Instability of Quantum Tomography with Bounded Operators

Chris Dock, joint work with Radu V. Balan

Tufts University

christopher.dock@tufts.edu

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Quantum Computing and Measurement
Quantum Computing

QCs aim to revolutionize physics simulation and potentially our lives, if we can only glean their answers!

Figure: A model of IBM’s 127 qbit computer.
Definition (The Dirac-von Neumann axioms for a quantum system)

The space $H$ is a $\mathbb{C}$-Hilbert space of finite or countably infinite dimension.

- The set of observables a quantum system is the set of self-adjoint operators on $H$.
- A quantum state is a unit vector $\psi$ in $\hat{H} = H/U(1)$, equivalently a ray of $\hat{H}$.
- The expectation of an observable $A$ when the system is in state $\psi$ is

$$E_\psi[A] \equiv \langle \psi, A\psi \rangle$$
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\mathbb{E}_\psi[A] \equiv \langle \psi, A\psi \rangle
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**Example:** $H = \mathbb{C}^2$. $A = \hbar \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\psi = c_1 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + c_2 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.

$$
\mathbb{E}_\psi[A] = |c_1|^2 \frac{\hbar}{2} + |c_2|^2 \frac{-\hbar}{2}
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Example: $H = \mathbb{C}^2$. $A = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\psi = c_1 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + c_2 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.

$$E_\psi[A] = |c_1|^2 \frac{\hbar}{2} + |c_2|^2 \frac{-\hbar}{2}$$

Remark: $(iii)$ induces a probability measure on $\sigma(A)$. Here $\sigma(A) = \{ \frac{\hbar}{2}, \frac{-\hbar}{2} \}$ and $P_\psi(\hbar/2) = |c_1|^2 = |\langle \psi, e_1 \rangle|^2$ and $P_\psi(-\hbar/2) = |c_2|^2 = |\langle \psi, e_2 \rangle|^2$. 
**Definition (The Dirac-von Neumann axioms for a quantum system)**

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**Remark:** The dimension of $H$ (the number of output qbits) is likely to grow rapidly as QCs improve, motivating study of the infinite dimensional case.
Definition (The Dirac-von Neumann axioms for a quantum system)

The space $H$ is a $\mathbb{C}$-Hilbert space of finite or countably infinite dimension.

(i) The set of observables a quantum system is the set of self-adjoint operators on $H$

(ii) A quantum state is a unit vector $\psi$ in $\hat{H} = H/U(1)$, equivalently a ray of $\hat{H}$.

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Remark: The dimension of $H$ (the number of output qbits) is likely to grow rapidly as QCs improve, motivating study of the infinite dimensional case.

Remark: We should not expect to have access to the exact value of $E_\psi[A]$, since in general there will be measurement noise and we are only capable of taking finitely many measurements.
Quantum tomography / quantum inference

Given $\psi \in \hat{H}$ we can compute the statistics of observables $\mathcal{A} = \{A_j\}_{j \in I}$. Commonly, we need to do the opposite:

**Problem (Quantum Tomography)**

*Given measurements of $\mathcal{A}$, can we deduce $\psi \in \hat{H}$?*

**Applications:**

1. Reading the output of a quantum computer.
2. Characterizing the gain/loss of optical devices.
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**Problem (Quantum Tomography)**

*Given measurements of $\mathcal{A}$, can we deduce $\psi \in \hat{H}$?*

We will impose on our observables the restriction $\text{redundancy}(\mathcal{A}) < \infty$ where:

$$\text{redundancy}(\mathcal{A}) := \inf \{ B > 0 : \sum_{j \in I} \langle v, |A_j|v\rangle \leq B \|v\|^2 \quad \forall v \in H \}$$

$$= \sup_{v \neq 0} \frac{1}{\|v\|^2} \sum_{j \in I} \langle v, |A_j|v\rangle$$

In particular, this implies that each $A_j$ is bounded.

$\text{redundancy}(\mathcal{A})$ is a measure of how much overlapping information there is across the various observables of $\mathcal{A}$.

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*Given measurements of $\mathcal{A}$, can we deduce $\psi \in \hat{H}$?*

**Remark:** The redundancy inequality

$$\sum_{j \in I} \langle v, |A_j| v \rangle \leq B ||v||^2 \quad \forall v \in H$$

means the frame operator series $S = \sum_{j \in I} A_j$ converges absolutely in the strong operator topology.
Quantum tomography / quantum inference

Given \( \psi \in \hat{H} \) we can compute the statistics of observables \( \mathcal{A} = \{A_j\}_{j \in I} \).

Commonly, we need to do the opposite:

**Problem (Quantum Tomography)**

*Given measurements of \( \mathcal{A} \), can we deduce \( \psi \in \hat{H} \)?*

**Example:** Suppose \( \mathcal{A} \) are compact and commuting \( \implies A_j = \sum_{i \geq 1} \lambda_{ij} e_i e_i^* \).

Then if \( \psi \) is a state with coefficients \( \psi_i = \langle e_i, \psi \rangle \):

\[
\sum_{j \geq 1} \langle \psi, |A_j| \psi \rangle = \sum_{j \geq 1} E_\psi [|A_j|] = \sum_{i,j \geq 1} |\lambda_{ij}| |\psi_i|^2 = \sum_i |\psi_i|^2 \sum_{j \geq 1} |\lambda_{ij}|
\]

Meanwhile \( \sum_{i \geq 1} |\psi_i|^2 = 1 \), so if \( B \) is finite one must have

\[
\sup_{i \geq 0} \sum_{j \geq 0} |\lambda_{ij}| < \infty
\]

E.g. the “mass” of an eigvec must be finite across \( \mathcal{A} \).
Quantum tomography / quantum inference

Given $\psi \in \hat{H}$ we can compute the statistics of observables $\mathcal{A} = \{A_j\}_{j \in I}$. Commonly, we need to do the opposite:

**Problem (Quantum Tomography)**

Given measurements of $\mathcal{A}$, can we deduce $\psi \in \hat{H}$?

Let $\beta_\mathcal{A} : \hat{H} \rightarrow \ell^1(I)$ be

$$ (\beta_\mathcal{A}(\psi))_j = E_\psi[A_j] = \langle \psi, A_j \psi \rangle \quad j \in I $$

Fix $\epsilon > 0$. Under some technical requirements on $\mathcal{A}$, if one has $M$ repeated observations of $\mathcal{A}$ then the sample mean $y \in \ell^1(I)$ will satisfy

$$ P(||y - \beta_\mathcal{A}(\psi)||_1 > \epsilon) \leq \frac{C}{M\epsilon^2} \quad \text{(Banach Space Chebyshev-Inequality)} $$

$\implies$ Would like $\beta_\mathcal{A}$ to be **injective** and **lower-Lipschitz** to retrieve $\psi \in \hat{H}$
Lipschitz stability and previous work
Suppose \( \beta_A : (\hat{H}, d) \rightarrow (\beta_A(\hat{H}), \| \cdot \|_{\ell_2}) \) is \( \ell \)-lower-Lipschitz:

\[
0 < \ell = \inf_{x, y \in H \atop [x] \neq [y]} \frac{\|\beta_A(x) - \beta_A(y)\|_2}{d(x, y)}
\]

Then **Kirzbraun's Theorem** provides \( \omega : \ell^2(I) \rightarrow H \), a \( 1/\ell \)-Lipschitz extension of \( \beta^{-1} : \beta(\hat{H}) \rightarrow \hat{H} \). If additionally \( d([x], [y]) \leq b\|x - y\| \) then

\[
\Rightarrow P(\|y - \beta(\psi)\|_1 > \varepsilon) \leq \frac{C}{M\varepsilon^2} \text{ implies } P(d(\psi, \omega(y)) > \frac{b\varepsilon}{\ell}) \leq \frac{C}{M\varepsilon^2}.
\]
Suppose $\beta_A : (\hat{H}, d) \to (\beta_A(\hat{H}), \| \cdot \|_{\ell_2})$ is $\ell$-lower-Lipschitz:

$$0 < \ell = \inf_{x, y \in H \atop [x] \neq [y]} \frac{\| \beta_A(x) - \beta_A(y) \|_2}{d(x, y)}$$

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$$\implies P(\|y - \beta(\psi)\|_1 > \epsilon) \leq \frac{C}{M\epsilon^2} \text{ implies } P(d(\psi, \omega(y)) > \frac{b\epsilon}{\ell}) \leq \frac{C}{M\epsilon^2}!$$

**Theorem (Kirzbraun’s Lipschitz Extension Theorem)**

Let $H_1, H_2$ be Hilbert spaces and let $U \subset H_1$. If $f : U \to H_2$ is an $L$-Lipschitz function then there is an $L$-Lipschitz function $\omega : H_1 \to H_2$ such that $\omega|_U = f$.

**Remark:** In general Kirzbraun requires the axiom of choice, but if $H_1$ and $H_2$ are separable then $\omega$ is constructible.

**Remark:** Here $H_2 = l^2(I) \supset \beta_A(\hat{H})$. 
Here $\beta_A$ is $\ell$ lower Lipschitz implies $P(d(\psi, \omega(y)) > \frac{b\epsilon}{\ell}) \leq \frac{C}{M\epsilon^2}$
One natural choice of metric on $\hat{H}$ is that induced by the embedding of $\hat{H}$ into $\text{Sym}(H)$ via $x \mapsto xx^*$:

**Definition (Embedding Metrics)**

The family of embedding metrics $d_p : \hat{H} \times \hat{H} \to \mathbb{R}$ are defined for $p \in [1, \infty]$ by

$$d_p(x, y) = \|xx^* - yy^*\|_p$$

Where $\| \cdot \|_p$ denotes the $p$th Schatten norm.

We will take $p = 1$, yielding the (squared) lower Lipschitz constant

$$a_0 = \inf_{x, y \in H \atop [x] \neq [y]} \frac{||\beta_A(x) - \beta_A(y)||^2_2}{||xx^* - yy^*||^2_1}$$
Lower-Lipschitz w.r.t what?

A second natural choice is the quotient metric on $\hat{H}$:

**Definition (Quotient Metric)**

Define $D : \hat{H} \times \hat{H} \to \mathbb{R}$ via

$$D(x, y) = \min_{\theta \in [0, 2\pi)} \| x - ye^{i\theta} y \|$$

Because $\beta_A(\lambda x) = |\lambda|^2 \beta_A(x)$ whereas $D(\lambda x, \lambda y) = |\lambda| D(x, y)$ one defines $\alpha_A = \beta^{\frac{1}{2}}$ and analyzes

$$A_0 = \inf_{x, y \in \mathcal{H}} \frac{\| \alpha_A(x) - \alpha_A(y) \|^2}{D(x, y)^2}$$
Known results

The case of $A_j = f_j f_j^*$ rank-one positive semi-definite is understood:

**Case I:** $H \simeq \mathbb{C}^n$ is finite dimensional.

**Theorem (Balan and Zou 2016 [1])**

Let $\mathcal{F} = \{f_j\}_{j=1}^m \subset \mathbb{C}^n$ such that $\beta_{\mathcal{F}}$ is injective. Then $\beta_{\mathcal{F}}$ is $\sqrt{a_0}$-lower-Lipschitz where:

Define $\mathcal{R} : \mathbb{R}^{2n} \to \text{Sym}(\mathbb{R}^{2n})$ via $\mathcal{R}(\xi) = \sum_{j=1}^m \Phi_j \xi \xi^T \Phi_j$ where

$\Phi_j = \phi_j \phi_j^T + J\phi_j \phi_j^T J^T$, $\phi_j = \begin{bmatrix} \Re f_j \\ \Im f_j \end{bmatrix}$ and $J$ is the symplectic form $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$.

$$a_0 = \inf_{x,y \in \mathbb{C}^n \atop xx^* \neq yy^*} \frac{||\beta(x) - \beta(y)||_2^2}{||xx^* - yy^*||_1^2} = \min_{\xi \in \mathbb{R}^{2n}} \lambda_{2n-1}(\mathcal{R}(\xi)) > 0 \quad (1)$$

**TLDR:** If $\beta$ is injective it is automatically lower-Lipschitz.
Known results

The case of $A_j = f_j f_j^*$ rank-one positive semi-definite is understood:

**Case II:** $H \simeq \ell^2(\mathbb{N})$ is infinite dimensional.

**Theorem (Cahill, Casazza, Daubechies 2016 [2])**

Suppose $\mathcal{F} = \{f_j\}_{j \geq 1}$ has an upper frame bound for the infinite dimensional Hilbert space $H$ and is such that $c := \inf_{j \geq 0} \|f_j\|_2 > 0$ and such that $\alpha_{\mathcal{F}}$ is injective. Then for every $\delta > 0$ there exists $v_1, v_2 \in H$ such that

$$\min_{\theta \in [0, 2\pi)} \|v_1 - e^{i\theta} v_2\| \geq 1 \text{ and } \|\alpha_{\mathcal{F}}(v_1) - \alpha_{\mathcal{F}}(v_2)\|_{l^2(I)} < \delta.$$ 

Here $\alpha_F = \beta_F^{\frac{1}{2}}$ and the quotient metric $D(x, y) = \min_{\theta \in [0, 2\pi)} \|x - e^{i\theta} y\|_2$ is used instead of $d_1$.

**Corollary:** The theorem holds under $\alpha_F \leftrightarrow \beta_F$ and $D \leftrightarrow d_1$, i.e. $a_0 = 0$.

**TLDR:** If $\mathcal{F}$ has an upper frame bound and uniformly lower-bounded norms then $\beta_{\mathcal{F}}$ injective is never lower-Lipschitz.
Injectivity implies stability for $H = \mathbb{C}^n$

The Balan and Zou result readily extends to arbitrary observables. An easy topological proof that $\beta_A$ injective $\implies a_0 > 0$ is:

$$0 = a_0 := \inf_{x, y \in \mathbb{C}^n \atop [x] \neq [y]} \frac{\sum_{j=1}^{m} |\langle x, A_j x \rangle - \langle y, A_j y \rangle|^2}{\|xx^* - yy^*\|_1^2}$$

$$= \inf_{x, y \in \mathbb{C}^n \atop [x] \neq [y]} \frac{\sum_{j=1}^{m} |\langle A_j, xx^* - yy^* \rangle|^2}{\|xx^* - yy^*\|_1^2}$$

$$= \inf_{W \in \Delta} \sum_{j=1}^{m} |\langle A_j, W \rangle| \quad \sum_{j=1}^{m} |\langle A_j, W \rangle|$$

Here $\Delta = \{X \in \mathbb{C}^{n \times n} : \text{rank}(X^+) \leq 1, \text{rank}(X^-) \leq 1\}$. 

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$$0 = a_0 := \inf_{\substack{x, y \in \mathbb{C}^n \\ [x] \neq [y]}} \sum_{j=1}^{m} |\langle x, A_j x \rangle - \langle y, A_j y \rangle|^2 / \|xx^* - yy^*\|_1^2$$

$$= \inf_{\substack{x, y \in \mathbb{C}^n \\ [x] \neq [y]}} \sum_{j=1}^{m} |\langle A_j, xx^* - yy^* \rangle|^2 / \|xx^* - yy^*\|_1^2$$

$$= \inf_{W \in \Delta} \sum_{j=1}^{m} |\langle A_j, W \rangle|^2$$

Here $\Delta = \{X \in \mathbb{C}^{n \times n} : \text{rank}(X^+) \leq 1, \text{rank}(X^-) \leq 1\}$. For $\|B\|_2$ sufficiently small $\text{rank}(A + B) \geq \text{rank}(A) \implies \Delta$ is closed.

$\implies \Delta \cap B_1(0, 1)$ is compact $\implies \exists W = xx^* - yy^*$ with $\|W\|_1 = 1$ s.t. $\sum_{j=1}^{m} |\langle A_j, W \rangle|^2 = 0$

$\implies \beta_A$ is not injective.
Extending Cahill, Casazza, Daubechies
We’ll prove the following strengthening of the Daubechies, Cahill, Casazza result:

**Theorem**

Let $H$ be a separable infinite dimensional Hilbert space and let $\mathcal{A} = \{A_j\}_{j \geq 1} \subset B(H)$ have redundancy$(\mathcal{A}) < \infty$. Then for all $\delta > 0$ there exist $w_1, w_2 \in H$ such that

\[ ||w_1 w_1^* - w_2 w_2^*||_1 \geq 1 \quad \text{and} \quad ||\beta_{\mathcal{A}}(w_1) - \beta_{\mathcal{A}}(w_2)||_1 < \delta \]

In particular, such a collection $\mathcal{A}$ never does stable quantum tomography.

**TLDRFWT (TLDR For Whole Talk):** No collection of bounded observables with finite redundancy permits stable quantum tomography globally on an infinite dimensional Hilbert space.
Extending the Cahill, Casazza, Daubechies result to finite rank observables

As might be expected, the instability result from Cahill, Casazza, Daubechies extends to $\mathcal{A}$ finite rank positive semi-definite:

**Theorem**

Suppose $\mathcal{F} = \{f_j\}_{j \geq 1}$, $\mathcal{A} = (A_j)_{j \geq 1}$ is finite rank and PSD and has an upper operator frame bound for the infinite dimensional Hilbert space $H$ and such that $c := \inf_{j \geq 0} \|f_j\|_2^2 \|A_j\|_{\infty} > 0$ and such that $\alpha_{\mathcal{A}}$ is injective. Then for every $\delta > 0$ there exists $v_1, v_2 \in H$ such that $\min_{\theta \in [0, 2\pi]} \|v_1 - e^{i\theta} v_2\| \geq 1$ and $\|\alpha_{\mathcal{A}}(v_1) - \alpha_{\mathcal{A}}(v_2)\|_{l^2(I)} < \delta$. 

A collection of PSD observables $\mathcal{A}$ is an operator frame if $\exists A, B > 0, \forall v \in H$ $A \|v\|^2 \leq \sum_{j \in I} \langle v, A_j v \rangle \leq B \|v\|^2$. The least upper operator frame bound is redundancy($\mathcal{A}$) since $\mathcal{A}$ is PSD.
Extending the Cahill, Casazza, Daubechies result to finite rank observables

As might be expected, the instability result from Cahill, Casazza, Daubechies extends to \( A \) finite rank positive semi-definite:

**Theorem**

Suppose \( \mathcal{F} = \{f_j\}_{j \geq 1} \) \( A = (A_j)_{j \geq 1} \) is finite rank and PSD and has an upper operator frame bound for the infinite dimensional Hilbert space \( H \) and such that \( c := \inf_{j \geq 0} \|f_j\|_2^2 \|A_j\|_\infty > 0 \) and such that \( \alpha_A \) is injective. Then for every \( \delta > 0 \) there exists \( v_1, v_2 \in H \) such that \( \min_{\theta \in [0, 2\pi]} \|v_1 - e^{i\theta}v_2\| \geq 1 \) and \( \|\alpha_A(v_1) - \alpha_A(v_2)\|_{1^2(I)} < \delta \).

A collection of PSD observables \( \mathcal{A} \) is an **operator frame** if \( \exists A, B > 0 \forall v \in H \)

\[
A\|v\|^2 \leq \sum_{j \in I} \langle v, A_j v \rangle \leq B\|v\|^2
\]

**Remark:** The least upper operator frame bound is \( \text{redundancy}(\mathcal{A}) \) since \( \mathcal{A} \) is PSD.
Extending the Cahill, Casazza, Daubechies result to compact operators

One can bootstrap up to compact PSDs:

**Theorem**

Suppose $\mathcal{F} = \{f_j\}_{j \geq 1}$, $A = (A_j)_{j \geq 1}$ is compact and PSD and has an upper operator frame bound for the infinite dimensional Hilbert space $H$ and such that $c := \inf_{j \geq 0} \|f_j\|^2_2 \|A_j\|_\infty > 0$ and such that $\alpha_A$ is injective. Then for every $\delta > 0$ there exists $v_1, v_2 \in H$ such that $\min_{\theta \in [0, 2\pi]} \|v_1 - e^{i\theta}v_2\| \geq 1$ and $\|\alpha_A(v_1) - \alpha_A(v_2)\|_{l^2(I)} < \delta$. 
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Fix two sequences of positive reals $(f_k)_{k \geq 1}$ with $\lim_{k \to \infty} f_k = 0$ and $(g_j)_{j \geq 1}$ summable. Let $(r_{j,k})_{j,k \in \mathbb{N}} \subset \mathbb{N}$ be a double sequence such that

$$\|A_j - (A_j)_{r_{j,k}}\|_\infty^2 < f_k g_j$$
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Fix two sequences of positive reals $(f_k)_{k \geq 1}$ with $\lim_{k \to \infty} f_k = 0$ and $(g_j)_{j \geq 1}$ summable. Let $(r_j, k)_{j, k \in \mathbb{N} \subset \mathbb{N}}$ be a double sequence such that

$$\|A_j - (A_j)_{r_j, k}\|_\infty^2 < f_k g_j$$

By the finite rank case $\exists(W^k_l)_{l \geq 1} \subset \Delta$ with $\|W^k_l\|_1 \geq 1$ and

$$\lim_{l \to \infty} \sum_{j \geq 1} |\langle W^k_l, (A_j)_{r_j, k} \rangle|^2 = 0$$
Fix two sequences of positive reals \((f_k)_{k \geq 1}\) with \(\lim_{k \to \infty} f_k = 0\) and \((g_j)_{j \geq 1}\) summable. Let \((r_{j,k})_{j,k \in \mathbb{N}} \subset \mathbb{N}\) be a double sequence such that

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By the finite rank case \(\exists (W^k_l)_{l \geq 1} \subset \Delta\) with \(4 \geq \| W^k_l \|_1 \geq 1\) and

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Fix two sequences of positive reals \((f_k)_{k \geq 1}\) with \(\lim_{k \to \infty} f_k = 0\) and \((g_j)_{j \geq 1}\) summable. Let \((r_j,k)_{j,k \in \mathbb{N} \subset \mathbb{N}}\) be a double sequence such that

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\|A_j - (A_j)_{r_j,k}\|_\infty^2 < f_k g_j
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By the finite rank case \(\exists (W_l^k)_{l \geq 1} \subset \Delta\) with \(4 \geq \|W_l^k\|_1 \geq 1\) and

\[
\lim_{l \to \infty} \sum_{j \geq 1} |\langle W_l^k, (A_j)_{r_j,k} \rangle|^2 = 0
\]

Pass to a diagonal subsequence \((W_{l_k}^k)_{k \geq 1}\) so that

\[
\lim_{k \to \infty} \sum_{j \geq 1} |\langle W_{l_k}^k, (A_j)_{r_j,k} \rangle|^2 = 0
\]
Extending the Cahill, Casazza, Daubechies result to compact operators

Thus we have a sequence \((W^k_i)_{k \geq 1} \subset \Delta\) satisfying \(4 \geq ||W^k_i||_1 \geq 1\) and:

\[
\lim_{k \to \infty} \sum_{j \geq 1} |\langle W^k_i, (A_j)_{r,j,k} \rangle|^2 = 0
\]

We will show that \((W^k_i)_{k \geq 1}\) gives the stability counter example:

\[
\lim_{k \to \infty} \sum_{j \geq 1} |\langle W^k_i, A_j \rangle|^2 \leq 2 \lim_{k \to \infty} \sum_{j \geq 1} |\langle W^k_i, (A_j)_{r,j,k} \rangle|^2 + \sum_{j \geq 1} |\langle W^k_i, A_j - (A_j)_{r,j,k} \rangle|^2
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Thus we have a sequence \((W_{l_k}^k)_{k \geq 1} \subset \Delta\) satisfying \(4 \geq \|W_i^k\|_1 \geq 1\) and:
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\lim_{k \to \infty} \sum_{j \geq 1} |\langle W_{l_k}^k, (A_j)_{r_j, k} \rangle|^2 = 0
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\]
\[
= 2 \lim_{k \to \infty} \sum_{j \geq 1} |\langle W_{l_k}^k, A_j - (A_j)_{r_j, k} \rangle|^2
\]
Extending the Cahill, Casazza, Daubechies result to compact operators

Thus we have a sequence \( (W_k^l)_{k \geq 1} \subset \Delta \) satisfying \( 4 \geq \|W^k_i\|_1 \geq 1 \) and:

\[
\lim_{k \to \infty} \sum_{j \geq 1} |\langle W^k_k, (A_j)_{r,j} \rangle|^2 = 0
\]

We will show that \( (W^k_k)_{k \geq 1} \) gives the stability counter example:

\[
\lim_{k \to \infty} \sum_{j \geq 1} |\langle W^k_k, A_j \rangle|^2 \leq 2 \lim_{k \to \infty} \sum_{j \geq 1} |\langle W^k_k, (A_j)_{r,j} \rangle|^2 + \sum_{j \geq 1} |\langle W^k_k, (A_j - (A_j)_{r,j} \rangle|^2
\]

\[
= 2 \lim_{k \to \infty} \sum_{j \geq 1} |\langle W^k_k, (A_j - (A_j)_{r,j} \rangle|^2
\]

\[
\leq 32 \lim_{k \to \infty} \sum_{j \geq 1} \|A_j - (A_j)_{r,j}\|_\infty^2
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Extending the Cahill, Casazza, Daubechies result to compact operators

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\]

\[
\leq 32 \lim_{k \to \infty} \sum_{j \geq 1} \|A_j - (A_j)_{r_j,k}\|_\infty^2
\]

\[
\leq 32 \lim_{k \to \infty} f_k \sum_{j \geq 1} g_j = 0
\]
Extending the Cahill, Casazza, Daubechies result to compact operators (cont.)

Thus $(W^k_{l_k})_{k \geq 1}$ provides a sequence in $\Delta$ such that $||W_k^k||_2 \geq 1$ and

$$\lim_{k \to \infty} \sum_{j \geq 1} |\langle W^k_{l_k}, A_j \rangle|^2 = 0$$
Thus \((W^k_{l_k})_{k \geq 1}\) provides a sequence in \(\Delta\) such that \(\|W^k_k\|_2 \geq 1\) and

\[
\lim_{k \to \infty} \sum_{j \geq 1} |\langle W^k_{l_k}, A_j \rangle|^2 = 0
\]

If \(K\) is large enough that \(\sum_{j \geq 1} |\langle W^K_{l_k}, A_j \rangle|^2 < \delta^2\) and \(W^K_{l_k} = w_1 w_1^* - w_2 w_2^*\) then \(\|w_1 w_1^* - w_2 w_2^*\|_1 \geq 1\) and \(\|\beta_A(w_1) - \beta_A(w_2)\|_2 < \delta\).
Unfortunately that’s as far as we can bootstrap (approximation of bdd operators in weak operator sense doesn’t cut it). We found a new proof, however, of:

**Theorem**

Let $H$ be a separable infinite dimensional Hilbert space and let $\mathcal{A} = \{A_j\}_{j \geq 1} \subset B(H)$ have redundancy($\mathcal{A}$) $< \infty$. Then for all $\delta > 0$ there exist $w_1, w_2 \in H$ such that

$$||w_1 w_1^* - w_2 w_2^*||_1 \geq 1 \quad \text{and} \quad ||\beta_\mathcal{A}(w_1) - \beta_\mathcal{A}(w_2)||_1 < \delta$$

In particular, such a collection $\mathcal{A}$ never does stable quantum tomography.

**Remark:** We have removed the artificial requirement that $\inf_{j \geq 1} ||A_j||_\infty > 0$.

**Remark:** This theorem reduces to a strengthening of the previous in the case where $\mathcal{A}$ is compact and PSD.
Proof of the theorem

Fix $\delta > 0$ and fix $v \in H$ as any unit vector. Find $m$ such that

$$\sum_{j > m} \langle v, |A_j|v \rangle < \delta^2 / (32B)$$

This is possible since $\text{redundancy}(\mathcal{A}) < \infty$. 
Proof of the theorem

Fix $\delta > 0$ and fix $\nu \in H$ as any unit vector. Find $m$ such that

$$\sum_{j > m} \langle \nu, |A_j|\nu \rangle < \frac{\delta^2}{(32B)}$$

Then choose $\psi \in \text{span}\{\nu, A_1\nu, \ldots, A_m\nu\}^\perp$ and let $w_1 = \nu + \psi$, $w_2 = \nu - \psi$.

Let $A_j = A^+_j - A^-_j$ where $A^\pm_j$ are PSD with Cholesky factors $B_j$ and $C_j$ resp.
Proof of the theorem

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$$\sum_{j > m} \langle \nu, |A_j|\nu \rangle < \delta^2 / (32B)$$

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Let $A_j = A_j^+ - A_j^-$ where $A_j^{\pm}$ are PSD with Cholesky factors $B_j$ and $C_j$ resp.

$$\|\beta_A(w_1) - \beta_A(w_2)\|_1 = \sum_{j \geq 1} |\langle (\nu + \psi), A_j(\nu + \psi) \rangle - \langle (\nu - \psi), A_j(\nu - \psi) \rangle|$$

$$= 4 \sum_{j > m} |\Re \langle \nu, A_j\psi \rangle|$$

$$\leq 4 \sum_{j > m} \|B_j^*\psi\|_2 \|B_j^*\nu\|_2 + \|C_j^*\psi\|_2 \|C_j^*\nu\|_2$$
Proof of the theorem

Fix $\delta > 0$ and fix $v \in H$ as any unit vector. Find $m$ such that

$$\sum_{j > m} \langle v, |A_j|v \rangle < \frac{\delta^2}{32B}$$

Then choose $\psi \in \text{span}\{v, A_1v, \ldots, A_mv\}^\perp$ and let $w_1 = v + \psi$, $w_2 = v - \psi$.

Let $A_j = A_j^+ - A_j^-$ where $A_j^\pm$ are PSD with Cholesky factors $B_j$ and $C_j$ resp.

$$\|\beta_A(w_1) - \beta_A(w_2)\|_1 \leq 4 \sum_{j > m} \|B_j^\ast \psi\|_2 \|B_j^\ast v\|_2 + \|C_j^\ast \psi\|_2 \|C_j^\ast v\|_2$$

$$\leq 4 \sqrt{\sum_{j > m} \langle \psi, A_j^+ \psi \rangle} \sqrt{\sum_{j > m} \langle v, A_j^+ v \rangle} + 4 \sqrt{\sum_{j > m} \langle \psi, A_j^- \psi \rangle} \sqrt{\sum_{j > m} \langle v, A_j^- v \rangle}$$

$$\leq 4\sqrt{B} \left( \sqrt{\sum_{j > m} \langle v, A_j^+ v \rangle} + \sqrt{\sum_{j > m} \langle v, A_j^- v \rangle} \right) \leq 4\sqrt{2B} \sqrt{\sum_{j > m} \langle v, |A_j|v \rangle} < \delta$$
Proof of the theorem

It remains to show $\|w_1 w_1^* - w_2 w_2^*\|_1 \geq 1$. Recall $v \perp \psi$.

\[
\|w_1 w_1^* - w_2 w_2^*\|_1 \geq \|w_1 w_1^* - w_2 w_2^*\|_2 \\
= \|(v + \psi)(v + \psi) - (v - \psi)(v - \psi)^*\|_2 \\
= 2\|\psi v^* + v \psi^*\|_2 = 2\sqrt{2}
\]

This concludes the proof of the theorem.
Proof of the theorem

It remains to show $\|w_1w_1^* - w_2w_2^*\|_1 \geq 1$. Recall $v \perp \psi$.

\[
\|w_1w_1^* - w_2w_2^*\|_1 \geq \|w_1w_1^* - w_2w_2^*\|_2 \\
= \|(v + \psi)(v + \psi) - (v - \psi)(v - \psi)^*\|_2 \\
= 2\|\psi v^* + v\psi^*\|_2 = 2\sqrt{2}
\]

This concludes the proof of the theorem.

**Remark:** With minimal changes this theorem can be adapted to $\beta_A \leftrightarrow \alpha_A$ and $d_1 \leftrightarrow D$ (the measure of stability used in Cahill, Casazza, Daubechies).
Interpretation of the theorem and future work

**Theorem**

Let $H$ be a separable infinite dimensional Hilbert space and let \( A = \{A_j\}_{j \geq 1} \subset B(H) \) have $B > 0$ such that $\sum_{j \geq 1} \langle v, |A_j| v \rangle \leq B ||v||^2$ for all $v \in H$. Then for $\delta > 0$ there exist $w_1, w_2 \in H$ such that

\[
||w_1 w_1^* - w_2 w_2^*||_2 \geq 1 \quad \text{and} \quad ||\beta_A(w_1) - \beta_A(w_2)||_1 < \delta
\]

In particular, such a collection $A$ never does stable quantum tomography.

We wanted to understand and extend the Cahill, Casazza, Daubechies result in the context of quantum tomography:

**TLDRFWT (TLDR For Whole Talk):** No collection of bounded observables with finite redundancy permits stable quantum tomography globally on quantum states in an infinite dimensional Hilbert space.

**Future Work:** Two obvious question: What additional constraints on $\psi \in \hat{H}$ would allow observables to be found allowing stable recovery? What about unbounded observables?
Thank you!

References

Radu Balan and Dongmian Zou.

Jameson Cahill, Peter Casazza, and Ingrid Daubechies.