z$ is conjugate to a diagonal matrix $\widetilde{F}_z$
- the centraliser of $\widetilde{F}_z$ contains all diagonal matrices in $\mathrm{SL}(2, \mathbb{Z}/d\mathbb{Z})$
- the components $v_j$, $j = 1, \ldots, d-1$, of the fiducial vector (with $v_0 = 1$) are on a single orbit with respect to the Galois group, i.e.,

$$v_{\theta^k} = \sigma^k(v_1)$$

  for generators $\theta$ and $\sigma$ of $(\mathbb{Z}/d\mathbb{Z})^\times$ and the Galois group, resp.
- for a permutation symmetry of order $3\ell$, we need only $m = \frac{d-1}{3\ell}$ numbers

**dream:**

find a *direct* way to determine the algebraic number $v_1$, as well as $\sigma$ and $\theta$

# Prime Dimensions $p = n^2 + 3$

[Appleby, Bengtsson, Grassl, Harrison, McConnell, "SIC-POVMs from Stark Units"]

**Conjecture:**

- for prime dimensions $p = n^2 + 3$ $(n > 0)$, there is an *almost flat* fiducial vector $\boldsymbol{v}$ with

$$v_j = \begin{cases} -2 - \sqrt{d+1} & j = 0 \\ \sqrt{v_0 e^{i\vartheta_j}} & j > 0 \end{cases}$$

- the components of $\boldsymbol{v}$ generate a "small" ray class field $\mathbb{K}^{\mathfrak{m}}$ with finite modulus $\sqrt{d+1} \pm 1$ and ramification at one infinite place
- the phases $e^{i\vartheta_j}$ are Galois conjugates of (real) *Stark units* for the ray class field $\mathbb{K}^{\mathfrak{m}}$

# Application of Stark's Conjectures

- for certain ray class fields $\mathbb{K}^{\mathfrak{m}}$ over the real quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, $D > 0$, one can compute numerical approximations of *Stark units* $\epsilon_\sigma$ via special values of derivatives of $L$-functions
- the Stark units are labelled by elements $\sigma$ of the Galois group $\mathrm{Gal}(\mathbb{K}^{\mathfrak{m}}/\mathbb{K})$ such that $\epsilon_\sigma = \sigma(\epsilon_0)$
- from numerical Stark units with sufficiently high precision, we can deduce their exact minimal polynomial over $\mathbb{K}$
- we have a heuristic that allows us to deduce the required precision from numerical Stark units with low precision
- the complexity of the calculation appears to be roughly $\mathcal{O}\left(\deg(\mathbb{K}^{\mathfrak{m}}/\mathbb{K}) \times (\#\text{digits})^{3.3}\right)$

# Runtime $L$-Functions

total CPU time to compute the numerical derivative of $L$-functions using Magma and PARI/GP (last three cases)

| $d$ | $\deg(\mathbb{K}^{\mathfrak{m}}/\mathbb{K})$ | log height | precision | CPU time |
|---|---|---|---|---|
| 487 | 324 | 424 | 1000 digits | 251 hours |
| 787 | 262 | 299 | 1000 digits | 118 hours |
| 2707 | 902 | 1861 | 3800 digits | 900 days |
| 4099 | 1366 | 974 | 2000 digits | 170 days |
| 5779 | 214 | 127 | 300 digits | 18 min |
| 1447 | 964 | 2158 | 4600 digits | 111 days |
| 19603 | 2178 | 1754 | 4000 digits | 82 days |
| 2503 | 3336 | 6464 | 13000 digits | 60.5 years |

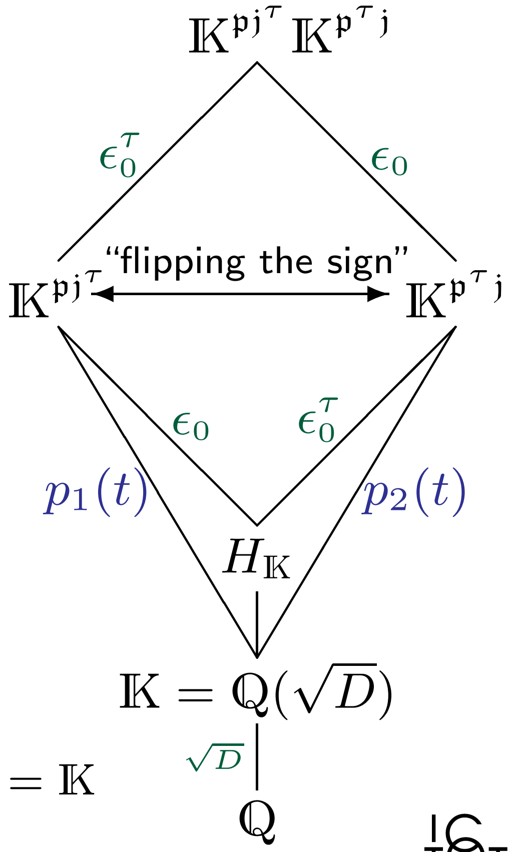
# Flipping the Sign

- real quadratic field $\mathbb{K}=\mathbb{Q}(\sqrt{D})$ with non-trivial automorphism $\tau\colon \sqrt{D}\mapsto -\sqrt{D}$

- embedding $\mathfrak{j}\colon \mathbb{K}\hookrightarrow\mathbb{R}, \mathfrak{j}(\sqrt{D})>0$, $\mathfrak{j}^\tau(\sqrt{D})=\mathfrak{j}((\sqrt{D})^\tau)<0$
  - "real" Stark units $\epsilon_\sigma$: $\mathfrak{j}(\epsilon_\sigma)>0$
  - "complex" Stark units $\epsilon_\sigma^\tau$: $\mathfrak{j}(\epsilon_\sigma^\tau)=\mathfrak{j}^\tau(\epsilon_\sigma)\in\mathbb{C}\setminus\mathbb{R}$

- minimal polynomial of $\epsilon_\sigma$: $p_1(t)\in\mathbb{K}[t]$
  $\Longrightarrow$ minimal polynomial of $\epsilon_\sigma^\tau$: $p_2(t)=p_1^\tau(t)$

- **obstacle:**
  operation of $\sigma$ on $\epsilon_0^\tau$ would require factoring $p_2(t)$




for simplicity, we assume in the following class number $h=1$, i.e., $H_\mathbb{K}=\mathbb{K}$

# The Galois Polynomial

- fixing some labelling, we know how $\sigma$ permutes the $m$ (numerical) Stark units $\epsilon_j$: $\sigma(\epsilon_j) = \sigma_{\pi_\sigma(j)}$ for some permutation $\pi_\sigma$
- there exists a unique polynomial $g_1$ of degree at most $m-1$ such that

$$g_1(\epsilon_j) = \epsilon_{\pi_\sigma(j)} \qquad \text{for } j = 1, \ldots, m \tag{1}$$

- using Newton interpolation, $g_1$ can be computed with $\mathcal{O}(m^2)$ arithmetic operations ($\mathcal{O}(m(\log m)^2)$ when using FFT-based methods)
- the coefficients of $g_1$ are in $\mathbb{K}$, as $(1)$ is invariant wrt. $\mathrm{Gal}(\mathbb{K}^{\mathfrak{p}\mathfrak{j}^\tau}/\mathbb{K})$
- $g_2(t) = g_1^\tau(t)$ corresponds to the action of $\sigma$ on $\epsilon_j^\tau$: $g_2(\epsilon_j^\tau) = \epsilon_{\pi_\sigma(j)}^\tau$
- **potential computational obstacle:**
  we don't know an *a priori* bound for the required precision
  (for $d = 19603$, the coefficients have more than 1 million digits)

# Solving the Sign Problem

Recall: We conjecture that the components of the fiducial vector are *square roots* of Galois conjugates of Stark units, i.e., $v_{\theta^k} = \sqrt{v_0 \sigma^k(\epsilon_0^\tau)}$.

**Problem:** there are two square roots $\pm\sqrt{v_0 \sigma^k(\epsilon_0^\tau)}$

**Solution:**

- it turns out that polynomial $p_2(t^2/v_0)$ with $v_0 = -2 - \sqrt{d+1}$ factors in $\mathbb{K}[t]$ as

$$v_0^m p_2(t^2/v_0) = p_4(t) p_4(-t)$$

- pick the factor $p_4(t)$ and check which of the square roots is a root of $p_4(t)$
- we are left with a global sign ambiguity, i.e., two possibilities
- note: it does not matter which of the Galois conjugates of the Stark units is assigned to $\epsilon_0^\tau$; all choices yield eventually fiducial vectors

# Final Step: Combinatorial Search

so far, we have

- exact minimal polynomials $p_1(t), p_2(t), p_4(t) \in \mathbb{K}[t]$ and exact Galois polynomials $g_1(t), g_2(t) \in \mathbb{K}[t]$
- numerical square roots $\sqrt{v_0 \epsilon_j^\tau}$ (up to a global sign) together with the permutation action of the (cyclic) Galois group $\mathrm{Gal}(\mathbb{K}^{\mathfrak{m}}/H_{\mathbb{K}})$ on them

final step:

- we have to identify which primitive element $\theta \in (\mathbb{Z}/d\mathbb{Z})^\times$ corresponds to the action of $\sigma$
- we have to fix the global sign (we can choose the sign of the first coordinate)
- compute a (numerical) vector $\boldsymbol{v}$ for all choices (less than $d$) and test the overlap $|\langle \boldsymbol{v}|X|\boldsymbol{v}\rangle|^2/||\boldsymbol{v}||^4 \stackrel{?}{=} \frac{1}{d+1}$
- we know that $\sigma^{m/2}$ corresponds to complex conjugation

# Exact Solution

We can also compute an exact representation of the fiducial vector without explicit factorisation in the extension field:

- define the field $\mathbb{L} = H_{\mathbb{K}}(\gamma)$ with $p_4(\gamma) = 0$
- compute the exact Galois polynomial $g_4(t) \in H_{\mathbb{K}}[t]$ from the numerical values $\sqrt{v_0 \epsilon_\sigma^\tau}$
- the action of the Galois automorphism $\sigma$ on $\mathbb{L}$ is defined by $\sigma\colon \gamma \mapsto g_4(\gamma)$
- we can compute the components of the fiducial vector using

$$v_0 = \pm(2 + \sqrt{d+1}), \qquad v_1 = \gamma, \qquad \text{and } v_{\theta j} = g_4(v_j) \text{ for } j > 0$$

**computational obstacles:** missing an *a priori* bound on the precision to compute the exact Galois polynomial $p_4(t)$ and arithmetic in the field $\mathbb{L}$ is slow when the degree is large (use tower of subfields if possible)

# Verification of the Solution

- second frame potential for a fiducial vector

$$f(|\psi\rangle) = \sum_{j,k=1}^{d} \Bigg| \underbrace{\sum_{\ell=1}^{d} \overline{\psi}_{j+\ell}\, \psi_\ell\, \overline{\psi}_{k+\ell}\, \psi_{j+k+\ell}}_{=:G(j,k)} \Bigg|^2 = \frac{2}{d+1}$$

- moreover $G(j,k) = \dfrac{\delta_{j,0} + \delta_{k,0}}{d+1}$
- $G(j,k)$ has an eightfold symmetry
- we don't need $d$-th roots of unity
- $\mathcal{O}(d^3)$ arithmetic operations

*verifying the solution takes longer than computing it*

# Runtime Verification

CPU time for the exact/numerical verification of the solution

| $d$ | $\deg(\mathbb{K}^{\mathfrak{m}}/\mathbb{K})$ | precision | CPU time | $G(j,k)$ |
|---|---|---|---|---|
| 103 | $2^2 \times 17$ | exact | 440 s | 1.3 s |
| 199 | $2 \times 11$ | exact | 310 s | 0.3 s |
| 487 | $2^2 \times 3^4$ | exact | 31 days | 315 s |
| 787 | $2 \times 131$ | 10000 digits | 3 hours | 65 min |
| 1447 | $2^2 \times 241$ | 10000 digits | 17.0 hours | |
| 2707 | $2 \times 11 \times 41$ | 2000 digits | 11.2 hours | |
| 4099 | $2 \times 683$ | 2000 digits | 36.5 hours | |
| 5779 | $2 \times 107$ | 2000 digits | 100 hours | 88 min |
| 19603 | $2 \times 3^2 \times 11^2$ | 1000 digits | 1367 days | |
| 39604 | $2^2 \times 3^2 \times 5^2$ | 100 digits | 684 days | $\approx$ 28 days |

# Solutions for $d = n^2 + 3$

- the method can be generalised to composite dimensions $d = n^2 + 3$
- even dimensions $d = n^2 + 3$ are divisible by $4$, but not by $8$; almost flat fiducial vector after change of basis
- for composite dimensions, one has to compute Stark units for certain subfields as well
- there are more possibilities to match the action of $(\mathbb{Z}/d\mathbb{Z})^\times$ and the action of the Galois group

so far, our method has been successfully applied in $34$ dimensions:

$d = 7, 12, 19, 28, 39, 52, 67, 84, 103, 124, 147, 172, 199, 259, 292, 327, 403, 487, 628, 787, 844, 964, 1027, 1228, 1299, 1447, 1684, 1852, 2404, 2707, 4099, 5779,$ $19603$, and $39604$

# Conclusions & Outlook

- deterministic procedure to compute SIC-POVMs from Stark units
- successfully applied in $34$ dimensions $d = n^2 + 3$; did not fail in any
- can we obtain a fiducial vector directly from the *real* Stark units, without "flipping the sign"?
- can we work with lower precision?
- can we avoid the combinatorial search in the final step?
- assuming Stark's conjectures to be true, can be prove that our construction always works?
- can we extend the method to other dimensions?

forthcoming publication:

Marcus Appleby, Ingemar Bengtsson, Markus Grassl, Michael Harrison, Gary McConnell, "SIC-POVMs from Stark Units"

# Thank you!
# Danke! Merci!
# Dziekuje!

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