Instability of Quantum Tomography with Bounded Operators

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Quantum Computing and Meaurement

Quantum Computing

QCs aim to revolutionize physics simulation and potentially our lives, if we can only glean their answers!

Figure: A model of IBM's 127 qbit computer.
https://www.nytimes.com/2023/06/14/science/ibm-quantum-computing.html

Measurement: The axioms of quantum mechanics

Definition (The Dirac-von Neumann axioms for a quantum system)

The space H is a $\mathbb{C}$-Hilbert space of finite or countably infinite dimension.

- (i) *The set of observables a quantum system is the set of self-adjoint operators on H*
- (ii) *A quantum state is a unit vector ψ in $\hat{H} = H/U(1)$, equivalently a ray of $\hat{H}$.*
- (iii) *The expectation of an observable A when the system is in state ψ is*

$$\mathbb{E}_{\psi}[A] \equiv \langle \psi, A\psi \rangle$$

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Example: $H = \mathbb{C}^2$. $A = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\psi = c_1 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + c_2 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.

$$\mathbb{E}_\psi[A] = |c_1|^2 \frac{\hbar}{2} + |c_2|^2 \frac{-\hbar}{2}$$

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$$\mathbb{E}_\psi[A] = |c_1|^2 \frac{\hbar}{2} + |c_2|^2 \frac{-\hbar}{2}$$

Remark: (*iii*) induces a probability measure on $\sigma(A)$. Here $\sigma(A) = \{\frac{\hbar}{2}, \frac{-\hbar}{2}\}$ and $P_\psi(\hbar/2) = |c_1|^2 = |\langle \psi, e_1 \rangle|^2$ and $P_\psi(-\hbar/2) = |c_2|^2 = |\langle \psi, e_2 \rangle|^2$.

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Remark: We should not expect to have access to the exact value of $\mathbb{E}_\psi[A]$, since in general there will be measurement noise and we are only capable of taking finitely many measurements.

Quantum tomography / quantum inference

Given $\psi \in \hat{H}$ we can compute the statistics of observables $\mathcal{A} = \{A_j\}_{j \in I}$.
Commonly, we need to do the opposite:

Problem (Quantum Tomography)

Given measurements of $\mathcal{A}$, can we deduce $\psi \in \hat{H}$?

Applications:

1. Reading the output of a quantum computer.
2. Characterizing the gain/loss of optical devices.

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We will impose on our observables the restriction redundancy$(\mathcal{A}) < \infty$ where:

$$\begin{aligned} \text{redundancy}(\mathcal{A}) &:= \inf\{B > 0 : \sum_{j \in I} \langle v, |A_j| v\rangle \leq B\|v\|^2 \qquad \forall v \in H\} \\ &= \sup_{v \neq 0} \frac{1}{\|v\|^2} \sum_{j \in I} \langle v, |A_j| v\rangle \end{aligned}$$

In particular, this implies that each A_j is bounded.

redundancy$(\mathcal{A})$ is a measure of how much overlapping information there is across the various observables of A.

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Remark: The redundancy inequality

$$\sum_{j \in I} \langle v, |A_j| v \rangle \leq B ||v||^2 \qquad \forall v \in H$$

means the frame operator series $S = \sum_{j \in I} A_j$ converges absolutely in the strong operator topology.

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Example: Suppose $\mathcal{A}$ are compact and commuting $\implies A_j = \sum_{i \geq 1} \lambda_{ij} e_i e_i^*$.
Then if ψ is a state with coefficients $\psi_i = \langle e_i, \psi \rangle$:

$$\sum_{j \geq 1} \langle \psi, |A_j| \psi \rangle = \sum_{j \geq 1} \mathbb{E}_\psi[|A_j|] = \sum_{i,j \geq 1} |\lambda_{ij}| |\psi_i|^2 = \sum_i |\psi_i|^2 \sum_{j \geq 1} |\lambda_{ij}|$$

Meanwhile $\sum_{i \geq 1} |\psi_i|^2 = 1$, so if B is finite one must have

$$\sup_{i \geq 0} \sum_{j \geq 0} |\lambda_{ij}| < \infty$$

E.g. the "mass" of an eigvec must be finite across $\mathcal{A}$.

Quantum tomography / quantum inference

Given $\psi \in \hat{H}$ we can compute the statistics of observables $\mathcal{A} = \{A_j\}_{j \in I}$.
Commonly, we need to do the opposite:

Problem (Quantum Tomography)

Given measurements of $\mathcal{A}$, can we deduce $\psi \in \hat{H}$?

Let $\beta_{\mathcal{A}} : \hat{H} \to \ell^1(I)$ be

$$(\beta_{\mathcal{A}}(\psi))_j = \mathbb{E}_\psi[A_j] = \langle \psi, A_j \psi \rangle \qquad j \in I$$

Fix $\epsilon > 0$. Under some technical requirements on $\mathcal{A}$, if one has M repeated observations of $\mathcal{A}$ then the sample mean $y \in \ell^1(I)$ will satisfy

$$P(||y - \beta_{\mathcal{A}}(\psi)||_1 > \epsilon) \leq \frac{C}{M\epsilon^2} \qquad \text{(Banach Space Chebyshev-Inequality)}$$

$\implies$ Would like $\beta_{\mathcal{A}}$ to be **injective** and **lower-Lipschitz** to retrieve $\psi \in \hat{H}$

Lipschitz stability and previous work

Stable reconstruction and Kirzbraun's theorem

Suppose $\beta_{\mathcal{A}} : (\hat{H}, d) \to (\beta_{\mathcal{A}}(\hat{H}), ||\cdot||_{\ell_2})$ is ℓ-lower-Lipschitz:

$$0 < \ell = \inf_{\substack{x,y \in H \\ [x] \neq [y]}} \frac{||\beta_{\mathcal{A}}(x) - \beta_{\mathcal{A}}(y)||_2}{d(x,y)}$$

Then **Kirzbraun's Theorem** provides $\omega : \ell^2(I) \to H$, a $1/\ell$-Lipschitz extension of $\beta^{-1} : \beta(\hat{H}) \to \hat{H}$. If additionally $d([x],[y]) \leq b||x-y||$ then

$\implies P(||y - \beta(\psi)||_1 > \epsilon) \leq \frac{C}{M\epsilon^2}$ implies $P(d(\psi, \omega(y)) > \frac{b\epsilon}{\ell}) \leq \frac{C}{M\epsilon^2}$ **!**

Stable reconstruction and Kirzbraun's theorem

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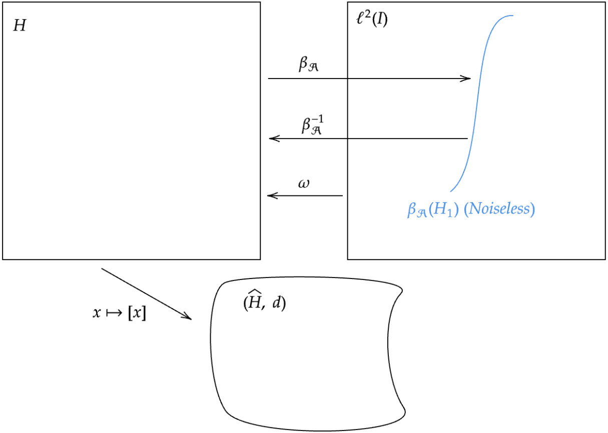
Theorem (Kirzbraun's Lipschitz Extension Theorem)

Let H_1, H_2 be Hilbert spaces and let $U \subset H_1$. If $f : U \to H_2$ is an L-Lipschitz function then there is an L-Lipschitz function $\omega : H_1 \to H_2$ such that $\omega|_U = f$.

Remark: In general Kirzbraun requires the axiom of choice, but if H_1 and H_2 are separable then ω is constructible.
Remark: Here $H_2 = l^2(I) \supset \beta_{\mathcal{A}}(\hat{H})$.

Stable reconstruction and Kirzbraun's theorem



Here $\beta_{\mathcal{A}}$ is ℓ lower Lipschitz implies $P(d(\psi, \omega(y)) > \frac{b\epsilon}{\ell}) \leq \frac{C}{M\epsilon^2}$

Lower-Lipschitz w.r.t what?

One natural choice of metric on $\hat{H}$ is that induced by the embedding of $\hat{H}$ into $\mathrm{Sym}(H)$ via $x \mapsto xx^*$:

Definition (Embedding Metrics)

The family of embedding metrics $d_p : \hat{H} \times \hat{H} \to \mathbb{R}$ are defined for $p \in [1, \infty]$ by

$$d_p(x, y) = ||xx^* - yy^*||_p$$

Where $|| \cdot ||_p$ denotes the pth Schatten norm.

We will take $p = 1$, yielding the (squared) lower Lipschitz constant

$$a_0 = \inf_{\substack{x,y \in H \\ [x] \neq [y]}} \frac{||\beta_{\mathcal{A}}(x) - \beta_{\mathcal{A}}(y)||_2^2}{||xx^* - yy^*||_1^2}$$

Lower-Lipschitz w.r.t what?

A second natural choice is the quotient metric on $\hat{H}$:

Definition (Quotient Metric)

Define $D : \hat{H} \times \hat{H} \to \mathbb{R}$ via

$$D(x,y) = \min_{\theta \in [0,2\pi)} ||x - ye^{i\theta}y||$$

Because $\beta_{\mathcal{A}}(\lambda x) = |\lambda|^2 \beta_{\mathcal{A}}(x)$ whereas $D(\lambda x, \lambda y) = |\lambda| D(x,y)$ one defines $\alpha_{\mathcal{A}} = \beta^{\odot \frac{1}{2}}$ and analyzes

$$A_0 = \inf_{\substack{x,y \in H \\ [x] \neq [y]}} \frac{||\alpha_{\mathcal{A}}(x) - \alpha_{\mathcal{A}}(y)||_2^2}{D(x,y)^2}$$

Known results

The case of $A_j = f_j f_j^*$ rank-one positive semi-definite is understood:

Case I: $H \simeq \mathbb{C}^n$ is finite dimensional.

Theorem (Balan and Zou 2016 [1])

Let $\mathcal{F} = \{f_j\}_{j=1}^m \subset \mathbb{C}^n$ such that $\beta_{\mathcal{F}}$ is injective. Then $\beta_{\mathcal{F}}$ is $\sqrt{a_0}$-lower-Lipschitz where:

Define $\mathcal{R} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n})$ via $\mathcal{R}(\xi) = \sum_{j=1}^m \Phi_j \xi \xi^T \Phi_j$ where
$\Phi_j = \phi_j \phi_j^T + J\phi_j\phi_j^T J^T$, $\phi_j = \begin{bmatrix} \Re f_j \\ \Im f_j \end{bmatrix}$ *and J is the symplectic form* $\begin{bmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}$.

$$a_0 = \inf_{\substack{x,y\in\mathbb{C}^n \\ xx^* \neq yy^*}} \frac{||\beta(x) - \beta(y)||_2^2}{||xx^* - yy^*||_1^2} = \min_{\substack{\xi\in\mathbb{R}^{2n} \\ ||\xi||_2=1}} \lambda_{2n-1}(\mathcal{R}(\xi)) > 0 \quad (1)$$

TLDR: If β is injective it is **automatically** lower-Lipschitz.

Known results

The case of $A_j = f_j f_j^*$ rank-one positive semi-definite is understood:

Case II: $H \simeq \ell^2(\mathbb{N})$ is infinite dimensional.

Theorem (Cahill, Casazza, Daubechies 2016 [2])

Suppose $\mathcal{F} = \{f_j\}_{j\geq 1}$ has an upper frame bound for the infinite dimensional Hilbert space H and is such that $c := \inf_{j\geq 0} ||f_j||_2 > 0$ and such that $\alpha_{\mathcal{F}}$ is injective. Then for every $\delta > 0$ there exists $v_1, v_2 \in H$ such that $\min_{\theta\in[0,2\pi)} ||v_1 - e^{i\theta} v_2|| \geq 1$ and $||\alpha_{\mathcal{F}}(v_1) - \alpha_{\mathcal{F}}(v_2)||_{l^2(I)} < \delta$.

Here $\alpha_F = \beta_F^{\odot \frac{1}{2}}$ and the quotient metric $D(x, y) = \min_{\theta\in[0,2\pi)} ||x - e^{i\theta} y||_2$ is used instead of d_1.

Corollary: The theorem holds under $\alpha_F \leftrightarrow \beta_F$ and $D \leftrightarrow d_1$, i.e. $a_0 = 0$.

TLDR: If $\mathcal{F}$ has an upper frame bound and uniformly lower-bounded norms then $\beta_{\mathcal{F}}$ injective is **never** lower-Lipschitz.

Injectivity implies stability for $H = \mathbb{C}^n$

The Balan and Zou result readily extends to arbitrary observables. An easy topological proof that $\beta_{\mathcal{A}}$ injective $\implies$ $a_0 > 0$ is:

$$\begin{aligned} 0 = a_0 &:= \inf_{\substack{x,y\mathbb{C}^n \\ [x]\neq[y]}} \frac{\sum_{j=1}^m |\langle x, A_j x\rangle - \langle y, A_j y\rangle|^2}{||xx^* - yy^*||_1^2} \\ &= \inf_{\substack{x,y\mathbb{C}^n \\ [x]\neq[y]}} \frac{\sum_{j=1}^m |\langle A_j, xx^* - yy^*\rangle|^2}{||xx^* - yy^*||_1^2} \\ &= \inf_{\substack{W\in\Delta \\ ||W||_1=1}} \sum_{j=1}^m |\langle A_j, W\rangle| \end{aligned}$$

Here $\Delta = \{X \in \mathbb{C}^{n\times n} : \mathrm{rank}(X^+) \leq 1, \mathrm{rank}(X^-) \leq 1\}$.

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Here $\Delta = \{X \in \mathbb{C}^{n\times n} : \mathrm{rank}(X^+) \leq 1, \mathrm{rank}(X^-) \leq 1\}$. For $||B||_2$ sufficiently

small $\mathrm{rank}(A + B) \geq \mathrm{rank}(A) \implies \Delta$ is closed.

$\implies \Delta \cap B_1(0, 1)$ is compact $\implies \exists W = xx^* - yy^*$ with $||W||_1 = 1$ s.t. $\sum_{j=1}^m |\langle A_j, W\rangle|^2 = 0$

$\implies \beta_{\mathcal{A}}$ is not injective.

Extending Cahill, Casazza, Daubechies

TLDRFWT (TLDR For Whole Talk)

We'll prove the following strengthening of the Daubechies, Cahill, Casazza result:

Theorem

Let H be a separable infinite dimensional Hilbert space and let $\mathcal{A} = \{A_j\}_{j\geq 1} \subset B(H)$ have redundancy$(\mathcal{A}) < \infty$. Then for all $\delta > 0$ there exist $w_1, w_2 \in H$ such that

$$||w_1 w_1^* - w_2 w_2^*||_1 \geq 1 \qquad \textit{and} \qquad ||\beta_{\mathcal{A}}(w_1) - \beta_{\mathcal{A}}(w_2)||_1 < \delta$$

In particular, such a collection $\mathcal{A}$ never does stable quantum tomography.

TLDRFWT (TLDR For Whole Talk): No collection of bounded observables with finite redundancy permits stable quantum tomography globally on an infinite dimensional Hilbert space.

Extending the Cahill, Casazza, Daubechies result to finite rank observables

As might be expected, the instability result from Cahill, Casazza, Daubechies extends to $\mathcal{A}$ finite rank positive semi-definite:

Theorem

Suppose $\cancel{\mathcal{F} = \{f_j\}_{j\geq 1}}$ $\mathcal{A} = (A_j)_{j\geq 1}$ *is finite rank and PSD and has an upper operator frame bound for the infinite dimensional Hilbert space H and such that* $c := \inf_{j\geq 0} \cancel{\|f_j\|_2^2} \, ||A_j||_\infty > 0$ *and such that* $\alpha_{\mathcal{A}}$ *is injective. Then for every* $\delta > 0$ *there exists* $v_1, v_2 \in H$ *such that* $\min_{\theta\in[0,2\pi)} ||v_1 - e^{i\theta} v_2|| \geq 1$ *and* $||\alpha_{\mathcal{A}}(v_1) - \alpha_{\mathcal{A}}(v_2)||_{l^2(I)} < \delta$.

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A collection of PSD observables $\mathcal{A}$ is an **operator frame** if $\exists A, B > 0 | \forall v \in H$

$$A||v||^2 \leq \sum_{j\in I} \langle v, A_j v\rangle \leq B||v||^2$$

Remark: The least upper operator frame bound is redundancy($\mathcal{A}$) since $\mathcal{A}$ is PSD.

Extending the Cahill, Casazza, Daubechies result to compact operators

One can bootstrap up to compact PSDs:

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Fix two sequences of positive reals $(f_k)_{k\geq 1}$ with $\lim_{k\to\infty} f_k = 0$ and $(g_j)_{j\geq 1}$ summable. Let $(r_{j,k})_{j,k\in\mathbb{N}} \subset \mathbb{N}$ be a double sequence such that

$$||A_j - (A_j)_{r_{j,k}}||_\infty^2 < f_k g_j$$

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By the finite rank case $\exists (W_l^k)_{l\geq 1} \subset \Delta$ with $||W_l^k||_1 \geq 1$ and

$$\lim_{l\to\infty} \sum_{j\geq 1} |\langle W_l^k, (A_j)_{r_{j,k}}\rangle|^2 = 0$$

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Pass to a diagonal subsequence $(W_{l_k}^k)_{k\geq 1}$ so that

$$\lim_{k\to\infty} \sum_{j\geq 1} |\langle W_{l_k}^k, (A_j)_{r_{j,k}}\rangle|^2 = 0$$

Extending the Cahill, Casazza, Daubechies result to compact operators

Thus we have a sequence $(W_{l_k}^k)_{k\geq 1} \subset \Delta$ satisfying $4 \geq ||W_l^k||_1 \geq 1$ and:

$$\lim_{k\to\infty} \sum_{j\geq 1} |\langle W_{l_k}^k, (A_j)_{r_{j,k}} \rangle|^2 = 0$$

We will show that $(W_{l_k}^k)_{k\geq 1}$ gives the stability counter example:

$$\lim_{k\to\infty} \sum_{j\geq 1} |\langle W_{l_k}^k, A_j \rangle|^2 \leq 2 \lim_{k\to\infty} \sum_{j\geq 1} |\langle W_{l_k}^k, (A_j)_{r_{j,k}} \rangle|^2 + \sum_{j\geq 1} |\langle W_{l_k}^k, A_j - (A_j)_{r_{j,k}} \rangle|^2$$

Extending the Cahill, Casazza, Daubechies result to compact operators

Thus we have a sequence $(W_{l_k}^k)_{k\geq 1} \subset \Delta$ satisfying $4 \geq ||W_l^k||_1 \geq 1$ and:

$$\lim_{k\to\infty} \sum_{j\geq 1} |\langle W_{l_k}^k, (A_j)_{r_{j,k}} \rangle|^2 = 0$$

We will show that $(W_{l_k}^k)_{k\geq 1}$ gives the stability counter example:

$$\lim_{k\to\infty} \sum_{j\geq 1} |\langle W_{l_k}^k, A_j \rangle|^2 \leq 2 \lim_{k\to\infty} \sum_{j\geq 1} |\langle W_{l_k}^k, (A_j)_{r_{j,k}} \rangle|^2 + \sum_{j\geq 1} |\langle W_{l_k}^k, A_j - (A_j)_{r_{j,k}} \rangle|^2$$

$$= 2 \lim_{k\to\infty} \sum_{j\geq 1} |\langle W_{l_k}^k, A_j - (A_j)_{r_{j,k}} \rangle|^2$$

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$$= 2 \lim_{k\to\infty} \sum_{j\geq 1} |\langle W_{l_k}^k, A_j - (A_j)_{r_{j,k}} \rangle|^2$$

$$\leq 32 \lim_{k\to\infty} \sum_{j\geq 1} ||A_j - (A_j)_{r_{j,k}}||_\infty^2$$

Extending the Cahill, Casazza, Daubechies result to compact operators

Thus we have a sequence $(W^k_{l_k})_{k\geq 1} \subset \Delta$ satisfying $4 \geq ||W^k_l||_1 \geq 1$ and:

$$\lim_{k\to\infty} \sum_{j\geq 1} |\langle W^k_{l_k}, (A_j)_{r_{j,k}} \rangle|^2 = 0$$

We will show that $(W^k_{l_k})_{k\geq 1}$ gives the stability counter example:

$$\lim_{k\to\infty} \sum_{j\geq 1} |\langle W^k_{l_k}, A_j \rangle|^2 \leq 2 \lim_{k\to\infty} \sum_{j\geq 1} |\langle W^k_{l_k}, (A_j)_{r_{j,k}} \rangle|^2 + \sum_{j\geq 1} |\langle W^k_{l_k}, A_j - (A_j)_{r_{j,k}} \rangle|^2$$

$$= 2 \lim_{k\to\infty} \sum_{j\geq 1} |\langle W^k_{l_k}, A_j - (A_j)_{r_{j,k}} \rangle|^2$$

$$\leq 32 \lim_{k\to\infty} \sum_{j\geq 1} ||A_j - (A_j)_{r_{j,k}}||^2_\infty$$

$$\leq 32 \lim_{k\to\infty} f_k \sum_{j\geq 1} g_j = 0$$

Extending the Cahill, Casazza, Daubechies result to compact operators (cont.)

Thus $(W^k_{l_k})_{k\geq 1}$ provides a sequence in Δ such that $||W^k_k||_2 \geq 1$ and

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Extending the Cahill, Casazza, Daubechies result to compact operators (cont.)

Thus $(W^k_{l_k})_{k\geq 1}$ provides a sequence in Δ such that $||W^k_k||_2 \geq 1$ and

$$\lim_{k\to\infty} \sum_{j\geq 1} |\langle W^k_{l_k}, A_j\rangle|^2 = 0$$

If K is large enough that $\sum_{j\geq 1} |\langle W^K_{l_K}, A_j\rangle|^2 < \delta^2$ and $W^K_{l_K} = w_1 w_1^* - w_2 w_2^*$ then $||w_1 w_1^* - w_2 w_2^*||_1 \geq 1$ and $||\beta_{\mathcal{A}}(w_1) - \beta_{\mathcal{A}}(w_2)||_2 < \delta$.

Bounded operators and a new, simple proof

Unfortunately that's as far as we can bootstrap (approximation of bdd operators in weak operator sense doesn't cut it). We found a new proof, however, of:

Theorem

Let H be a separable infinite dimensional Hilbert space and let $\mathcal{A} = \{A_j\}_{j\geq 1} \subset B(H)$ have redundancy$(\mathcal{A}) < \infty$. Then for all $\delta > 0$ there exist $w_1, w_2 \in H$ such that

$$||w_1 w_1^* - w_2 w_2^*||_1 \geq 1 \qquad \textit{and} \qquad ||\beta_{\mathcal{A}}(w_1) - \beta_{\mathcal{A}}(w_2)||_1 < \delta$$

In particular, such a collection $\mathcal{A}$ never does stable quantum tomography.

Remark: We have removed the artificial requirement that $\inf_{j\geq 1} ||A_j||_\infty > 0$.

Remark: This theorem reduces to a strengthening of the previous in the case where $\mathcal{A}$ is compact and PSD.

Proof of the theorem

Fix $\delta > 0$ and fix $v \in H$ as any unit vector. Find m such that

$$\sum_{j>m} \langle v, |A_j| v \rangle < \delta^2/(32B)$$

This is possible since redundancy($\mathcal{A}$) $< \infty$.

Proof of the theorem

Fix $\delta > 0$ and fix $v \in H$ as any unit vector. Find m such that

$$\sum_{j>m} \langle v, |A_j| v\rangle < \delta^2/(32B)$$

Then choose $\psi \in \text{span}\{v, A_1 v, \ldots, A_m v\}^\perp$ and let $w_1 = v + \psi$, $w_2 = v - \psi$.

Let $A_j = A_j^+ - A_j^-$ where $A_j^\pm$ are PSD with Cholesky factors B_j and C_j resp.

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Let $A_j = A_j^+ - A_j^-$ where $A_j^{\pm}$ are PSD with Cholesky factors B_j and C_j resp.

$$||\beta_{\mathcal{A}}(w_1) - \beta_{\mathcal{A}}(w_2)||_1 = \sum_{j\geq 1} |\langle (v+\psi), A_j(v+\psi)\rangle - \langle (v-\psi), A_j(v-\psi)\rangle|$$

$$= 4 \sum_{j>m} |\Re\langle v, A_j \psi\rangle|$$

$$\leq 4 \sum_{j>m} ||B_j^* \psi||_2 ||B_j^* v||_2 + ||C_j^* \psi||_2 ||C_j^* v||_2$$

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Let $A_j = A_j^+ - A_j^-$ where $A_j^\pm$ are PSD with Cholesky factors B_j and C_j resp.

$$||\beta_{\mathcal{A}}(w_1) - \beta_{\mathcal{A}}(w_2)||_1 \leq 4 \sum_{j>m} ||B_j^* \psi||_2 ||B_j^* v||_2 + ||C_j^* \psi||_2 ||C_j^* v||_2$$

$$\leq 4 \sqrt{\sum_{j>m} \langle \psi, A_j^+ \psi\rangle} \sqrt{\sum_{j>m} \langle v, A_j^+ v\rangle} + 4 \sqrt{\sum_{j>m} \langle \psi, A_j^- \psi\rangle} \sqrt{\sum_{j>m} \langle v, A_j^- v\rangle}$$

$$\leq 4\sqrt{B} \left(\sqrt{\sum_{j>m} \langle v, A_j^+ v\rangle} + \sqrt{\sum_{j>m} \langle v, A_j^- v\rangle} \right) \leq 4\sqrt{2B} \sqrt{\sum_{j>m} \langle v, |A_j| v\rangle} < \delta$$

Proof of the theorem

It remains to show $||w_1 w_1^* - w_2 w_2^*||_1 \geq 1$. Recall $v \perp \psi$.

$$\begin{aligned} ||w_1 w_1^* - w_2 w_2^*||_1 &\geq ||w_1 w_1^* - w_2 w_2^*||_2 \\ &= ||(v+\psi)(v+\psi) - (v-\psi)(v-\psi)^*||_2 \\ &= 2||\psi v^* + v\psi^*||_2 = 2\sqrt{2} \end{aligned}$$

This concludes the proof of the theorem.

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This concludes the proof of the theorem.

Remark: With minimal changes this theorem can be adapted to $\beta_{\mathcal{A}} \leftrightarrow \alpha_{\mathcal{A}}$ and $d_1 \leftrightarrow D$ (the measure of stability used in Cahill, Casazza, Daubechies).

Interpetation of the theorem and future work

Theorem

Let H be a separable infinite dimensional Hilbert space and let $\mathcal{A} = \{A_j\}_{j\geq 1} \subset B(H)$ have $B > 0$ such that $\sum_{j\geq 1}\langle v, |A_j| v\rangle \leq B||v||^2$ for all $v \in H$. Then for $\delta > 0$ there exist $w_1, w_2 \in H$ such that

$$||w_1 w_1^* - w_2 w_2^*||_2 \geq 1 \qquad \text{and} \qquad ||\beta_{\mathcal{A}}(w_1) - \beta_{\mathcal{A}}(w_2)||_1 < \delta$$

In particular, such a collection $\mathcal{A}$ never does stable quantum tomography.

We wanted to understand and extend the Cahill, Casazza, Daubechies result in the context of quantum tomography:

TLDRFWT (TLDR For Whole Talk): No collection of bounded observables with finite redundancy permits stable quantum tomography globally on quantum states in an infinite dimensional Hilbert space.

Future Work: Two obvious question: What additional constraints on $\psi \in \hat{H}$ would allow observables to be found allowing stable recovery? What about unbounded observables?

Thank you!

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