Multiplication-invariant operators and the classification abelian group frames

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Abstract

In this talk we discuss the properties of multiplication invariant (MI) operators acting on subspaces of the vector-valued space $L^2(X;\mathcal{H})$. We show that there is a natural isomorphism between the category of MI spaces (with MI operators as morphisms) and the category of measurable range functions whose morphisms are measurable range operators. We investigate how global properties of an MI operator are reflected by local pointwise properties of its corresponding range operator. We present several results about frames generated by multiplications in $L^2(X;\mathcal{H})$. This includes the classification of frames of multiplications with respect to unitary equivalence by measurable fields of Gramians. The talk is based on a joint work with Joey Iverson.

MI spaces

Assumption

We fix a separable Hilbert spaces $\mathcal{H}$ and a positive, σ-finite, and complete measure space $(X, \mathcal{M}, \mu)$ for which $L^2(X)$ is separable.

For $\phi \in L^\infty(X)$, a *multiplication operator* $M_\phi \in B(L^2(X;\mathcal{H}))$ is given by

$$(M_\phi \varphi)(x) = \phi(x)\varphi(x) \qquad \varphi \in L^2(X;\mathcal{H}), \text{ a.e. } x \in X.$$

Definition

A closed subspace $V \subseteq L^2(X;\mathcal{H})$ is called *multiplication-invariant* (MI) if $M_\phi V \subseteq V$ for every $\phi \in L^\infty(X)$.
Given two MI spaces $V, V' \subseteq L^2(X;\mathcal{H})$, a *multiplication-invariant operator* (*MI operator*) is a bounded linear operator $T\colon V \to V'$ such that

$$TM_\phi = M_\phi T \qquad \text{for all } \phi \in L^\infty(X).$$

Definition

A *determining set* for $L^1(X)$ is a subset $\mathcal{D}$ of the dual $L^\infty(X)$ which separates points in $L^1(X)$:

$$\forall f_1 \neq f_2 \in L^1(X)\ \exists g \in \mathcal{D} \qquad \int_X f_1 \overline{g}\, d\mu \neq \int_X f_2 \overline{g}\, d\mu.$$

Equivalently, span $\mathcal{D}$ is weak-$*$ dense in $L^\infty(X) \cong L^1(X)^*$.

Definition

A *range function* on X is a mapping

$$J \colon X \to \{\text{closed subspaces of } \mathcal{H}\}.$$

Let $P_J(x)$ be the projection onto $J(x)$. J *measurable* if P_J is weakly measurable:

$$\forall u, v \in \mathcal{H} \qquad x \mapsto \langle P_J(x)u, v\rangle \text{ is measurable on } X.$$

Characterization of MI spaces

Theorem (Helson, MB-Ross)

Let $V \subseteq L^2(X;\mathcal{H})$ be a closed subspace. For every determining set $\mathcal{D}$, the following are equivalent:

1. *V is an MI space.*
2. *For every $\phi \in \mathcal{D}$, $M_\phi V \subseteq V$.*
3. *There is a measurable range function J such that*

$$V = V_J := \{\varphi \in L^2(X;\mathcal{H}) \colon \varphi(x) \in J(x) \text{ for a.e. } x \in X\}.$$

More precisely, if an MI space V is an MI space is generated by $\{\varphi_i\}_{i\in I}$, then the corresponding range function J satisfies

$$J(x) = \overline{\operatorname{span}}\{\varphi_i(x) : i \in \mathbb{N}\} \qquad \text{for a.e. } x \in X.$$

The mapping $J \mapsto V_J$ is a bijection between measurable range functions (up to equality a.e.) and MI subspaces of $L^2(X;\mathcal{H})$.

Examples of MI spaces

1. Helson, Srinivasan (1964). Doubly invariant subspaces of $L^2(X;\mathcal{H})$, where $X=[0,1]$, $\mathcal{D}=\{e^{2\pi ikx}:k\in\mathbb{Z}\}$.
2. de Boor, DeVore, Ron (1994), MB (2000). $V\subseteq L^2(\mathbb{R}^n)$ is shift-invariant (SI) if $f(\cdot-k)\in V$ for every $k\in\mathbb{Z}^n$, $f\in V$. The fiberization operator $\mathcal{T}\colon L^2(\mathbb{R}^n)\to L^2([0,1)^n;\ell^2(\mathbb{Z}^n))$

   $$(\mathcal{T}f)(x)=\left\{\hat{f}(x-k)\right\}_{k\in\mathbb{Z}^n} \qquad f\in L^2(\mathbb{R}^n),\ x\in[0,1)^n$$

   SI space $V\leftrightarrow$ *MI* space $\mathcal{T}(V)$ with $X=[0,1]^n$, $\mathcal{D}=\{e^{2\pi i\langle k,x\rangle}:k\in\mathbb{Z}^n\}$, and $\mathcal{H}=\ell^2(\mathbb{Z}^n)$.
3. Cabrelli, Paternostro (2010). SI spaces for discrete co-compact subgroup G of a second countable locally compact abelian (LCA) group $\mathcal{G}$.
4. MB, Ross (2015). SI spaces for co-compact subgroup G of a second countable LCA group $\mathcal{G}$.
5. Iverson (2018). SI spaces for an abelian subgroup G of a second countable locally compact (not necessarily abelian) group $\mathcal{G}$. Use generalized Zak transform instead of fiberization

Range operators

Definition

Given measurable range functions

$$J, J' \colon X \to \{\text{closed subspaces of } \mathcal{H}\},$$

a *range operator* $R\colon J \to J'$ is a choice of linear operators $R(x)\colon J(x) \to J'(x)$ for each $x \in X$. We say R is *bounded* if

$$\|R\| := \operatorname{ess\,sup}_{x \in X} \|R(x)\|_{\text{op}} < \infty.$$

R is *measurable* if for every $u \in \mathcal{H}$, $v \in \mathcal{H}'$ the function $x \mapsto \langle R(x)P_J(x)u, v\rangle$ is measurable on X.

Every bounded, measurable range operator $R\colon J \to J'$ defines a bounded MI operator $\int_X^{\oplus} R(x)\, d\mu(x)\colon V_J \to V_{J'}$ by

$$\left[\int_X^{\oplus} R(x) d\mu(x)\, \varphi\right](y) = R(y)[\varphi(y)] \qquad \varphi \in V_J,\ y \in X.$$

Characterization of MI operators

Theorem (MB, Iverson)

Let

$$J, J' \colon X \to \{\textit{closed subspaces of } \mathcal{H}\}$$

be measurable range functions, and let $T \colon V_J \to V_{J'}$ *be a bounded linear operator. For every determining set* $\mathcal{D}$*, the following are equivalent:*

1. T *is an MI operator.*
2. *For every* $\phi \in \mathcal{D}$, $TM_\phi = M_\phi T$.
3. *There is a bounded measurable range operator* $R \colon J \to J'$ *such that* $T = \int_X^\oplus R(x)\, d\mu(x)$.

Moreover, the mapping $R \mapsto \int_X^\oplus R(x)\, d\mu(x)$ *gives a one-to-one correspondence between bounded measurable range operators and MI operators, provided we identify range operators that agree a.e. on* X.

Pointwise properties of MI operators

Let $V, V' \subseteq L^2(X;\mathcal{H})$ be two MI spaces with range functions J and J'. Let $T : V \to V'$ be an MI operator with the corresponding range operator $R : J \to J'$. Then, the following are true.

1. $||T\varphi|| \geq C||\varphi||$ for all $\varphi \in V \iff$
$$||R(x)v|| \geq C||v|| \qquad \text{for all } v \in J(x), \text{ a.e. } x.$$
2. T is invertible $\iff$ $R(x)$ is invertible for a.e. $x \in X$ and $\operatorname{ess\,sup}_{x\in X} \left\|R(x)^{-1}\right\|_{op} < \infty$.
3. the adjoint $T^* : V' \to V$ is an MI operator with $R^* : J' \to J$ given by $R^*(x) = (R(x))^*$ for a.e. $x \in X$.
4. T is unitary $\iff$ $R(x)$ is unitary for a.e. $x \in X$.
5. T is normal $\iff$ $R(x)$ is normal for a.e. $x \in X$.
6. T is 1-to-1 $\iff$ $R(x)$ is 1-to-1 for a.e. $x \in X$.
7. T is an isometry $\iff$ $R(x)$ is an isometry a.e. x.
8. T is a partial isometry $\iff$ $R(x)$ is a partial isometry a.e. x.

Theorem

Suppose that $V \subseteq L^2(X;\mathcal{H})$ is an MI space and $T : V \to V$ is an MI operator. Let R be its corresponding range operator so that $T = \int_X^{\oplus} R(x) d\mu(x)$. Then, the following are true.

1. *Let $A \leq B$ be two real numbers. Then, T is self-adjoint with spectrum $\sigma(T) \subseteq [A, B] \iff R(x)$ is self-adjoint with spectrum $\sigma(R(x)) \subseteq [A, B]$ for a.e. $x \in X$,*
2. *Let $K \subseteq \mathbb{C}$ be a compact set. Then, T is normal with spectrum $\sigma(T) \subseteq K \iff R(x)$ is normal with spectrum $\sigma(R(x)) \subseteq K$ for a.e. $x \in X$.*
3. *Assume that either:*
   a. *h is a holomorphic function on some neighborhood of $\sigma(T)$, or*
   b. *h is a bounded complex Borel fun. on $\sigma(T)$ and T is normal.*

   Then, $h(T)$ is also an MI operator and its corresponding range operator is $x \mapsto h(R(x))$.

Definition

A *Parseval determining set* for $L^1(X)$ consists of another σ-finite measure space (Y, ν) and a family $\{g_t\}_{t \in Y}$ in $L^\infty(X)$ such that:

1. For every $f \in L^1(X)$, the mapping $t \mapsto \int_X f \overline{g_t}\, d\mu$ is measurable on Y; and
2. For every $f \in L^1(X)$,

$$\int_Y \left| \int_X f(x) \overline{g_t(x)}\, d\mu(x) \right|^2 d\nu(t) = \int_X |f(x)|^2 d\mu(x). \tag{1}$$

(Possibly both sides are infinite.)

Example

$X = G$ is a locally compact abelian group, the dual $Y = \hat{G}$ is equipped with Plancherel measure, then characters $\{g_t\}_{t \in Y}$ form a Parseval determining set for $L^1(G)$. In this case, (1) amounts to Plancherel's Theorem.

Frames of multiplications

Let $\mathscr{A} = \{\varphi_i\}_{i\in I}$ be a countable sequence in $L^2(X;\mathcal{H})$. Define

$$E(\mathscr{A}) = \{M_{g_t}\varphi_i\}_{t\in Y,\, i\in I},$$

the system of all multiplications of $\mathscr{A}$ by $\mathcal{D}$. We write

$$S(\mathscr{A}) = \overline{\operatorname{span}}\{M_{g_t}\varphi_i \colon t \in Y,\ i \in I\}$$

for the MI space generated by $\mathscr{A}$. We have

$$S(\mathscr{A}) = \int_X^{\oplus} J(x)\, d\mu(x), \qquad \text{where } J(x) = \overline{\operatorname{span}}\{\varphi_i(x) \colon i \in I\}.$$

Theorem (Iverson (2015))

For constants $A, B > 0$, the following are equivalent:

1. *$E(\mathscr{A})$ is an A, B-frame for $S(\mathscr{A})$,*
2. *For a.e. $x \in X$, $\{\varphi_i(x)\}_{i\in I}$ is an A, B-frame for $J(x)$.*

Corollary

Under the circumstances of previous theorem, the frame operator $S\colon S(\mathscr{A}) \to S(\mathscr{A})$ *of* $E(\mathscr{A})$ *is multiplication invariant, with*

$$S = \int_X^{\oplus} S(x)\, d\mu(x).$$

Here, $S(x)\colon J(x) \to J(x)$ *is the frame operator of* $\{\varphi_i(x)\}_{i \in I}$ *for a.e.* $x \in X$*.*

Corollary

Let $\mathscr{A} = \{\varphi_i\}_{i\in I}$ *and* $\mathscr{A}' = \{\varphi_i'\}_{i\in I}$ *be sequences in* $L^2(X;\mathcal{H})$ *with* $S(\mathscr{A}) = S(\mathscr{A}')$*, and suppose that* $E(\mathscr{A})$ *and* $E(\mathscr{A}')$ *are both frames for* $S(\mathscr{A})$*. Then the following are equivalent:*

1. $E(\mathscr{A})$ *and* $E(\mathscr{A}')$ *are dual frames.*
2. *For a.e.* $x \in X$, $\{\varphi_i(x)\}_{x\in X}$ *and* $\{\varphi_i'(x)\}_{x\in X}$ *are dual frames.*

Classification of unitarily equivalent frames

Theorem

Assume $\{g_t\}_{t\in Y}$ is a Parseval determining set for $L^1(X)$. Let $\mathscr{A} = \{\varphi_i\}_{i\in I}$ and $\mathscr{A}' = \{\varphi'_i\}_{i\in I}$ be sequences in $L^2(X;\mathcal{H})$ indexed by I, and let $U\colon S(\mathscr{A}) \to S(\mathscr{A}')$ be a bounded operator. Then the following are equivalent:

1. $U(M_{g_t}\varphi_i) = M_{g_t}\varphi'_i$ *for every* $t \in Y$ *and every* $i \in I$.
2. U *is an MI operator* $U = \int_X^{\oplus} U(x)\, d\mu(x)$ *whose fibers satisfy* $U(x)\varphi_i(x) = \varphi'_i(x)$ *a.e. on* X.

Corollary

The following are equivalent:

1. $E(\mathscr{A})$ *is unitarily equivalent to* $E(\mathscr{A}')$.
2. $\{\varphi_i(x)\}_{i\in I}$ *is unitarily equivalent to* $\{\varphi'_i(x)\}_{i\in I}$ *a.e.*
3. $\{\varphi_i(x)\}_{i\in I}$ *and* $\{\varphi'_i(x)\}_{i\in I}$ *have the same Gramian a.e.*

Definition

We say that a positive MI operator $\int_X^\oplus \mathrm{Gr}(x)\,d\mu(x)\colon L^2(X;\ell^2(I)) \to L^2(X;\ell^2(I))$ *integrable* if

$$\int_X \langle \mathrm{Gr}(x)\delta_i, \delta_i\rangle\, d\mu(x) < \infty \qquad \text{for every } i \in I.$$

Here, $\delta_i \in \ell^2(I)$ is the canonical basis element, $i \in I$.

Theorem

Unitary equivalence classes of Bessel systems $E(\{\varphi_i\}_{i\in I})$ in $L^2(X;\mathcal{H})$ are in one-to-one correspondence with positive, integrable MI operators

$$\int_X^\oplus \mathrm{Gr}(x)\,d\mu(x)\colon L^2(X;\ell^2(I)) \to L^2(X;\ell^2(I))$$

having $\operatorname{rank}\mathrm{Gr}(x) \le \dim\mathcal{H}$ *a.e.* $x \in X$. *In this correspondence,* $\mathrm{Gr}(x)$ *is the Gramian of* $\{\varphi_i(x)\}_{i\in I}$ *a.e.* $x \in X$.

Classification of frames of multiplications

Definition

Given a Hilbert space $\mathcal{K}$, a positive operator $T \in B(\mathcal{K})$ is called *locally invertible* if there exists $\delta > 0$ such that $\sigma(T) \subseteq \{0\} \cup [\delta, \infty)$.

Corollary

Unitary equivalence classes of systems $E(\{\varphi_i\}_{i\in I})$ in $L^2(X;\mathcal{H})$ that are frames for $S(\{\varphi_i\}_{i\in I})$ are in one-to-one correspondence with locally invertible, integrable MI operators

$$\int_X^{\oplus} \mathrm{Gr}(x)\, d\mu(x) \colon L^2(X;\ell^2(I)) \to L^2(X;\ell^2(I))$$

having $\operatorname{rank} \mathrm{Gr}(x) \le \dim \mathcal{H}$ *a.e.* $x \in X$*. In this correspondence,* $\mathrm{Gr}(x)$ *is the Gramian of* $\{\varphi_i(x)\}_{i\in I}$ *a.e.* $x \in X$.

Classification of Parseval frames of multiplications

Definition

A measurable range function $J\colon X \to \{\text{closed subspaces of } \ell^2(I)\}$ is *integrable* if

$$\int_X \langle P_J(x)\delta_i, \delta_i\rangle \, d\mu(x) < \infty \qquad \text{for every } i \in I.$$

Corollary

Unitary equivalence classes of systems $E(\{\varphi_i\}_{i\in I})$ in $L^2(X;\mathcal{H})$ that are Parseval frames for $S(\{\varphi_i\}_{i\in I})$ are in one-to-one correspondence with integrable range functions

$$J\colon X \to \{\textit{closed subspaces of } \ell^2(I)\}$$

having $\dim J(x) \le \dim \mathcal{H}$ *a.e.* $x \in X$*. In this correspondence, the Gramian of* $\{\varphi_i(x)\}_{i\in I}$ *is orthogonal projection onto* $J(x)$ *a.e.* x*.*

Applications to LCA groups

- $Y = G$ a locally compact abelian group,
- $X = \hat{G}$,
- $\mathcal{D} = \{\hat{x}\}_{x \in G}$, where $\hat{x} \colon \hat{G} \to \mathbb{T}$, $\hat{x}(\alpha) = \overline{\alpha(x)}$, $\alpha \in \hat{G}$.

By Pontryagin duality, $\mathcal{D}$ is the set of characters on $\hat{G}$. By Plancherel's Theorem, $\mathcal{D}$ is a Parseval determining set for $L^1(\hat{G})$.

Theorem

The following are equivalent for every representation $\pi \colon G \to U(\mathcal{H})$ with $\mathcal{H}$ separable:

1. *There is a sequence $\mathscr{A}$ in $\mathcal{H}$ such that $E(\mathscr{A})$ is a complete Bessel system in $\mathcal{H}$*
2. *There is a separable Hilbert space $\mathcal{K}$ and a linear isometry $U \colon \mathcal{H} \to L^2(\hat{G}; \mathcal{K})$ that intertwines π with modulation.*
3. *There is a sequence $\mathscr{A}$ in $\mathcal{H}$ such that $E(\mathscr{A})$ is a Parseval frame for $\mathcal{H}$.*

We call such representation π admissible. U is an "interpreter" isometry.

Definition (Group frames)

Given a representation $\pi\colon G \to U(\mathcal{H})$ and a sequence $\mathscr{A} = \{u_i\}_{i\in I}$ in $\mathcal{H}$, we write $E(\mathscr{A}) = \{\pi(x)u_i\}_{x\in G,\, i\in I}$. We call $E(\mathscr{A})$:

- *complete*, if $\overline{\operatorname{span}}\{\pi(x)u_i : x \in G,\ i \in I\} = \mathcal{H}$;
- a *Bessel G-system*, if there is a constant $B > 0$ such that

$$\sum_{i\in I} \int_G |\langle v, \pi(x)u_i\rangle|^2\, dx \le B\,\|v\|^2 \qquad \text{for all } v \in \mathcal{H};$$

- a *G-frame*, if there are constants $A, B > 0$ such that

$$A\,\|v\|^2 \le \sum_{i\in I} \int_G |\langle v, \pi(x)u_i\rangle|^2\, dx \le B\,\|v\|^2 \qquad \text{for all } v \in \mathcal{H}.$$

Classifications of abelian group frames

For $N \in \mathbb{N}$, let $\ell^2_N := \ell^2(\{1,\dots,N\})$ and $\ell^2_\infty := \ell^2(\{1,2,\dots\})$.
The following are the LCA version of our measure-theoretic classification results.

Corollary

For $N \in \{1,2,\dots,\infty\}$, unitary equivalence classes of Bessel G-systems (resp. G-frames) having N generators are in one-to-one correspondence with positive (resp. locally invertible), integrable MI operators on $L^2(\hat{G};\ell^2_N)$.

Corollary

For $N \in \{1,2,\dots,\infty\}$, unitary equivalence classes of Parseval G-frames having N generators are in one-to-one correspondence with integrable range functions

$$J\colon \hat{G} \to \{\textit{closed subspaces of } \ell^2_N\}.$$

THANKS FOR YOUR ATTENTION