A survey on complex conference matrices

Boumediene Et-Taoui

IRIMAS Mulhouse

Mulhouse , 28th september 2021

1. Real conference matrices
2. Paley symmetric conference matrices
3. Complex Hermitian conference matrices
4. Complex equiangular thigt frames
5. Complex symmetric conference matrices of odd orders
6. Equi-isoclinic planes in Euclidean spaces
7. Complex skew-symmetric conference matrices
8. Open problems

A complex $n \times n$ *conference matrix* C is a matrix with $C_{ii} = 0$ and $|C_{ij}| = 1$, $i \neq j$ that satisfies

$$C^* C = (n-1) I_n.$$

Real conference matrices have been heavily studied in the literature in connection with combinatorial designs in geometry, engineering, statistics, and algebra.
Complex conference matrices have received considerable attention in the past few years due to their application in quantum information theory and in geometry.

Here two matrices are equivalent if one can be obtained from the other by simultaneous permutation of rows and corresponding columns, and or multiplication of some rows or columns with complex unit numbers, in other words two complex conference matrices A and A' are equivalent if there exist unitary diagonal matrices D_1 and D_2 and a permutation matrix S such that $A' = D_1 SAS^T D_2$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is symmetric but equivalent to the skew-symmetric matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

.

Here is the unique skew-symmetric conference matrix of order 4:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}.$$

It is known from Sylvester that if C is a real skew-symmetric conference matrix of order n then by construction the matrix

$$C_{2n} = \begin{pmatrix} C & C - I_n \\ C + I_n & -C \end{pmatrix},$$

is a real skew-symmetric conference matrix of order $2n$.

Here is the unique 6-order real symmetric conference matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

Here is the smallest complex conference matrix

$$\begin{pmatrix} 0 & \omega & \omega^2 & \omega^2 & \omega \\ \omega & 0 & \omega & \omega^2 & \omega^2 \\ \omega^2 & \omega & 0 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega & 0 & \omega \\ \omega & \omega^2 & \omega^2 & \omega & 0 \end{pmatrix},$$

with $\omega = e^{\frac{i2\pi}{3}}$

Real conference matrices

Questions in the theory of polytopes, posed by Coxeter , led Paley to the construction of real symmetric *conference* matrices. Paley in 1933 constructed real symmetric conference matrices with orders $p^\alpha + 1 \equiv 2 \pmod 4$, p odd prime, α non-negative integer.

The following necessary conditions for the existence of a real symmetric (respectively skew-symmetric) conference matrix of order n are known :

$n \equiv 2 (\mathrm{mod}\, 4)$ and $n - 1 = a^2 + b^2$, a and b integers (respectively $n = 2$ or $n \equiv 0 \pmod 4$).

Real conference matrices

Questions in the theory of polytopes, posed by Coxeter , led Paley to the construction of real symmetric *conference* matrices. Paley in 1933 constructed real symmetric conference matrices with orders $p^\alpha + 1 \equiv 2 \pmod 4$, p odd prime, α non-negative integer.
The following necessary conditions for the existence of a real symmetric (respectively skew-symmetric) conference matrix of order n are known :
$n \equiv 2 (\bmod 4)$ and $n - 1 = a^2 + b^2$, a and b integers (respectively $n = 2$ or $n \equiv 0 \pmod 4$).

The only real conference matrices that have been constructed so far are symmetric matrices of order $n = p^{\alpha} + 1 \equiv 2 \pmod 4$, p prime, α non-negative integer (Paley) or $n = (q-1)^2 + 1$, where q is the order of a conference symmetric or skew symmetric matrix (Goethals and Seidel) or $n = (q+2)q^2 + 1$, where $q = 4t - 1 = p^{\alpha}$, p prime and $q + 3$ is the order of a conference symmetric matrix (Mathon), or $n = 5 \cdot 9^{2\alpha+1} + 1$, α non-negative integer (Seberry and Whiteman), and skew symmetric matrices of order $n = 2^s \prod_{i=1}^{r} (p_i^{\alpha_i} + 1)$, $p_i^{\alpha_i} + 1 \equiv 0 \pmod 4$, p_i primes, s, r and α_i non-negative integers (Williamson).

In fact Delsarte, Goethals and Seidel proved that essentially there are no other real conference matrices. Precisely they prove that any real conference matrix of order $n > 2$ is equivalent, under multiplication of rows and columns by -1, to a conference symmetric or to a skew symmetric matrix according as n satisfies $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$.

In addition we observe that n must be even. This is not the case for complex *conference* matrices. There is no complex conference matrix of order 3; however we can find such a matrix of order 5. Some complex conference matrices of even orders can be easily constructed. If C is a real symmetric *conference* matrix then iC is a complex symmetric *conference* matrix of even order (but equivalent to the real one). However only one method to construct complex conference matrices of odd orders is known.

Observe that complex Hermitian or skew-symmetric conference matrices exist only if their orders are even.

In addition we observe that n must be even. This is not the case for complex *conference* matrices. There is no complex conference matrix of order 3; however we can find such a matrix of order 5.
Some complex conference matrices of even orders can be easily constructed. If C is a real symmetric *conference* matrix then iC is a complex symmetric *conference* matrix of even order (but equivalent to the real one). However only one method to construct complex conference matrices of odd orders is known.
Observe that complex Hermitian or skew-symmetric conference matrices exist only if their orders are even.

In addition we observe that n must be even. This is not the case for complex *conference* matrices. There is no complex conference matrix of order 3; however we can find such a matrix of order 5. Some complex conference matrices of even orders can be easily constructed. If C is a real symmetric *conference* matrix then iC is a complex symmetric *conference* matrix of even order (but equivalent to the real one). However only one method to construct complex conference matrices of odd orders is known.
Observe that complex Hermitian or skew-symmetric conference matrices exist only if their orders are even.

Paley symmetric conference matrices

Let $GF(q)$ be the Galois field of order $q = p^\alpha$, $p^\alpha \equiv 1 \pmod 4$, p odd prime, α non-negative integer. Let $\varkappa$ denote the Legendre symbol, defined by $\varkappa(0) = 0$, $\varkappa(x) = 1$ or -1 according as x is or not a square in $GF(p^\alpha)$. It is well known that $\varkappa(-1) = 1$, $\varkappa(xy) = \varkappa(x)\varkappa(y)$, $\varkappa(x^{-1}) = (\varkappa(x))^{-1}$, $\varkappa(-x) = \varkappa(x)$ and $\sum\limits_{x \in GF(q)} \varkappa(x) = 0$.

The Paley matrix P of order $q = 2k - 1$ is defined by

$$P_{\alpha\alpha} = 0,\ \alpha = 1, ..., q, \text{ and } P_{\alpha\beta} = \varkappa(a_\alpha - a_\beta),\ \alpha \neq \beta,\\ \alpha, \beta = 1, ..., q.$$

Theorem

(Jacobsthal) For any $b \in GF(q)^*$ we have

$$\sum_{a \in GF(q)} \varkappa(a)\varkappa(a + b) = -1.$$

Paley proved that $P^2 = (2k - 1)I_{2k-1} - J_{2k-1}$ and $PJ_{2k-1} = 0$ by use of Jacobsthal's theorem.

Paley extended the matrix P to obtain a real symmetric conference matrix of order $2k$ as follows:

$$C = \begin{pmatrix} 0 & j^\top \\ j & P \end{pmatrix},$$

where j is the $(2k-1) \times 1$ matrix consisting solely of $1's$.

Goethals and Seidel proved that any Paley matrix of order $n = 2k = p^\alpha + 1 \equiv 2 \pmod 4$ is equivalent to a matrix of the form

$$C = \begin{pmatrix} 0 & 1 & j^T & j^T \\ 1 & 0 & -j^T & j^T \\ j & -j & A & B \\ j & j & B^T & -A \end{pmatrix},$$

where A and B are square matrices of order $k-1$, j is the $(k-1)\times 1$ matrix consisting solely of $1's$.

The matrices A and B satisfy :

$$A^T = A, AJ = J, BJ = JB = 0, \tag{1}$$

$$AB = BA, BB^T = B^T B, \tag{2}$$

$$A^2 + BB^T = (2k-1)I - 2J, \tag{3}$$

Theorem

(Et-Taoui)
Let $k \geq 3$ be an odd integer such that $2k = p^{\alpha} + 1$. There exists an infinite family of complex Hermitian conference matrices of order $2k$ depending on one complex parameter b of modulus 1.

Consider the matrix

$$C(b) = \begin{pmatrix} 0 & 1 & j^T & j^T \\ 1 & 0 & -j^T & j^T \\ j & -j & A & bB \\ j & j & \bar{b}B^T & -A \end{pmatrix}.$$

Theorem

(Et-Taoui)
Let $k \geq 3$ be an odd integer such that $2k = p^{\alpha} + 1$. There exists an infinite family of complex Hermitian conference matrices of order $2k$ depending on one complex parameter b of modulus 1.

Consider the matrix

$$C(b) = \begin{pmatrix} 0 & 1 & j^T & j^T \\ 1 & 0 & -j^T & j^T \\ j & -j & A & bB \\ j & j & \overline{b}B^T & -A \end{pmatrix}.$$

Important for coding and quantum information theories are real and complex equiangular tight frames. In a Hilbert space $\mathcal{H}$, a subset $F = \{f_i\}_{i \in I} \subset \mathcal{H}$ is called a *frame* for $\mathcal{H}$ provided that there are two constants $C, D > 0$ such that

$$C \|x\|^2 \leq \sum_{i \in I} |\langle x, f_i \rangle|^2 \leq D \|x\|^2 ,$$

holds for every $x \in \mathcal{H}$. If $C = D = 1$ then the set is called *normalized tight* or a *Parseval frame*.

Throughout this paragraph we use the term (n, k) frame to refer to a Parseval frame of n vectors in $\mathbb{C}^k$ equipped with the usual inner product. The ratio $\frac{n}{k}$ is called the *redundancy* of the (n, k) frame. It is well- known that any Parseval frame induces an isometric embedding of $\mathbb{C}^k$ into $\mathbb{C}^n$ which maps $x \in \mathbb{C}^k$ to its frame coefficients $(Vx)_j = \langle x, f_j \rangle$, called the analysis operator of the frame. Because V is linear, we may identify V with an $n \times k$ matrix and the vectors $\{f_1, ..., f_n\}$ denote the columns of V^*, the Hermitian conjugate of V. From Holmes and Paulsen we know that a (n, k) frame is determined up to a unitary equivalence by its Gram matrix VV^*, which is a self-adjoint projection of rank k.

If in addition the frame is uniform and equiangular that is, $\|f_i\|^2$ and $|\langle f_i, f_j\rangle|$ are constants for all $1 \leq i \leq n$ and for all $i \neq j$, $1 \leq i, j \leq n$, respectively, then follows

$$VV^* = \frac{k}{n} I_n + \sqrt{\frac{k(n-k)}{n^2(n-1)}} Q,$$

where Q is a self-adjoint matrix with diagonal entries all 0 and off-diagonal entries all of modulus 1, and I_n is the identity matrix of order n. The matrix Q is called the *Seidel matrix* or *signature matrix* associated with the (n, k) frame. The existence of an equiangular Parseval frame is known from Holmes and Paulsen to be equivalent to the existence of a Seidel matrix with two eigenvalues.

Theorem

(Et-Taoui)
For any integer $k \geq 3$ *such that* $2k = p^\alpha + 1 \equiv 2 \pmod 4$ *there exists a* $(2k, k)$ *CETF.*

However Zauner constructed in his Phd thesis $(q+1,(q+1)/2)$ *CETFs* for any odd prime power q. But I proved that the associated Seidel matrices of these frames are real symmetric conference matrices or the product by i of real skew-symmetric conference matrices.

First we recall the construction. For any odd prime power $q = p^m$ let $GF(q)$ be the Galois field of order q, χ be the legendre symbol which is a multiplicative character of $GF(q)^*$ and let ψ be the additive character defined by $\psi(a) = e^{2i\pi Tr(a)/p}$ where the Tr is the linear mapping from $GF(q)$ to F_p such that $Tr(a) = a^p + ... + a^{p^m}$. Now let $a_1, ..., a_q$ be the elements of $GF(q)$, $b_1, ..., b_{(q-1)/2}$ be the non-zero squares and $b'_1, ..., b'_{(q-1)/2}$ be the non-zero non squares. The following vectors are given in Zauner's thesis.

$$x_1 = (1, 0, ..., 0),$$

$$x_2 = (1/\sqrt{q}, \sqrt{2/q}\psi(b_1 a_1), ..., \sqrt{2/q}\psi(b_{(q-1)/2} a_1), ...,$$

$$x_{q+1} = (1/\sqrt{q}, \sqrt{2/q}\psi(b_1 a_q), ..., \sqrt{2/q}\psi(b_{(q-1)/2} a_q)).$$

Based on a formula on additive characters Zauner showed in that his vectors are unit and that the absolute value of any Hermitian product $\langle x_k, x_l \rangle$ with $k \neq l$, is equal to $\frac{1}{\sqrt{q}}$. In the following the Hermitian products $\langle x_k, x_l \rangle$ with $k \neq l$ are computed. For any $2 \leq k < l \leq q+1$ the Hermitian product $\langle x_k, x_l \rangle$ is equal to

$$\frac{1}{q} + \frac{2}{q} \sum_{s=1}^{\frac{q-1}{2}} \psi(b_s(a_k - a_l)).$$

On the one hand

$$\sum_{s=1}^{\frac{q-1}{2}} \psi(b_s(a_k - a_l)) + \sum_{s=1}^{\frac{q-1}{2}} \psi(b'_s(a_k - a_l)) = \sum_{\alpha \in GF(q)} \psi(\alpha) - 1 = -1,$$

because $\sum_{\alpha \in GF(q)} \psi(\alpha) = 0.$

On the other hand

$$\sum_{s=1}^{\frac{q-1}{2}} \psi(b_s(a_k - a_l)) - \sum_{s=1}^{\frac{q-1}{2}} \psi(b'_s(a_k - a_l)) = \frac{1}{\chi(a_k - a_l)} \sum_{\alpha \in GF(q)} \chi(\alpha)\psi(\alpha).$$

However the last sum is a general Gauss sum which was computed by Berndt and Evans. It turns out that this sum is equal to

$$(-1)^{m-1}\sqrt{q} \text{ if } p \equiv 1 \pmod 4 \text{ and}$$

$$-(-i)^m\sqrt{q} \text{ if } p \equiv -1 \pmod 4.$$

From this follows clearly that

$$\langle x_k, x_l \rangle = (-1)^{m-1}\frac{1}{\sqrt{q}}\chi(a_k - a_l) \text{ if } p \equiv 1 (\mathrm{mod}\, 4) \text{ and}$$

$$\langle x_k, x_l \rangle = -(-i)^{m}\frac{1}{\sqrt{q}}\chi(a_k - a_l) \text{ if } p \equiv -1 (\mathrm{mod}\, 4).$$

We see from the Gram matrices that with this construction we find again Paley matrices and no other complex conference matrices.

That is Zauner's construction does not lead to new $(2k, k)$ *CETFs* in comparison with our previous theorem. In the same time it is interesting to see how Zauner obtained again the Paley matrices using an additive character on $GF(q)^*$ instead of the Legendre symbol which is a multiplicative character.
Any real conference skew-symmetric matrix C of order $2k$ leads to a Seidel matrix iC with two eigenvalues and then to a $(2k, k)$ *CETF*. Note that the $2k$ vectors of this frame generate a set of equiangular lines called in my Phd thesis an *F-regular 2k-tuple* in $\mathbb{CP}^{k-1}$. This is a tuple in which all triples of lines are pairwise congruent.

Let $q = 2k - 1$, ω be any complex number of modulus one, (a_α), $\alpha = 1, \ldots, q$ be the elements of $GF(q)$, and define the square matrix $C(\omega)$ of order q by

$$c_{\alpha\alpha} = 0,\ \alpha = 1, \ldots, q, \text{ and}$$

$$c_{\alpha\beta} = \omega^{\varkappa(a_\alpha - a_\beta)},\ \alpha \neq \beta,\ \alpha, \beta = 1, \ldots, q.$$

Let $q = 2k - 1$, ω be any complex number of modulus one, (a_α), $\alpha = 1, ..., q$ be the elements of $GF(q)$, and define the square matrix $C(\omega)$ of order q by

$$c_{\alpha\alpha} = 0,\ \alpha = 1, ..., q, \text{ and}$$

$$c_{\alpha\beta} = \omega^{\varkappa(a_\alpha - a_\beta)},\ \alpha \neq \beta,\ \alpha, \beta = 1, ..., q.$$

An important formula

Here is an analogous theorem to Jacobsthal's theorem.

Theorem

(Et-Taoui) *For any $b \in GF(q)^*$ we have*

$$\sum_{a \in GF(q)^* \setminus \{-b\}} \omega^{\varkappa(a) - \varkappa(a+b)} = k - 2 + (k-1)\Re(\omega^2).$$

An important formula

Here is an analogous theorem to Jacobsthal's theorem.

Theorem

(Et-Taoui) For any $b \in GF(q)^*$ we have

$$\sum_{a \in GF(q)^* \setminus \{-b\}} \omega^{\varkappa(a)-\varkappa(a+b)} = k - 2 + (k-1)\Re(\omega^2).$$

Complex symmetric conference matrices of odd orders

Theorem

(Et-Taoui) The complex symmetric matrix $C(\omega)$ of order $2k-1$ satisfies

$$C^*C = (2k-2-c)I + cJ, \text{ with}$$

$$c = k - 2 + (k-1)\Re(\omega^2).$$

Complex symmetric conference matrices of odd orders

Theorem

(Et-Taoui) The complex symmetric matrix $C(\omega)$ of order $2k-1$ satisfies

$$C^*C = (2k-2-c)I + cJ, \text{ with}$$

$$c = k-2+(k-1)\Re(\omega^2).$$

As a consequence we obtain the following result.

Corollary

The complex symmetric matrix $C(\omega)$ of order $2k-1$ such that $\Re(\omega^2)=\frac{2-k}{k-1}$ satisfies

$$C^*C=(2k-2)I.$$

Examples

Here are the 5-order and the 9-order complex symmetric conference matrices :

$$\begin{pmatrix} 0 & \omega & \omega^2 & \omega^2 & \omega \\ \omega & 0 & \omega & \omega^2 & \omega^2 \\ \omega^2 & \omega & 0 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega & 0 & \omega \\ \omega & \omega^2 & \omega^2 & \omega & 0 \end{pmatrix},$$

with $\omega = e^{\frac{i2\pi}{3}}$, and

$$\begin{pmatrix}
0 & \omega & \omega & \omega & \omega & \omega^{-1} & \omega^{-1} & \omega^{-1} & \omega^{-1} \\
\omega & 0 & \omega^{-1} & \omega^{-1} & \omega & \omega & \omega & \omega^{-1} & \omega^{-1} \\
\omega & \omega^{-1} & 0 & \omega & \omega^{-1} & \omega & \omega^{-1} & \omega & \omega^{-1} \\
\omega & \omega^{-1} & \omega & 0 & \omega^{-1} & \omega^{-1} & \omega & \omega^{-1} & \omega \\
\omega & \omega & \omega^{-1} & \omega^{-1} & 0 & \omega^{-1} & \omega^{-1} & \omega & \omega \\
\omega^{-1} & \omega & \omega & \omega^{-1} & \omega^{-1} & 0 & \omega & \omega & \omega^{-1} \\
\omega^{-1} & \omega & \omega^{-1} & \omega & \omega^{-1} & \omega & 0 & \omega^{-1} & \omega \\
\omega^{-1} & \omega^{-1} & \omega & \omega^{-1} & \omega & \omega & \omega^{-1} & 0 & \omega \\
\omega^{-1} & \omega^{-1} & \omega^{-1} & \omega & \omega & \omega^{-1} & \omega & \omega & 0
\end{pmatrix},$$

where ω is a unit complex number such that $\Re(\omega^2) = -\frac{3}{4}$.

The following operations on the set of complex symmetric conference matrices of the same order q:

1. multiplication by a unit complex number of any row and the corresponding column,
2. interchange of rows and, simultaneously, of the corresponding columns,

generate a relation , called *equivalence*. concerning this equivalence relation we have the following result.

Theorem

(Et-Taoui) For any $q = 2k - 1 = p^{\alpha}$, p odd prime, $k \geq 3$ there exist four complex symmetric conference matrices of order q and all are equivalent.

The following operations on the set of complex symmetric conference matrices of the same order q:

1. multiplication by a unit complex number of any row and the corresponding column,
2. interchange of rows and, simultaneously, of the corresponding columns,

generate a relation , called *equivalence*. concerning this equivalence relation we have the following result.

Theorem

(Et-Taoui) For any $q = 2k - 1 = p^{\alpha}$, *p odd prime,* $k \geq 3$ *there exist four complex symmetric conference matrices of order q and all are equivalent.*

We know from Lemmens and Seidel , that there exist n equi-isoclinic planes which span $\mathbb{R}^r$ with the parameter λ if and only if there exists a real symmetric matrix of order $2n$, partitioned into square blocks of order 2 with zero blocks on the diagonal and orthonormal blocks elsewhere, whose smallest eigenvalue equals $-\lambda^{\frac{-1}{2}}$ and has multiplicity $2n - r$. The same authors pose the problem of finding the maximum number $v(2, r)$ of equi-isoclinic planes that can be imbedded in $\mathbb{R}^r$, and more precisely, the maximal number $v_\lambda(2, r)$ of equi-isoclinic planes in $\mathbb{R}^r$ with the parameter λ, that is of pairwise isoclinic planes with the same angle ϕ, $\cos^2 \phi = \lambda$.

Seidel matrices and a second formula

Let us denote $\omega = e^{i\theta}$ with $\cos(2\theta) = \frac{2-k}{k-1}$ and define the block matrix S of order $2(2k-1)$ as follows :

$$S_{\alpha\alpha} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$\alpha = 1, ..., q$, and

$$S_{\alpha\beta} = \begin{pmatrix} \cos(\theta\varkappa(a_\alpha - a_\beta)) & \sin(\theta\varkappa(a_\alpha - a_\beta)) \\ \sin(\theta\varkappa(a_\alpha - a_\beta)) & -\cos(\theta\varkappa(a_\alpha - a_\beta)) \end{pmatrix},$$

$\alpha \neq \beta$, $\alpha, \beta = 1, ..., q$.

As in case of complex symmetric conference matrices of odd orders we need the following formula.

Theorem

(Et-Taoui) Let θ be any real number, r_η the plane rotation with angle η and $b \in GF(q)^*$. Then

$$\sum_{a \in GF(q)^* \setminus \{-b\}} r_{\theta(\varkappa(a)-\varkappa(a+b))} = (k-2+(k-1)\cos(2\theta))I_2.$$

Theorem

(Et-Taoui) The matrix S of order $4k-2$ is symmetric and satisfies

$$S^2 = (2k-2)I_{4k-2}.$$

As in case of complex symmetric conference matrices of odd orders we need the following formula.

Theorem

(Et-Taoui) Let θ be any real number, r_η the plane rotation with angle η and $b \in GF(q)^*$. Then

$$\sum_{a \in GF(q)^* \setminus \{-b\}} r_{\theta(\varkappa(a)-\varkappa(a+b))} = (k-2+(k-1)\cos(2\theta))I_2.$$

Theorem

(Et-Taoui) The matrix S of order $4k-2$ is symmetric and satisfies

$$S^2 = (2k-2)I_{4k-2}.$$

As in case of complex symmetric conference matrices of odd orders we need the following formula.

Theorem

(Et-Taoui) Let θ be any real number, r_η the plane rotation with angle η and $b \in GF(q)^*$. Then

$$\sum_{a \in GF(q)^* \setminus \{-b\}} r_{\theta(\varkappa(a)-\varkappa(a+b))} = (k-2+(k-1)\cos(2\theta))I_2.$$

Theorem

(Et-Taoui) The matrix S of order $4k-2$ is symmetric and satisfies

$$S^2 = (2k-2)I_{4k-2}.$$

Corollary

S has two eigenvalues with equal multiplicities.

Corollary

For any integer $k \geq 3$ such that $2k = p^\alpha + 1 \equiv 2 \pmod 4$, p odd prime, α non-negative integer, $v_{\frac{1}{2k-2}}(2, 2k-1) = 2k-1$.

Corollary

S has two eigenvalues with equal multiplicities.

Corollary

For any integer $k \geq 3$ such that $2k = p^{\alpha} + 1 \equiv 2 \pmod 4$, p odd prime, α non-negative integer, $v_{\frac{1}{2k-2}}(2, 2k-1) = 2k - 1$.

Corollary

$v_{\frac{1}{4}}(2,5) = 5,\ v_{\frac{1}{8}}(2,9) = 9,\ v_{\frac{1}{12}}(2,13) = 13....$

In order to give more examples, we look for all odd $k \geq 3$ and $k \leq 51$ such that $2k = p^{\alpha} + 1$, p odd prime, α non-negative integer, we obtain the following corollary.

Corollary

If k is odd, $3 \leq k \leq 51$, we may construct the $(2k-1)$-order complex symmetric conference matrix and thus the maximal $(2k-1)$-tuple of equi-isoclinic planes with parameter $\frac{1}{2k-2}$ in $\mathbb{R}^{2k-1}$, except possibly in the cases $k = 11, 17, 23, 29, 33, 35, 39, 43, 47$.

Corollary

$v_{\frac{1}{4}}(2,5) = 5,\ v_{\frac{1}{8}}(2,9) = 9,\ v_{\frac{1}{12}}(2,13) = 13 \ldots$

In order to give more examples, we look for all odd $k \geq 3$ and $k \leq 51$ such that $2k = p^{\alpha} + 1$, p odd prime, α non-negative integer, we obtain the following corollary.

Corollary

If k is odd, $3 \leq k \leq 51$, we may construct the $(2k-1)$-order complex symmetric conference matrix and thus the maximal $(2k-1)$-tuple of equi-isoclinic planes with parameter $\frac{1}{2k-2}$ in $\mathbb{R}^{2k-1}$, except possibly in the cases $k = 11, 17, 23, 29, 33, 35, 39, 43, 47$.

Corollary

$v_{\frac{1}{4}}(2,5)=5,\ v_{\frac{1}{8}}(2,9)=9,\ v_{\frac{1}{12}}(2,13)=13\ldots.$

In order to give more examples, we look for all odd $k \geq 3$ and $k \leq 51$ such that $2k = p^{\alpha}+1$, p odd prime, α non-negative integer, we obtain the following corollary.

Corollary

If k is odd, $3 \leq k \leq 51$, we may construct the $(2k-1)$-order complex symmetric conference matrix and thus the maximal $(2k-1)$-tuple of equi-isoclinic planes with parameter $\frac{1}{2k-2}$ in $\mathbb{R}^{2k-1}$, except possibly in the cases $k = 11, 17, 23, 29, 33, 35, 39, 43, 47$.

Complex Hadamard matrices

Complex *conference* matrices are also important in complex Hadamard matrix theory because if C_n is a complex conference matrix of order n then by construction the matrix

$$H_{2n} = \begin{pmatrix} C_n + I_n & C_n^* - I_n \\ C_n - I_n & -C_n^* - I_n \end{pmatrix},$$

is a complex Hadamard matrix of order $2n$. A matrix of order n with unimodular entries and satisfying $HH^* = nI_n$ is called *complex Hadamard*.

Diedral complex symmetric conference matrices

Recently with Blokhuis and Brehm, we generalise the previous result. We proved that

Theorem

If there exists a real symmetric conference matrix of order n then there exists a complex symmetric conference matrix of order $n-1$.

We also constructed other complex symmetric conference matrices called diedral and which are not obtained by the previous construction.

Diedral complex symmetric conference matrices

Recently with Blokhuis and Brehm, we generalise the previous result. We proved that

Theorem

If there exists a real symmetric conference matrix of order n then there exists a complex symmetric conference matrix of order $n-1$.

We also constructed other complex symmetric conference matrices called diedral and which are not obtained by the previous construction.

Complex skew-symmetric conference matrices

Recently, with Makhlouf we constructed complex skew-symmetric conference matrices, from which we can construct complex symmetric conference matrices. These matrices can be used for our geometric problem. Let C be a complex conference matrix of order n. We may assume

$$C = \begin{pmatrix} 0 & j^\top \\ j & P \end{pmatrix},$$

and C satisfies $C^*C = (n-1)I_n$. Thus P of order $n-1$ satisfies

$$P^*P = (n-1)I_{n-1} - J_{n-1}, \tag{4}$$

$$P^*J = JP = O. \tag{5}$$

Complex skew-symmetric conference matrices

Recently, with Makhlouf we constructed complex skew-symmetric conference matrices, from which we can construct complex symmetric conference matrices. These matrices can be used for our geometric problem. Let C be a complex conference matrix of order n. We may assume

$$C = \begin{pmatrix} 0 & j^\top \\ j & P \end{pmatrix},$$

and C satisfies $C^*C = (n-1)I_n$. Thus P of order $n-1$ satisfies

$$P^*P = (n-1)I_{n-1} - J_{n-1}, \tag{4}$$

$$P^*J = JP = O. \tag{5}$$

The matrix of order $(n-1)^2$ defined by

$$T = P \otimes P + I_{n-1} \otimes J_{n-1} - J_{n-1} \otimes I_{n-1},$$

is symmetric (if P is symmetric or skew-symmetric) and satisfies

$$T^*T = (n-1)^2 I_{(n-1)^2} - J_{(n-1)^2}, \tag{6}$$

$$T^*J = JT = O. \tag{7}$$

Hence T could be extended to a complex symmetric conference matrix of order $(n-1)^2 + 1$

The matrix of order $(n-1)^2$ defined by

$$T = P \otimes P + I_{n-1} \otimes J_{n-1} - J_{n-1} \otimes I_{n-1},$$

is symmetric (if P is symmetric or skew-symmetric) and satisfies

$$T^*T = (n-1)^2 I_{(n-1)^2} - J_{(n-1)^2}, \quad (6)$$

$$T^*J = JT = O. \quad (7)$$

Hence T could be extended to a complex symmetric conference matrix of order $(n-1)^2+1$

The matrix of order $(n-1)^2$ defined by

$$T = P \otimes P + I_{n-1} \otimes J_{n-1} - J_{n-1} \otimes I_{n-1},$$

is symmetric (if P is symmetric or skew-symmetric) and satisfies

$$T^*T = (n-1)^2 I_{(n-1)^2} - J_{(n-1)^2}, \quad (6)$$

$$T^*J = JT = O. \quad (7)$$

Hence T could be extended to a complex symmetric conference matrix of order $(n-1)^2 + 1$

We also classified all complex conference matrices up to order 5 and we fully classify 6-order complex symmetric, skew-symmetric and Hermitian complex conference matrices.

1. Is there any other complex conference matrices of order 6 which are not equivalent to one of those obtained in our classification?
2. Can one find a complex conference matrix of order 7?
3. Find a necessary condition for the existence of a complex conference matrix of order n.

References

1. B.C. Berndt, R.J. Evans, *The determination of Gauss sums*, Bul. A.M.S., Vol. **5** (2) (1981), 107-129.
2. V. Belevitch, *Theory of $2n$-terminal networks with applications to conference telephony*, Elect. commun., **27** (1950), 231-244.
3. H. S. M. Coxeter, *Regular compound polytopes in more than four dimensions*, J. Math. and Phys.12 (1933), 334-345.
4. A. Blokhuis, U. Brehm and B. Et-Taoui, *Complex conference Matrices and Equi-isoclinic Planes in Euclidean Spaces*, Beit. Alg. Geom, (2017), 491-500.
5. B. Et-Taoui, *Equiangular lines in C^r*, Indag. Math., N.S., **11** (2) (2000), 201–207.

[6] B. Et-Taoui, *Equi-isoclinic planes in Euclidean spaces,* Indag. Math. N.S., **17** (2) (2006), 205–219.
[7] B. Et-Taoui, *Equi-isoclinic planes in Euclidean even dimensional spaces,* Adv. Geom. 7 (2007), 379–384.
[8] B. Et-Taoui, A. Fruchard, *Sous-espaces equi-isoclins de l'espace euclidien,* Adv. Geom. 9 (4) (2009), 471–515.
[9] B. Et-Taoui, *Infinite family of equi-isoclinic planes in Euclidean odd dimensional spaces and of complex conference matrices of odd orders*, Lin. Alg. and its applications (2018) 373-380.

[10] B. Et-Taoui, *Complex conference matrices, complex Hadamard matrices and equiangular tight frames,*, chapter in Convexity and Discrete Geometry Including Graph Theory, edited by A. Adiprasito, I. Barany and C. Vilcu, Springer (2016), arXiv : 1409.5720.

[11] B. Et-Taoui, A Makhlouf, *Complex skew-symmetric conference matrices*, to appear in Lin. Mult. Alg. (2021).

[12] P. W. H. Lemmens, J. J. Seidel, *Equi-isoclinic subspaces of Euclidean spaces,* Indag. Math. 35(2) (1973), 98-107.

[13] R. Mathon, *Symmetric conference matrices of order* pq^2+1, Can. J. Math. **30** (1978), 321–331.
[14] R. E. A. c. Paley, *On orthogonal matrices, J. Math. Phys.* **12** (1933), 311-320.
[15] J.J. Seidel, *Discrete Non-Euclidean Geometry*, chapter 15, 843-919, in Handbook of Incidence Geometry, Buekenhout Ed., Elsevier, 1995.
[16] D. Raghavarao, *Some aspects of weighing designs,* Ann. Math. Statist. , **31** (1960), 878-884.

[17] J. Seberry, A.L. Whiteman, *New Hadamard matrices and conference matrices obtained via Mathon's construction*, Graphs and Combinatorics, **4** (1988), 355-377.
[18] J. Williamson, *Hadamard's determinant theorem and the sum of four squares*, Duke Math. J. **11** (1944), 65-81.
[19] G. Zauner, *Quantum designs*, IJQI, Vol. 9, Issue 1 (2011), 445-507.