Frames of translates and the bandwidth intuition

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Outline

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1. Frames of translates
2. Density conditions and the bandwidth intuition
3. GSI systems in LCA groups (w. Jakob Lemvig)

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4. Frames of translates in nonabelian groups (w. Vignon Oussa)

Overview

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2. Density conditions and the bandwidth intuition
3. GSI systems in LCA groups (w. Jakob Lemvig)
4. Frames of translates in nonabelian groups (w. Vignon Oussa)

Setup

Throughout this talk: G is a second countable locally compact group with Haar measure μ_G and associated L^2-space $L^2(G)$.

Definition

Given a closed left-invariant subspace $\mathcal{H} \subset L^2(G)$, a frame of translates for $\mathcal{H}$ is a family $(L_x g)_{x \in \Gamma} \subset L^2(G)$ satisfying

$$\forall f \in L^2(G) : A\|f\|_2^2 \le \sum_{x \in \Gamma} |\langle f, L_x g \rangle|^2 \le B\|f\|_2^2 ,$$

where $L_x g(y) = g(x^{-1}y)$. The frame is called tight if $A = B$ holds. $\Gamma \subset G$ is called the translation set. This set is called regular if it's a subgroup.

Basic questions

- Which left-invariant subspaces $\mathcal{H} \subset L^2(G)$ admit a frame of translates?
- Given $\mathcal{H}$, which structural requirements do translation sets $\Gamma \subset G$ generating a frame of $\mathcal{H}$ have to fulfill?
- When does $\mathcal{H} = L^2(G)$ itself admit a frame of translates?

An overview of previous literature (incomplete & subjective)

- Whittaker 1928, Shannon 1949 (sampling)
- Duffin/Schaeffer 1952 (frames of translates over the reals)
- Kluvánek 1965 (over LCA groups)
- Landau 1967 (critical density results for sampling)
- Feichtinger/Gröchenig and numerous coauthors, since late eighties (coorbit theory, over general LC groups)
- Heil and various coauthors, since early nineties (wavelets, time-frequency analysis)
- Weiss and various coauthors, since early nineties (wavelets)
- Christensen and various coauthors, since early nineties (wavelets, time-frequency analysis)
- Waldron and various coauthors, '08-today (over finite groups)
- A. Höfler '14 (frames and density results over nilpotent Lie groups)
- Currey, Mayeli, Oussa (over nilpotent and solvable Lie groups)
- Iverson '15,'18 (left actions of abelian or compact subgroups of Lie groups)
- ...

Overview

Regular translations on the reals

Theorem

All closed invariant subspaces $\mathcal{H} \subset \mathbb{R}$ are of the form

$$\mathcal{H} = \mathcal{H}_\Omega = \{ \in L^2(\mathbb{R}) : \widehat{f} \cdot \mathbf{1}_\Omega = \widehat{f} \} ~,$$

with $\Omega \subset \mathbb{R}$ a Borel set.

Definition

Let $\Omega \subset \mathcal{R}$ be a Borel set, and $\Gamma \subset \mathbb{R}$ a lattice. We call Ω a Γ-tile if

$$\mathbb{R} = \dot{\bigcup_{\gamma \in \Gamma}} \gamma + \Omega$$

up to sets of measure zero.
We call Ω a Γ-subtile if $\mu_{\mathbb{R}}(\Omega \cap \gamma + \Omega) = 0$, forall $\gamma \in \Gamma \setminus \{0\}$.

Theorem (Kluvánek)

Let $\mathcal{H} = \mathcal{H}_\Omega$, with Ω measurable, and $\alpha > 0$. Then there exists a frame generator for $\alpha\mathbb{Z}$ in $\mathcal{H}$ iff Ω is a $\alpha^{-1}\mathbb{Z}$ subtile.

Irregular translation sets on the reals

Let $\Gamma \subset \mathbb{R}$ be given. The upper Beurling density of Γ is defined as

$$D^+(\Gamma) = \limsup_{r \to \infty} \sup_{x \in \mathbb{R}} \frac{\sharp(\Gamma \cap [x, x+r])}{r}$$

Example

For $\alpha > 0$, one has $D^+(\alpha\mathbb{Z}) = \alpha^{-1}$.

Theorem (Landau)

Let $\Gamma \subset \mathbb{R}$ be given, and $\Omega = [-R/2, R/2]$. Let $g \in \mathcal{H}_\Omega$ be defined by $\widehat{g} = 1_\Omega$. If $(L_x g)_{x \in \Gamma}$ is a frame for $\mathcal{H}_\Omega$, then $D^+(\Gamma) \geq R$.

Compare to regular case

There exists a frame generator in $\mathcal{H}_{[-R/2,R/2]}$ for $\Gamma = \alpha\mathbb{Z}$ iff $D^+(\Gamma) \geq R$.

Regular versus irregular translation sets

Example

Let $\Omega \subset \mathbb{R}$ be open, dense and of finite measure. There exists no $\alpha > 0$ such that Ω is an $\alpha^{-1}\mathbb{Z}$-subtile, hence there exists no lattice $\Gamma = \alpha\mathbb{Z}$ giving rise to a frame of translates of $\mathcal{H}_\Omega$.
On the other hand, defining $g \in \mathcal{H}_\Omega$ via $\widehat{g} = \mathbf{1}_\Omega$ gives rise to a continuous frame of translates for $\mathcal{H}_\Omega$.
By a Theorem due to Freeman and Speegle (2019), there exists $\Gamma' \subset \mathbb{R}$ such that $(L_x g)_{x \in \Gamma'}$ is a frame of $\mathcal{H}_\Omega$.

Theorem (Christensen/Deng/Heil '99, Olson and Zalik '92)

$L^2(\mathbb{R})$ does not admit a frame of translates.

Summary: The bandwidth intuition over the reals

Answers to basic questions

- Which left-invariant subspaces $\mathcal{H} \subset L^2(G)$ admit a frame of translates?
  $\mathcal{H} = \mathcal{H}_\Omega$ for finite measure set Ω.
- Given $\mathcal{H}$, which structural requirements do translation sets $\Gamma \subset G$ generating a frame of $\mathcal{H}$ have to fulfill?
  In combination with varying additional assumptions: Density $\geq$ bandwidth.
- Does $\mathcal{H} = L^2(\mathbb{R})$ itself admit a frame of translates?
  $\Omega = \mathbb{R}$ requires infinite density, so: No.

Plan for the following

- Expand investigation in two directions:
  - Multiple windows over lattice systems in LCA groups
  - Frames of translates in nonabelian LC groups
- Exhibit cases where bandwidth intuition holds up, as well as counterexamples.

Overview

GSI systems

Definition

- A lattice system in G is a family $\mathcal{G} = (\Gamma_j)_{j \in J}$ of lattices (discrete, cocompact subgroups) in G.
- A generalized shift-invariant (GSI) system associated to $\mathcal{G}$ is a family $(L_x g_j)_{x \in \Gamma_j, j \in J} \subset L^2(G)$.
- The g_j are called the generators of the system.
- A (tight) GSI frame associated to $\mathcal{G}$ is a GSI system that is also a (tight) frame of $L^2(G)$.

Main questions (for this section)

- Which lattice systems $\mathcal{G}$ allow the existence of tight frame generators?
- Are there lattice systems $\mathcal{G} = (\Gamma_j)_{j \in J}$ with infinite index set J that don't allow tight frame generators?
- Is there a meaningful bandwidth intuition helping to decide this question?

Basic Fourier-analytic facts about LCA groups

- We let $\widehat{G}$ denote the character group of G, i.e., the group of all continuous homomorphisms $\chi : G \to (\mathbb{T}, \cdot)$.
- The Fourier transform of a function $f \in L^1(G)$ is defined by

$$\widehat{f} : \widehat{G} \to \mathbb{C} \ , \ \widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx \ .$$

- $\widehat{G}$ is an LCA group as well, and with suitable normalization of the Haar measure $\mu_{\widehat{G}}$, the Fourier transform induces a unitary map $L^2(G) \to L^2(\widehat{G})$.
- For a lattice Γ, the dual lattice $\Gamma^\perp \subset \widehat{G}$ is defined as

$$\Gamma^\perp = \{\chi \in \widehat{\Gamma} : \chi|_\Gamma \equiv 1\} \ .$$

- As before: All closed left-invariant subspaces of $L^2(G)$ are of the form

$$\mathcal{H} = \mathcal{H}_\Omega = \{\in L^2(G) : \widehat{f} \cdot \mathbf{1}_\Omega = f\} \ ,$$

with $\Omega \subset \widehat{G}$ a Borel set.

Frames for single lattices

- We define the terms tile and subtile for lattices $\Gamma \subset G$ in the same way as over the reals.
- The density of a lattice Γ is defined as $D(\Gamma) = \frac{1}{\mu_G(A)}$, where A is a Γ-tile.
- By normalization of Haar measures, $D(\Gamma)D(\Gamma^\perp) = 1$.

Theorem (Kluvanek)

Let $\mathcal{H} = \mathcal{H}_\Omega \subset L^2(G)$, with Ω measurable, and $\Gamma < G$ a lattice. There exists a frame generator for Γ in $\mathcal{H}$ iff Ω is a $\Gamma^\perp$ subtile. As a consequence, $\mu_{\widehat{G}}(\Omega) \leq D(\Gamma)$.

$D(\Gamma)$ can be regarded as the maximal bandwidth that a single generator Γ-shiftinvariant system can cover.

A simple-minded construction of GSI frames via bandwidth intuition

Given $\mathcal{G} = (\Gamma_j)_{j \in J}$.

- Necessary condition for GSI frame generators g_j ($j \in J$):

$$0 < \sum_{j \in J} |\widehat{g_j}(\xi)|^2 \text{ a.e.}$$

- Basic idea: Use $\widehat{g_j} = \mathbf{1}_{\Omega_j}$, for a suitable Γ_j-subtile Ω_j
  $\Rightarrow (L_x g_j)_{j \in J}$ is a tight frame for $\mathcal{H}_{\Omega_j}$!
- The necessary condition for generators yields the covering requirement

$$\widehat{G} = \bigcup_{j \in J} \Omega_j$$

  by Γ_j-subtiles Ω_j. Making the union disjoint results in $L^2(G) = \bigoplus_{j \in J} \mathcal{H}_{\Omega_j}$, and the union of the tight frames for each subspace (suitably normalized) is a tight frame!
- If the Ω_j are essentially disjoint Γ_j-tiles, appropriate normalization yields an ONB.
- GSI bandwidth intuition: $(\Gamma_j)_{j \in J}$ has a family of tight frame generators if $\widehat{G}$ has a covering by $\Gamma_j^\perp$-subtiles Ω_j.
- A necessary condition for this approach to work: $\sum_{j \in J} D(\Gamma_j) \geq \mu_{\widehat{G}}(\widehat{G})$.
  We call $\sum_{j \in J} D(\Gamma_j)$ the bandwidth of the system of lattices.

A class where the bandwidth criterion is sharp

Definition

A lattice system $(\Gamma_j)_{j \in J}$ is called independent if for all families $(x_j)_{j \in J'}$ with finite $J' \subset J$ and $x_j \in \Gamma_j$, we have the implication

$$\sum_{j \in J'} x_j = 0 \Rightarrow \forall j \in J' : \; x_j = 0 \,.$$

Theorem (HF, J. Lemvig '19)

Let $\mathcal{G} = (\Gamma_j)_{j \in J}$ be a lattice system in the LCA group G. Assume that the family $(\Gamma_j^\perp)_{j \in J}$ of dual lattices is independent.
Then there exists a system of tight frame generators for $\mathcal{G}$ iff there exists a family of $\Gamma_j^\perp$-subtiles Ω_j fulfilling

$$\bigcup_{j \in J} \Omega_j = \widehat{G} \,.$$

In particular, $\sum_{j \in J} D(\Gamma_j) \geq \mu_{\widehat{G}}(\widehat{G})$ is a necessary criterion for the existence of tight frame generators.

Sketch of proof ("only if")

- Assume that $(g_j)_{j \in J}$ are tight frame generators. Let $f \in L^2(\mathbb{R})$ be such that $\widehat{f}$ is bounded and supported in a bounded set, with $\|f\|_2 = 1$.
- For $j \in J$, define the function $w_{f;g,j} : G \to \mathbb{C}$ by

$$w_{f;g,j}(x) = \sum_{\gamma \in \Gamma_j} |\langle T_x f, T_\gamma g_j \rangle|^2$$

- By assumption, we have

$$\sum_{j \in J} w_{f;g,j}(x) = 1 ,$$

  with absolute pointwise convergence.
- Independence of $(\Gamma_j^\perp)_{j \in J}$ allows to show uniform convergence of the series. We can then regard both sides of the equation as almost periodic functions, and compare Fourier coefficients.
- Running through all possible choices of f results in the Calderón-type condition

$$\sum_{j \in J : \alpha \in \Gamma_j^\perp} \frac{1}{D(\Gamma_j)} \overline{\hat{g}_j(\omega)} \hat{g}_j(\omega + \alpha) = \delta_{\alpha,0} \quad a.e.\ \omega \in \widehat{G}$$

  for all $\alpha \in \bigcup_j \Gamma_j^\perp$

Sketch of proof, concluded

- The Calderón-type condition

$$\sum_{j \in J : \alpha \in \Gamma_j^\perp} \frac{1}{D(\Gamma_j)} \overline{\widehat{g}_j(\omega)} \widehat{g}_j(\omega + \alpha) = \delta_{\alpha,0} \quad a.e.\ \omega \in \widehat{G}$$

for all $\alpha \in \bigcup_j \Gamma_j^\perp$ further simplifies under the assumption that the $\Gamma_j^\perp$ are independent: Here we have $\Gamma_{j_1}^\perp \cap \Gamma_{j_2}^\perp = \{0\}$ for $j_1 \neq j_2$, and we get the two equivalent conditions

$$\sum_{j \in J} \frac{1}{D(\Gamma_j)} |\widehat{g}_j(\omega)|^2 = 1 \ (a.e.)$$

and

$$\forall j \in J \forall \alpha \in \Gamma_j \setminus \{0\} \ : \ \overline{\widehat{g}_j(\omega)} \widehat{g}_j(\omega + \alpha) = 0 \ (a.e.)$$

- The second condition means that $\Omega_j = \widehat{g}_j^{-1}(\mathbb{C} \setminus \{0\})$ is a Γ_j-subtile, with $\bigcup_{j \in J} \Omega_j = \widehat{G}$.

Examples provided by the theorem

An example over $\mathbb{R}^2$

Fix a transcendental number $c > 1$, and let $\Gamma_j = C_j \mathbb{Z}^2$, where

$$C_j = \begin{pmatrix} c^{-j} & 0 \\ 0 & c^j \end{pmatrix} \quad \text{for } j \in \mathbb{N}.$$

This is a system of lattices with independent dual lattices, and $\sum_{j \in \mathbb{N}} D(\Gamma_j) = \infty$. But there does not exist a covering by Γ_j-subtiles.
In particular, there does not exist a family $(g_j)_{j \in \mathbb{N}}$ of tight frame generators.

An example over $\mathbb{Z}$

Fix an injective family $(c_j)_{j \in \mathbb{N}}$ of prime integers such that $\sum_{j \in \mathbb{N}} c_j^{-1} < 1$. Then the lattices $\Gamma_j = c_j \mathbb{Z}$ have independent dual lattices. Hence, there does not exist a family of tight frame generators in $\ell^2(\mathbb{Z})$ for them.

A class where the bandwidth criterion fails

Theorem (HF, J. Lemvig '19)

Suppose there exists a sequence $(\Gamma_n)_{n\in\mathbb{N}_0}$ of strictly decreasing lattices $\Gamma_0 \supsetneq \Gamma_1 \supsetneq \dots$ in G. Then there exists an associated system of generators $(g_n)_{n\in\mathbb{N}_0}$ such that the associated GSI system is an orthonormal basis of $L^2(G)$.

Corollary (HF, J. Lemvig '19)

Suppose there exists a sequence $\mathcal{G} = (\Gamma_n)_{n\in\mathbb{N}_0}$ of strictly decreasing lattices $\Gamma_0 \supsetneq \Gamma_1 \supsetneq \dots$ in G. Then, given any $\epsilon > 0$, there exists k such that $\mathcal{L} = (\Gamma_n)_{n\geq k}$

$$\sum_{n\geq k} D(\Gamma_n) < \epsilon \ ,$$

but there exists an orthonormal GSI basis of $L^2(G)$ associated to $\mathcal{L}$.
In particular, for $\epsilon < \mu_{\widehat{G}}(\widehat{G})$ there is no covering of $\widehat{G}$ by $\Gamma_n^\perp$-subtiles, $n \geq k$.

Proof of Corollary

The computation $D(\Gamma_{n+1}) = \frac{D(\Gamma_n)}{[\Gamma_n:\Gamma_{n+1}]} \leq \frac{D(\Gamma_n)}{2}$ shows for the system bandwidth:

$$\sum^{\infty} D(\Gamma_n) \leq 2D(\Gamma_k) \to 0 \ , \text{ as } k \to \infty \ .$$

Proving the theorem

Definition

Let $\mathcal{G} = (\Gamma_j)_{j \in J}$ and $\mathcal{L} = (\Lambda_i)_{i \in I}$ denote lattice systems. We say that $\mathcal{L}$ is a *refinement* of $\mathcal{G}$ if there exists a partition $(I_j)_{j \in J}$ of I and group elements $(\gamma_i)_{i \in I}$ with the property

$$\forall j \in J \ : \ \Gamma_j = \dot{\bigcup_{i \in I_j}} \gamma_i + \Lambda_i \ .$$

Refinement Lemma (M. Bownik/Z. Rzeszotnik)

Let $\mathcal{G} = (\Gamma_j)_{j \in J}$ be a system of lattices, and $\mathcal{L} = (\Lambda_i)_{i \in I}$ a refinement of $\mathcal{G}$ with associated shifts γ_i. Given a system of functions $(g_j)_{j \in J}$, and define

$$h_i = T_{\gamma_i} g_j \ , i \in I_j \ .$$

Then the GSI system $(T_\lambda h_i)_{i \in I, \lambda \in \Lambda_i}$ is obtained by reindexing the GSI system $(T_\gamma g_j)_{j \in J, \gamma \in \Gamma_j}$. In particular, all frame properties (constants included) are preserved.

Observation

If $\mathcal{L} = (\Lambda_i)_{i \in I}$ is a refinement of $\mathcal{G} = (\Gamma_j)_{j \in J}$, one has $\sum_{i \in I} D(\Lambda_i) \leq \sum_{j \in J} D(\Gamma_j)$. The inequality can be strict!

Proving the theorem

Lemma

Let $\Gamma_0 \supsetneq \Gamma_1 \supsetneq \dots$ denote a strictly decreasing sequence of lattices in G. Then $(\Gamma_n)_{n\geq 1}$ is a refinement of the single lattice system $(\Gamma_n)_{n=0}$.

Finishing the proof of the Theorem

- We reindex the decreasing sequence $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \dots$ to obtain infinitely many disjoint subsequences of strictly decreasing lattices in Γ_0.
- Applying the previous lemma to each sequence of decreasing lattices, we obtain that $(\Gamma_n)_{n\geq 0}$ is a refinement of $\mathcal{A} = (\Gamma_0)_{j\in\mathbb{N}_0}$ (infinite repetition of Γ_0).
- Applying the above simple-minded construction of GSI frames yields an GSI ONB associated to $\mathcal{A}$. (Cover G disjointly by $\Gamma_0^\perp$-tiles.)
- Applying the refinement lemma reindexes this ONB over $(\Gamma_n)_{n\geq 0}$, and we have system bandwidth

$$\sum_{n=0}^{\infty} D(\Gamma_n) \leq 2D(\Gamma_0) < \infty.$$

Open questions

Let G be a finite abelian groups, $\mathcal{G} = (\Gamma_i)_{i=1}^n$ a family of subgroups satisfying

$$\sum_{i=1}^{n} \sharp\Gamma_i \geq \sharp G \ .$$

Does there exist a family of (tight) frame generators for $\mathcal{G}$?

Let G be an arbitrary LCA group. Are there systems $\mathcal{G} = (\Gamma_i)_{i=1}^n$ of lattices admitting frame generators, but no tight frame generators?

Overview

Basic setup and results

Basic setup

Throughout this section: G second countable, locally compact, $\Gamma \subset G$. When does there exist $g \in L^2(G)$ such that $(L_x g)_{x \in \Gamma}$ is a frame for $L^2(G)$?
We call G an FT group if it allows a frame of translates.

Example

G discrete $\Rightarrow$ G is FT group. (Are there other ones?)

Definition

$\Gamma \subset G$ is relatively separated if $\sup_{x \in G} \sharp(\Gamma \cap xU) < \infty$ for some and hence all relatively compact neighborhoods U of the identity.

Lemma (HF/V .Oussa)

Let $\Gamma \subset G$, and suppose there exists a frame $(L_x g)_{x \in \Gamma}$ of $L^2(G)$, for suitable $g \in L^2(G)$.

- Γ is relatively separated.
- If G is nondiscrete, then

$$\Gamma^{-1} = \{x^{-1} : x \in \Gamma\}$$

  is not relatively separated.

Applications of the Lemma

- G nondiscrete $\Rightarrow$ only irregular sets Γ can generate frames of translates.
- As predicted by bandwidth intuition: G abelian, nondiscrete $\Rightarrow$ G is not an FT group. (Inverse sets of relatively separated sets are relatively separated in G.)
- More generally: If G is nondiscrete and such that its commutator subgroup

$$\langle x^{-1}y^{-1}xy : x, y \in G \rangle \subset G$$

is relatively compact, then G is not an FT group.
- Concrete example: Denote by $\mathbb{H}$ the Heisenberg group,

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} .$$

Let $Z \subset \mathbb{H}$ denote the central subgroup generated by the matrix

$$Z = \left\{ \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\} ,$$

and let $\mathbb{H}_r = \mathbb{H}/Z$, the reduced Heisenberg group. Then $\mathbb{H}_r$ is not an FT group.

Properties of FT groups

Theorem (HF/V. Oussa)

Let $(L_x g)_{x \in \Gamma}$ denote a frame of translates for G. Let $\mathcal{H} \subset L^2(G)$ denote a closed leftinvariant subspace with convolution reproducing kernel, i.e.,

$$\exists S \in \mathcal{H} \ , \ f \mapsto f * S \text{ is orthogonal projection onto } \mathcal{H}$$

Let

$$h = \int_G g(x) L_x S dx \in \mathcal{H} \ ,$$

which converges in the weak operator sense. Then $(L_x h)_{x \in \Gamma}$ is a frame for $\mathcal{H}$.

Comparison to bandwidth intuition over the reals

Convolution reproducing kernels S on $\mathbb{R}$ are characterized by

$$\widehat{S} = \mathbf{1}_\Omega$$

with Ω a Borel set of finite measure. Bandwidth intuition requires that the density of translates providing a frame for S is $\geq \mu_{\mathbb{R}}(\Omega)$.
By contrast, FT groups have universal frame sets for these spaces.

Sufficient criteria for FT groups

Theorem (HF/V. Oussa)

Let G be nonunimodular and of type I. Then G is an FT group.

Sketch of proof

Already known: $L^2(G)$ has a continuous frame of translates (HF, '02). By the already mentioned discretization result of Freeman and Speegle ('19), the result follows.

Theorem (HF/V. Oussa)

Let $H < G$ denote a closed subgroup that has FT, and such that $\lambda_H \simeq \infty \cdot \lambda_H$. Then G is FT. In fact, there exists a vector $g \in L^2(G)$ and $\Gamma \subset H$ such that $(L_x g)_{x \in \Gamma} \subset L^2(G)$ is a frame.
Here λ_H is the left regular representation of H, $y \mapsto L_y$.

Sketch of proof

(Following an idea due to Iverson)

- Using a measurable set of representatives $C \subset G$ mod H, there exists a Borel isomorphism
$$\varphi : H \times C \ni (h,x) \mapsto hx \in G ~,$$
intertwining the canonical left actions on H on both spaces.
- Weyl's integral formula provides a measure ν on C such that μ_G becomes the image measure of $\mu_H \times \nu$ under φ
- As a result, we obtain a unitary map
$$L^2(G) \to L^2(H) \otimes L^2(C)$$
resulting in an equivalence of representations
$$\lambda_H \simeq \lambda_H \otimes \mathbf{1}_{L^2(C)} \simeq \lambda_G|_H ~.$$
- The unitary equivalence allows to transfer a frame of translates from $L^2(H)$ to $L^2(G)$.

Example (classes) of FT groups

(HF/V. Oussa)

- Let $G = \mathbb{R} \rtimes \mathbb{R}^*$, the $ax+b$-group. Then G is nonunimodular, type I, hence FT. Furthermore, $\lambda_G \simeq \infty \cdot \lambda_G$.
- $G = SL(2,\mathbb{R})$ has a closed subgroup isomorphic to the $ax+b$-group. Hence it is FT.
- More general: $G = SL(n,\mathbb{R})$ is FT.
- $G = SO(p,q)$, with $p,q > 1 \Rightarrow G$ is FT.
- Let G be an exponential solvable Lie group that is not nilpotent. Then G is FT.

Open cases

1. Which nonabelian simply connected, connected nilpotent Lie groups are FT?
2. Is the Heisenberg group $\mathbb{H}$ an FT-group?

Note: If the answer to question 2) is "yes", the answer to question 1) is "all".

Conjecture: The answer to question 1) is: "none".

Thank you !