Mutually Unbiased Equiangular Tight Frames

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Projective Codes

Spherical codes: Tammes's problem (1930)

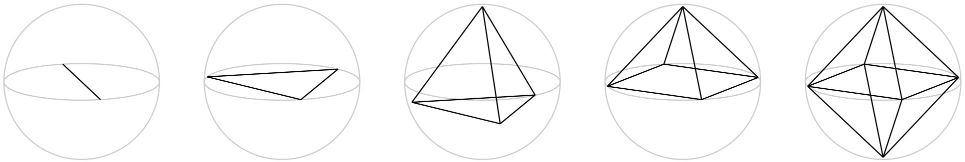
Problem: Find N unit vectors $\{\mathbf{x}_n\}_{n=1}^N$ in $\mathbb{R}^D$ that maximize

$$\min_{n_1 \neq n_2} \|\mathbf{x}_{n_1} - \mathbf{x}_{n_2}\|.$$

Equivalently, since $\|\mathbf{x}_{n_1} - \mathbf{x}_{n_2}\|^2 = 2(1 - \min_{n_1 \neq n_2} \langle \mathbf{x}_{n_1}, \mathbf{x}_{n_2} \rangle)$, minimize

$$\max_{n \neq n'} \langle \mathbf{x}_n, \mathbf{x}_{n'} \rangle.$$

Example: $D = 3$, $N = 2, 3, 4, 5, 6$:



Rankin's bounds (1955)

Theorem: When $N \leq D + 1$, {optimal spherical codes} = {regular simplices}.

Proof: For any unit vectors $\{\mathbf{x}_n\}_{n=1}^{N}$ in $\mathbb{R}^D$,

$$0 \leq \left\| \sum_{n=1}^{N} \mathbf{x}_n \right\|^2 = \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \langle \mathbf{x}_{n_1}, \mathbf{x}_{n_2} \rangle \leq N + N(N-1) \max_{n_1 \neq n_2} \langle \mathbf{x}_{n_1}, \mathbf{x}_{n_2} \rangle.$$

Theorem: $\max_{n_1 \neq n_2} \langle \mathbf{x}_{n_1}, \mathbf{x}_{n_2} \rangle \geq 0$ when $N \geq D + 2$. It's achievable when $N \leq 2D$.

Proof: By induction: if $\{x_n\}_{n=1}^{N}$ are mutually obtuse in $\mathbb{R}^D$ then projecting $\{\mathbf{x}_n\}_{n=1}^{N-1}$ onto $\mathbf{x}_N^{\perp}$ yields $N-1$ mutually obtuse vectors in $\mathbb{R}^{D-1}$.
To achieve equality when $N \leq 2D$, choose $\{\mathbf{x}_n\}_{n=1}^{N}$ as a subset of the **orthoplex**.

Projective spaces

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. The **projective space** of $\mathbb{F}^D$ is the set of its one-dimensional subspaces. We identify each with its rank-one projection:

$$\mathrm{P}(\mathbb{F}^D) := \{\varphi\varphi^* : \varphi \in \mathbb{F}^D, \|\varphi\| = 1\}.$$

The Frobenius inner product of two such projections is

$$\langle \varphi_1\varphi_1^*, \varphi_2\varphi_2^* \rangle_{\mathrm{Fro}} = \mathrm{Tr}(\varphi_1\varphi_1^*\varphi_2\varphi_2^*) = \mathrm{Tr}(\varphi_2^*\varphi_1\varphi_1^*\varphi_2) = |\langle \varphi_1, \varphi_2 \rangle|^2$$

The squared Frobenius distance between two such projections is thus

$$\|\varphi_1\varphi_1^* - \varphi_2\varphi_2^*\|_{\mathrm{Fro}}^2 = 2(1 - \langle \varphi_1\varphi_1^*, \varphi_2\varphi_2^* \rangle_{\mathrm{Fro}}) = 2(1 - |\langle \varphi_1, \varphi_2 \rangle|^2).$$

Projective codes

Problem: Design N unit vectors $\{\varphi_n\}_{n=1}^N$ in $\mathbb{F}^D$ that maximize

$$\min_{n_1 \neq n_2} \|\varphi_{n_1}\varphi_{n_1}^* - \varphi_{n_2}\varphi_{n_2}^*\|^2 = \min_{n_1 \neq n_2} 2(1 - |\langle \varphi_{n_1}, \varphi_{n_2} \rangle|^2)$$

or equivalently have minimal **coherence**:

$$\mathrm{coh}(\{\varphi_n\}_{n=1}^N) := \max_{n \neq n'} |\langle \varphi_n, \varphi_{n'} \rangle|.$$

That is, design a $D \times N$ matrix $\mathbf{\Phi} = \begin{bmatrix} \varphi_1 & \cdots & \varphi_N \end{bmatrix}$ so that its **Gram matrix**

$$\mathbf{\Phi}^*\mathbf{\Phi} = \begin{bmatrix} \varphi_1^* \\ \vdots \\ \varphi_N^* \end{bmatrix} \begin{bmatrix} \varphi_1 & \cdots & \varphi_N \end{bmatrix} = \begin{bmatrix} \langle \varphi_1, \varphi_1 \rangle & \cdots & \langle \varphi_1, \varphi_N \rangle \\ \vdots & \ddots & \vdots \\ \langle \varphi_N, \varphi_1 \rangle & \cdots & \langle \varphi_1, \varphi_N \rangle \end{bmatrix}$$

has off-diagonal entries of minimal ∞-norm, subject to having 1's in its diagonal.

Motivating applications

Spherical codes:

- Error-detecting and -correcting binary codes (± 1-valued real vectors)

Projective codes:

- Waveform design for communication, radar, sonar, etc.:
  - **autocorrelation** function gives coherence with own translates
  - **ambiguity** function gives coherence with own translates and modulates
  - pings, chirps, OFDM, CDMA, etc.
- Compressed sensing
- Quantum tomography: recover a $D \times D$ self-adjoint matrix $\mathbf{A}$ from

$$\{\varphi_n^* \mathbf{A} \varphi_n\}_{n=1}^N = \langle \varphi_n \varphi_n^*, \mathbf{A} \rangle_{\mathrm{Fro}}$$

- Design of experiments (± 1-valued real vectors in $\mathbf{1}^\perp$)

Embedding projective spaces into Euclidean spaces

Idea: Since $\langle \varphi_1\varphi_1^*, \varphi_2\varphi_2^*\rangle_{\mathrm{Fro}} = |\langle \varphi_1, \varphi_2\rangle|^2$, consider the mapping

$$\varphi \in \mathbb{F}^D \;\longmapsto\; x := \varphi\varphi^* \in \{\mathbf{A} \in \mathbb{F}^{D\times D} : \mathbf{A} = \mathbf{A}^*\} \cong \begin{cases} \mathbb{R}^{\frac{D(D+1)}{2}}, & \mathbb{F} = \mathbb{R}, \\ \mathbb{R}^{D^2}, & \mathbb{F} = \mathbb{C}. \end{cases}$$

In particular, $\{\varphi_n\}_{n=1}^N$ has minimal coherence if $\{\mathbf{x}_n\}_{n=1}^N$, $x_n := \varphi_n\varphi_n^*$ is an optimal spherical code in $\{\mathbf{A} \in \mathbb{F}^{D\times D} : \mathbf{A} = \mathbf{A}^*\}$. Unfortunately, it never will be.

Nevertheless, this gives the **(proto)-Gerzon bound:**

$$\mathrm{rank}(\mathbf{X}) = \mathrm{rank}(\mathbf{X}^*\mathbf{X}) = \mathrm{rank}(|\mathbf{\Phi}^*\mathbf{\Phi}|^2) \le \begin{cases} \frac{D(D+1)}{2}, & \mathbb{F} = \mathbb{R}, \\ D^2, & \mathbb{F} = \mathbb{C}, \end{cases}$$

where equality holds $\Leftrightarrow$ any self-adjoint $\mathbf{A}$ can be recovered from $\{\varphi_n^*\mathbf{A}\varphi_n\}_{n=1}^N$.

Embedding projective spaces into Euclidean spaces (redux)

Idea: [Conway, Hardin, Sloane 96] Instead map any $\varphi \in \mathbb{F}^D$ to

$$x := \left(\tfrac{D}{D-1}\right)^{\frac{1}{2}}\left(\varphi\varphi^* - \tfrac{1}{D}\mathbf{I}\right)$$

in $\{\mathbf{A} \in \mathbb{F}^{D\times D} : \mathbf{A} = \mathbf{A}^*, \mathrm{Tr}(\mathbf{A}) = 0\} \cong \begin{cases} \mathbb{R}^{\frac{D(D+1)}{2}-1}, & \mathbb{F} = \mathbb{R}, \\ \mathbb{R}^{D^2-1}, & \mathbb{F} = \mathbb{C}. \end{cases}$

For any unit vectors $\varphi_1, \varphi_2 \in \mathbb{F}^D$,

$$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle_{\mathrm{Fro}} = \tfrac{D}{D-1}\,\mathrm{Tr}\left[\left(\varphi_1\varphi_1^* - \tfrac{1}{D}\mathbf{I}\right)\left(\varphi_2\varphi_2^* - \tfrac{1}{D}\mathbf{I}\right)\right] = \tfrac{D}{D-1}\left(|\langle \varphi_1, \varphi_2 \rangle|^2 - 1\right).$$

Problem: Design $\{\varphi_n\}_{n=1}^N$ so that $\{\mathbf{x}_n\}_{n=1}^N$ is an optimal spherical code.

Equiangular Tight Frames (ETFs)
and
Mutually Unbiased Bases (MUBs)

The Welch bound and equiangular tight frames (ETFs)

Theorem: [Rankin 56, Welch 74, Conway Hardin Sloane 96]:
Let $N > 1$, $N \geq D \geq 1$. For any N unit vectors $\{\varphi_n\}_{n=1}^N$ in $\mathbb{F}^D$,

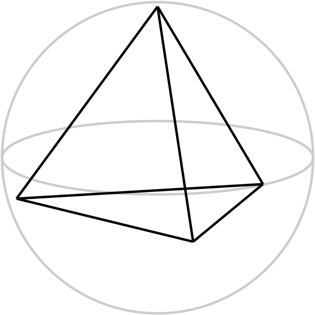
$$\max_{n_1 \neq n_2} |\langle \varphi_{n_1}, \varphi_{n_2} \rangle| \geq \left[\tfrac{N-D}{D(N-1)}\right]^{\frac{1}{2}},$$

where equality holds $\Leftrightarrow$ $\{\varphi_n\}_{n=1}^N$ is an $\mathrm{ETF}(D, N)$ for $\mathbb{F}^D$, namely

- $\exists A > 0$ such that $\mathbf{\Phi\Phi}^* = \sum_{n=1}^N \varphi_n \varphi_n^* = A\mathbf{I}$,
- $\exists C \geq 0$ such that $|(\mathbf{\Phi}^*\mathbf{\Phi})(n_1, n_2)| = |\langle \varphi_{n_1}, \varphi_{n_2} \rangle| = C$ for all $n_1 \neq n_2$.

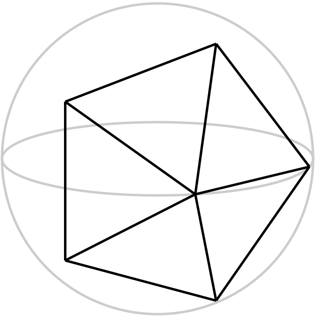
Gerzon's bound: If $\exists\, \mathrm{ETF}(D, N)$ then $N \leq \begin{cases} \frac{D(D+1)}{2}, & \mathbb{F} = \mathbb{R}, \\ D^2, & \mathbb{F} = \mathbb{C}. \end{cases}$

Example: $\mathbb{R}$ETF(3, 4) (vertices of tetrahedron)



$$\mathbf{\Phi} = \frac{1}{\sqrt{3}} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}, \quad \mathbf{\Phi}\mathbf{\Phi}^* = \frac{1}{3} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{\Phi}^*\mathbf{\Phi} = \frac{1}{3} \begin{bmatrix} 3 & - & - & - \\ - & 3 & - & - \\ - & - & 3 & - \\ - & - & - & 3 \end{bmatrix}.$$

Example: $\mathbb{R}$ETF(3, 6) (antipodes of icosahedron)



$$\mathbf{\Phi}\mathbf{\Phi}^* = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{\Phi}^*\mathbf{\Phi} = \mathbf{I} + \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & + & + & + & + & + \\ + & 0 & + & - & - & + \\ + & + & 0 & + & - & - \\ + & - & + & 0 & + & - \\ + & - & - & + & 0 & + \\ + & + & - & - & + & 0 \end{bmatrix}$$

Example: $\mathbb{C}$ETF$(3,7)$ (quadratic residues modulo 7)

$$\mathbf{\Phi} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^1 & \omega^3 & \omega^5 \\ 1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 \end{bmatrix}$$

$$\mathbf{\Phi}\mathbf{\Phi}^* = \frac{1}{3} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\mathbf{\Phi}^*\mathbf{\Phi} = \frac{1}{3} \begin{bmatrix} 1 & \zeta & \zeta & \overline{\zeta} & \zeta & \overline{\zeta} & \overline{\zeta} \\ \overline{\zeta} & 1 & \zeta & \zeta & \overline{\zeta} & \zeta & \overline{\zeta} \\ \overline{\zeta} & \overline{\zeta} & 1 & \zeta & \zeta & \overline{\zeta} & \zeta \\ \zeta & \overline{\zeta} & \overline{\zeta} & 1 & \zeta & \zeta & \overline{\zeta} \\ \overline{\zeta} & \zeta & \overline{\zeta} & \overline{\zeta} & 1 & \zeta & \zeta \\ \zeta & \overline{\zeta} & \zeta & \overline{\zeta} & \overline{\zeta} & 1 & \zeta \\ \zeta & \zeta & \overline{\zeta} & \zeta & \overline{\zeta} & \overline{\zeta} & 1 \end{bmatrix}$$

$$\omega = \exp\left(\tfrac{2\pi \mathrm{i}}{7}\right)$$

$$\zeta = 1 + \omega + \omega^4$$

Example: $\mathbb{C}$ETF(3, 9) (SIC-POVM)

$$\mathbf{\Phi} = \frac{1}{\sqrt{2}} \left[\begin{array}{ccc|ccc|ccc} 1 & \omega & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & \omega \\ 1 & \omega^2 & \omega & 1 & \omega & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \omega^2 & \omega & 1 & \omega & \omega^2 \end{array}\right] \qquad \mathbf{\Phi}\mathbf{\Phi}^* = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\mathbf{\Phi}^*\mathbf{\Phi} = \frac{1}{2} \left[\begin{array}{ccc|ccc|ccc} 2 & - & - & 1 & \omega & \omega^2 & 1 & \omega^2 & \omega \\ - & 2 & - & \omega & \omega^2 & 1 & \omega^2 & \omega & 1 \\ - & - & 2 & \omega^2 & 1 & \omega & \omega & 1 & \omega^2 \\ \hline 1 & \omega^2 & \omega & 2 & - & - & 1 & \omega & \omega^2 \\ \omega^2 & \omega & 1 & - & 2 & - & \omega & \omega^2 & 1 \\ \omega & 1 & \omega^2 & - & - & 2 & \omega^2 & 1 & \omega \\ \hline 1 & \omega & \omega^2 & 1 & \omega^2 & \omega & 2 & - & - \\ \omega & \omega^2 & 1 & \omega^2 & \omega & 1 & - & 2 & - \\ \omega^2 & 1 & \omega & \omega & 1 & \omega^2 & - & - & 2 \end{array}\right]$$

The orthoplex bound

Theorem: [Conway Hardin Sloane 96]: Let $N > \begin{cases} \frac{D(D+1)}{2}, & \mathbb{F} = \mathbb{R}, \\ D^2, & \mathbb{F} = \mathbb{C}. \end{cases}$

Then $\max_{n_1 \neq n_2} |\langle \boldsymbol{\varphi}_{n_1}, \boldsymbol{\varphi}_{n_2} \rangle| \geq \frac{1}{\sqrt{D}}$ for any N unit vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ in $\mathbb{F}^D$.
If $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ achieves equality, it's an **orthopletic Grassmannian frame (OGF)**.

Examples: (from the infinite families of OGFs of [Bodmann Hass 16]):

$$\exists\, \mathbb{R}\mathrm{OGF}(3,7): \quad \boldsymbol{\Phi} = \frac{1}{\sqrt{3}} \left[\begin{array}{cccc|ccc} + & - & + & - & \sqrt{3} & 0 & 0 \\ + & + & - & - & 0 & \sqrt{3} & 0 \\ + & - & - & + & 0 & 0 & \sqrt{3} \end{array}\right]$$

$$\exists\, \mathbb{C}\mathrm{OGF}(3,10): \quad \boldsymbol{\Phi} = \frac{1}{\sqrt{3}} \left[\begin{array}{ccccccc|ccc} 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \sqrt{3} & 0 & 0 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^1 & \omega^3 & \omega^5 & 0 & \sqrt{3} & 0 \\ 1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 & 0 & 0 & \sqrt{3} \end{array}\right]$$

Mutually unbiased bases (MUBs)

Definition: A sequence of M orthonormal bases $\{\varphi_{m,d}\}_{m=1,\ d=1}^{M\quad D}$ for $\mathbb{F}^D$ are **mutually unbiased** if $|\langle\varphi_{m_1,d_1},\varphi_{m_2,d_2}\rangle| = \frac{1}{\sqrt{D}}$ for all $m_1, m_2, d_1, d_2, m_1 \neq m_2$.

Example: $\exists\,\mathbb{R}\mathrm{MUB}(2,2) : \mathbf{\Phi} = \frac{1}{\sqrt{2}}\left[\begin{array}{cc|cc} + & + & \sqrt{2} & 0 \\ + & - & 0 & \sqrt{2} \end{array}\right]$

Example: $\exists\mathbb{C}\mathrm{MUB}(3,4) : \mathbf{\Phi} = \frac{1}{\sqrt{3}}\left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \sqrt{3} & 0 & 0 \\ 1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega^2 & 1 & \omega & 0 & \sqrt{3} & 0 \\ 1 & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega^2 & \omega & 1 & 0 & 0 & \sqrt{3} \end{array}\right]$

Corollary of Proto-Gerzon: If $\exists\,\mathbb{F}\mathrm{MUB}(D,M)$ then $M \leq \begin{cases} \frac{D}{2}+1, & \mathbb{F}=\mathbb{R}, \\ D+1, & \mathbb{F}=\mathbb{C}. \end{cases}$

Moreover, if equality is achieved then the MUB is an OGF and any self-adjoint $\mathbf{A}$ can be recovered from $\{\varphi_{m,d}^*\mathbf{A}\varphi_{m,d}\}_{m=1,\ d=1}^{M\quad D}$.

Some open problems about ETFs and MUB existence

- **When does an** $\mathrm{ETF}(D, N)$ **exist?**
  - **Zauner's conjecture:** Does a $\mathbb{C}\mathrm{ETF}(D, D^2)$ (SIC-POVM) exist for all D?
  - Necessary integrality conditions on $\mathbb{C}\mathrm{ETF}(D, N)$?
    - [Szöllősi 14] $\nexists\, \mathbb{C}\mathrm{ETF}(3, 8)$
    - For **all known** $\mathrm{ETF}(D, N)$, one member of $\{D, N - D, N - 1\}$ divides the product of the other two.

- **When does an** $\mathrm{MUB}(D, M)$ **exist?**
  - $\exists\, \mathbb{C}\mathrm{MUB}(Q, Q + 1)$ for prime power Q and $\exists\, \mathbb{R}\mathrm{MUB}(Q, \frac{Q}{2} + 1)$ for $Q = 2, 4^j$
  - if $\exists\, \mathbb{F}\mathrm{MUB}(D_1, M)$ and $\exists\, \mathbb{F}\mathrm{MUB}(D_2, M)$ then $\exists\, \mathbb{F}\mathrm{MUB}(D_1 D_2, M)$
  - [Wocjan Beth 05] MUBs from MOLS
  - Does a $\mathbb{C}\mathrm{MUB}(6, M)$ exist for $M = 3, 4, 5, 6, 7$?

Mutually Unbiased Equiangular Tight Frames (MUETFs)

MUETFs defined

Definition: A sequence $\{\psi_{m,n}\}_{m=1}^{M}\,_{n=1}^{N}$ of unit vectors in $\mathbb{F}^D$ is an MUETF(D, N, M) for $\mathbb{F}^D$ if

$$|\langle \psi_{m_1,n_1}, \psi_{m_2,n_2}\rangle|^2 = \begin{cases} \frac{N-D}{D(N-1)}, & m_1 = m_2, n_1 \neq n_2, \\ \frac{1}{D}, & m_1 \neq m_2. \end{cases}$$

Theorem: [F Mayo 20] (Proto-Gerzon corollary) If $\exists\,\mathbb{F}$MUETF(D, N, M) then

$$M \leq \begin{cases} \frac{D^2-1}{N-1}, & \mathbb{F} = \mathbb{C}, \\ \frac{(D-1)(D+2)}{2(N-1)}, & \mathbb{F} = \mathbb{R}. \end{cases}$$

- Reduces to Gerzon bound when $M = 1$
- Reduces to MUB bound when $N = D$

Example: $\mathbb{C}\mathrm{MUETF}(4,5,3)$ (F Schmitt 20)

$$\mathbf{\Phi} = \frac{1}{2}\left[\begin{array}{ccccc|ccccc|ccccc}
1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega & \omega^4 & \omega^7 & \omega^{10} & \omega^{13} & \omega^2 & \omega^5 & \omega^8 & \omega^{11} & \omega^{14} \\
1 & \omega^6 & \omega^{12} & \omega^3 & \omega^9 & \omega^2 & \omega^8 & \omega^{14} & \omega^5 & \omega^{11} & \omega^4 & \omega^{10} & \omega & \omega^7 & \omega^{13} \\
1 & \omega^9 & \omega^3 & \omega^{12} & \omega^6 & \omega^8 & \omega^2 & \omega^{11} & \omega^5 & \omega^{14} & \omega & \omega^{10} & \omega^4 & \omega^{13} & \omega^7 \\
1 & \omega^{12} & \omega^9 & \omega^6 & \omega^3 & \omega^4 & \omega & \omega^{13} & \omega^{10} & \omega^7 & \omega^8 & \omega^5 & \omega^2 & \omega^{14} & \omega^{11}
\end{array}\right] \qquad \omega = \exp\left(\frac{2\pi \mathrm{i}}{15}\right)$$

$$\mathbf{\Phi}^*\mathbf{\Phi} = \frac{1}{4}\left[\begin{array}{ccccc|ccccc|ccccc}
4 & -1 & -1 & -1 & -1 & \zeta & \zeta & \bar\zeta & -2 & \bar\zeta & \zeta & -2 & \zeta & \bar\zeta & \bar\zeta \\
-1 & 4 & -1 & -1 & -1 & \bar\zeta & \zeta & \zeta & \bar\zeta & -2 & \bar\zeta & \zeta & -2 & \zeta & \bar\zeta \\
-1 & -1 & 4 & -1 & -1 & -2 & \bar\zeta & \zeta & \zeta & \bar\zeta & \bar\zeta & \bar\zeta & \zeta & -2 & \zeta \\
-1 & -1 & -1 & 4 & -1 & \bar\zeta & -2 & \bar\zeta & \zeta & \zeta & \zeta & \bar\zeta & \bar\zeta & \zeta & -2 \\
-1 & -1 & -1 & -1 & 4 & \zeta & \bar\zeta & -2 & \bar\zeta & \zeta & -2 & \zeta & \bar\zeta & \bar\zeta & \zeta \\
\hline
\bar\zeta & \zeta & -2 & \zeta & \bar\zeta & 4 & -1 & -1 & -1 & -1 & \zeta & \zeta & \bar\zeta & -2 & \bar\zeta \\
\bar\zeta & \bar\zeta & \zeta & -2 & \zeta & -1 & 4 & -1 & -1 & -1 & \bar\zeta & \zeta & \zeta & \bar\zeta & -2 \\
\zeta & \bar\zeta & \bar\zeta & \zeta & -2 & -1 & -1 & 4 & -1 & -1 & -2 & \bar\zeta & \zeta & \zeta & \bar\zeta \\
-2 & \zeta & \bar\zeta & \bar\zeta & \zeta & -1 & -1 & -1 & 4 & -1 & \bar\zeta & -2 & \bar\zeta & \zeta & \zeta \\
\zeta & -2 & \zeta & \bar\zeta & \bar\zeta & -1 & -1 & -1 & -1 & 4 & \zeta & \bar\zeta & -2 & \bar\zeta & \zeta \\
\hline
\bar\zeta & \zeta & \zeta & \bar\zeta & -2 & \bar\zeta & \zeta & -2 & \zeta & \bar\zeta & 4 & -1 & -1 & -1 & -1 \\
-2 & \bar\zeta & \zeta & \zeta & \bar\zeta & \bar\zeta & \bar\zeta & \zeta & -2 & \zeta & -1 & 4 & -1 & -1 & -1 \\
\bar\zeta & -2 & \bar\zeta & \zeta & \zeta & \zeta & \bar\zeta & \bar\zeta & \zeta & -2 & -1 & -1 & 4 & -1 & -1 \\
\zeta & \bar\zeta & -2 & \bar\zeta & \zeta & -2 & \zeta & \bar\zeta & \bar\zeta & \zeta & -1 & -1 & -1 & 4 & -1 \\
\zeta & \zeta & \bar\zeta & -2 & \bar\zeta & \zeta & -2 & \zeta & \bar\zeta & \bar\zeta & -1 & -1 & -1 & -1 & 4
\end{array}\right] \qquad \zeta = \tfrac{1}{2}(1+\sqrt{15}i)$$

Why MUETFs matter

Theorem: [F Mayo 20] If $\{\varphi_{n_1}\}_{n_1=1}^{N_1}$ is an ETF(D_1, N_1) for $\mathbb{F}^{D_1}$ and $\{\psi_{n_1,n_2}\}_{n_1=1,\ n_2=1}^{N_1\quad N_2}$ is an MUETF(D_2, N_2, N_1) for $\mathbb{F}^{D_2}$, and

$$\tfrac{N_1-D_1}{D_1(N_1-1)} = \tfrac{N_2-D_2}{N_2-1},$$

then $\{\varphi_{n_1} \otimes \psi_{n_1,n_2}\}_{n_1=1,\ n_2=1}^{N_1\quad N_2}$ is an ETF(D_1D_2, N_1N_2) for $\mathbb{F}^{D_1D_2}$.

Example: Let $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ be an ETF$(2,3)$ and an MUETF$(4,5,3)$:

$$\boldsymbol{\Phi} = \left[\,\varphi_1\ \varphi_2\ \varphi_3\,\right] \in \mathbb{C}^{2\times 3}, \quad \boldsymbol{\Psi} = \left[\,\boldsymbol{\Psi}_1\ \boldsymbol{\Psi}_2\ \boldsymbol{\Psi}_3\,\right] \in \mathbb{C}^{4\times 12}.$$

Since $\frac{3-2}{2(3-1)} = \frac{1}{4} = \frac{5-4}{5-1}$,

$$\left[\ \varphi_1 \otimes \boldsymbol{\Psi}_1 \ \ \varphi_2 \otimes \boldsymbol{\Psi}_2 \ \ \varphi_3 \otimes \boldsymbol{\Psi}_3\ \right] \text{ is a } \mathbb{C}\mathrm{ETF}(8, 15).$$

Towards an MUETF construction: harmonic frames

Definition: Let $\mathbf{\Gamma}$ be the $\mathcal{G} \times \hat{\mathcal{G}}$ character table of a finite abelian group $\mathcal{G}$.

Restricting the characters of $\mathcal{G}$ to $\mathcal{D}$ and normalizing gives the **harmonic frame**

$$\{\boldsymbol{\varphi}_\gamma\}_{\gamma \in \hat{\mathcal{G}}}, \quad \boldsymbol{\varphi}_\gamma(d) = \tfrac{1}{\sqrt{D}}\gamma(d), \quad \text{i.e.,} \quad \sqrt{D}\mathbf{\Phi} \text{ is the } \mathcal{D} \times \hat{\mathcal{G}} \text{ submatrix of } \mathbf{\Gamma}.$$

Facts:

- Harmonic frames are tight: $(\mathbf{\Phi}\mathbf{\Phi}^*)(d_1, d_2) = \frac{1}{D}(\mathbf{\Gamma}\mathbf{\Gamma}^*)(d_1, d_2) = \frac{N}{D}\mathbf{I}(d_1, d_2)$.
- Harmonic frames are unit norm with

$$|\langle \boldsymbol{\varphi}_{\gamma_1}, \boldsymbol{\varphi}_{\gamma_2} \rangle|^2 = \tfrac{1}{D}|(\mathbf{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}})(\gamma_1\gamma_2^{-1})|^2 = \tfrac{1}{D}\mathbf{\Gamma}^*(\boldsymbol{\chi}_{\mathcal{D}} * \boldsymbol{\chi}_{-\mathcal{D}})(\gamma_1\gamma_2^{-1}).$$

- Here, $\boldsymbol{\chi}_{\mathcal{D}} * \boldsymbol{\chi}_{-\mathcal{D}}$ is the autocorrelation of the characteristic function of $\mathcal{D}$:

$$(\boldsymbol{\chi}_{\mathcal{D}} * \boldsymbol{\chi}_{-\mathcal{D}})(g) = \#[\mathcal{D} \cap (g + \mathcal{D})] = \{(d_1, d_2) \in \mathcal{D} \times \mathcal{D} : g = d_1 - d_2\}.$$

Towards an MUETF construction: relative difference sets

Definition: A subset $\mathcal{D}$ of $\mathcal{G}$ is a **difference set for $\mathcal{G}$ relative to a subgroup** $\mathcal{H} \leq \mathcal{G}$ if $\boldsymbol{\chi}_{\mathcal{D}} * \tilde{\boldsymbol{\chi}}_{\mathcal{D}} = \Lambda(\boldsymbol{\chi}_{\mathcal{G}} - \boldsymbol{\chi}_{\mathcal{H}}) + D\boldsymbol{\delta}_0$ for some $\Lambda \geq 0$, i.e.,

$$\{(d_1, d_2) \in \mathcal{D} \times \mathcal{D} : g = d_1 - d_2\} = \begin{cases} 0, & g \in \mathcal{H}, \\ \Lambda, & g \notin \mathcal{H}. \end{cases}$$

A **difference set** is an RDS with $\mathcal{H} = \{0\}$.

Examples: $\mathcal{D} = \{1, 2, 8, 4\}$ is an RDS$(5, 3, 4, 1)$ for $\mathcal{G} = \mathbb{Z}_{15}$, $\mathcal{H} = \{0, 5, 10\}$;
$\mathcal{D} = \{00, 11, 12\}$ is an RDS$(3, 3, 3, 1)$ for $\mathcal{G} = \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathcal{H} = \{0\} \times \mathbb{Z}_3$:

−	1	2	8	4
1	0	14	8	12
2	1	0	9	13
8	7	6	0	4
4	3	2	11	0

−	00	11	21
00	00	22	12
11	11	00	01
21	21	20	00

Harmonic MUETFs "=" relative difference sets

Theorem: [F & Mayo 20] Let $\mathcal{H}^\perp = \{\gamma \in \hat{\mathcal{G}} : \gamma(h) = 1 \;\forall\, h \in \mathcal{H}\}$. Then

$$\{\psi_{\alpha,\beta}\}_{\alpha\in\hat{\mathcal{G}}/\mathcal{H}^\perp,\, \beta\in\mathcal{H}^\perp} \subseteq \mathbb{C}^{\mathcal{D}}, \quad \psi_{\alpha,\beta}(d) := \tfrac{1}{\sqrt{D}}\alpha(d)\beta(d),$$

is an MUETF$(D, \frac{G}{H}, H)$ for $\mathbb{C}^{\mathcal{D}}$ if and only if $\mathcal{D}$ is an $\mathcal{H}$-RDS$(\frac{G}{H}, H, D, \Lambda)$ for $\mathcal{G}$.

This is a unification and generalization of:

- [König 99, Strohmer Heath 03, Xia Zhou Giannakis 05, Ding Feng 07] ETF(D, G) = MUETF$(D, G, 1)$ from DS(G, D, Λ) = RDS$(G, 1, D, \Lambda)$.
- [Godsil Roy 09] MUB(D, H) = MUETF(D, D, H) from RDS$(\frac{G}{H}, H, D, \Lambda)$ where $G = HD$.

Putting it together

Theorem: [Gordon Mills Welch 62] For any prime power Q and integer $J \geq 2$, $\exists\,\mathrm{RDS}(\frac{Q^J-1}{Q-1}, Q-1, Q^{J-1}, Q^{J-2})$ for $\mathcal{G} = \mathbb{F}_{Q^J}^{\times}$ relative to $\mathcal{H} = \mathbb{F}_{Q}^{\times}$.

Theorem: [F Mayo 20] If an ETF(D, N) exists where $D < N < 2D$ and

$$Q = \tfrac{D(N-1)}{N-D}$$

is a prime power, then for all integers $J \geq 1$, $\exists\,\mathbb{C}\mathrm{ETF}(D^{(J)}, N^{(J)})$ where

$$D^{(J)} = DQ^{J-1}, \qquad N^{(J)} = N(\tfrac{Q^J-1}{Q-1}).$$

Note: When applied to a harmonic ETF arising from the complement of a Singer difference set, this result yields another harmonic ETF of this same type. This theorem generalizes a difference set construction of [Gordon Mills Welch 62].

Example: A new infinite family of ETFs from the ETF$(3, 9)$

The Naimark complement of an ETF$(3, 9)$ is an ETF$(6, 9)$ and

$$Q = \tfrac{D(N-1)}{N-D} = \tfrac{6(9-1)}{9-6} = 16$$

is a prime power. For any $J \geq 2$, [Gordon Mills Welch 62] gives an

$$\mathrm{RDS}(\tfrac{1}{15}(16^J - 1), 15, 16^{J-1}, 16^{J-2}),\ \text{i.e., an MUETF}(16^{J-1}, \tfrac{1}{5}(16^J - 1), 15).$$

Tensoring the ETF$(6, 9)$ with 9 of these 15 yields an ETF$(D^{(J)}, N^{(J)})$ with

$$(D^{(J)}, N^{(J)}) = (6(16^{J-1}), \tfrac{9}{15}(16^J - 1)) = (96, 153),\ (1536, 2457), \ldots$$

Their Naimark complements have parameters

$$(D^{(J)}, N^{(J)}) = (57, 153),\ (921, 2457), \ldots$$

Takeaways and future work

- ETFs and MUBs are different types of optimal projective codes.
- MUETFs unify and generalize ETFs and MUBs into a common framework, with a common "Gerzon" bound and "difference set" equivalence.
- Tensoring an ETF with a "compatible" MUETF yields another ETF.
- Harmonic MUETFs equate to relative difference sets, and classical examples of these allow us to build an infinite family of ETF from any single ETF(D, N) where $Q = \frac{D(N-1)}{N-D}$ is a prime power. ("Most" of these are new.)
- When do MUETF(D, N, M) with $D < N$ (non-MUBs) exist?
  - **Every known** example of such an MUETF is complex.
    MUBs can be real, are not necessarily constructed harmonically.
  - Beyond [Gordon Mills Welch 62] we only have a $\mathbb{C}$MUETF$(16, 21, 6)$.

(Thanks! Questions?)