On point configurations and frame theory

Alex Iosevich

CodEx, June 2020

Dedicated to the memory of Jean Bourgain

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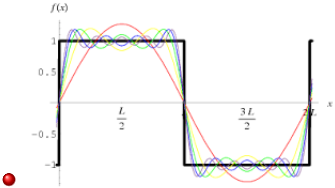
- 

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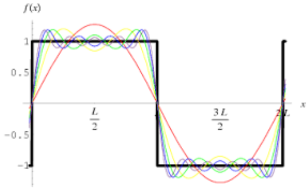
- 

- **"Fourier's theorem has all the simplicity and yet more power than other familiar explanations in science. Stated simply, any complex patterns, whether in time or space, can be described as a series of overlapping sine waves of multiple frequencies and various amplitudes** - Bruce Hood (clinical psychologist)

Basic questions

- Given a bounded domain $\Omega \subset \mathbb{R}^d$, does $L^2(\Omega)$ possess an orthogonal (or Riesz) exponential basis, i.e a basis of the form

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- In this context, a frame means that there exist $c, C > 0$ such that

$$c||f||^2_{L^2(\mu)} \leq \sum_{\lambda \in \Lambda} \left|\widehat{f\mu}(\lambda)\right|^2 \leq C||f||^2_{L^2(\mu)}.$$

Basic questions-Gabor

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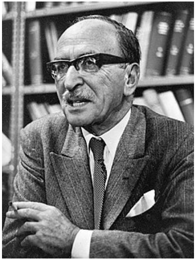
- 

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- The **Fuglede Conjecture** was disproved by Terry Tao in 2003, yet it holds in many cases and continues to inspire compelling research combining combinatorial, arithmetic and analytic techniques.

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- The Fuglede Conjecture holds for $\mathbb{Z}_p^2$, p prime (A.I., Mayeli and Pakianathan 2017) and Tiling $\to$ Spectral is known in $\mathbb{Z}_p^3$. Some partial results are available in the opposite direction (Birklbauer, Fallon, Mayeli, Villani).

Gabor bases: a key (mostly) open question

- The following question is largely unresolved: for which sets $E \subset \mathbb{R}^d$ does there exist $S \subset \mathbb{R}^{2d}$ such that $\{\chi_E(x-a)e^{2\pi i x \cdot b}\}_{(a,b)\in S}$ is an orthogonal basis for $L^2(\mathbb{R}^d)$?

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(Iosevich-Mayeli (Discrete Analysis 2018)) Let $g(x) = \chi_K(x)$, $K \subset \mathbb{R}^d$, $d \neq 1 \mod 4$, is a bounded symmetric convex set with a smooth boundary and everywhere non-vanishing Gaussian curvature. Then there **does not** *exist $S \subset \mathbb{R}^{2d}$ such that $\{g(x-a)e^{2\pi i x \cdot b}\}_{(a,b)\in S}$ is an orthogonal basis for $L^2(\mathbb{R}^d)$.*

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- If K is a non-symmetric convex polytope, the existence of an orthogonal Gabor basis with χ_K as the window function was previously ruled out by Chung and Lai (2018).

Connections with other interesting problems

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- Orthogonality implies that for any $\lambda \neq \lambda' \in \Lambda$,

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- Since zeroes of $J_{\frac{d}{2}}$ are uniformly separated, we use the density of Λ to conclude that $\#\{\Lambda \cap [-R,R]^d\} \approx R^d$, while

$$\#\{|\lambda-\lambda'| : \lambda, \lambda' \in \Lambda \cap [-R,R]^d\} \leq CR.$$

Connections with the Erdős Distance Problem

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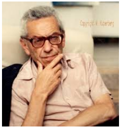

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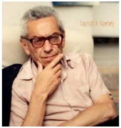
- 
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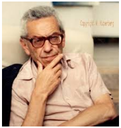
- 

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- This gives us a contradiction and proves that B_d, $d \geq 2$, does not possess an orthogonal basis of exponentials.

The Erdős Integer Distance Principle

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Theorem

(A.I. and M. Rudnev, IMRN (2003)) Let K be a bounded convex symmetric body with a smooth boundary and everywhere non-vanishing Gaussian curvature and let $\{e^{2\pi i x \cdot a}\}_{a \in A}$ denote a set of orthogonal exponentials in $L^2(K)$. If $d \neq 1 \mod 4$, then A is finite. If $d = 1 \mod 4$, A may be infinite. If A is infinite, it is a subset of a line.

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The Erdős Integer Distance Principle-the point

- If K is a symmetric with a C^∞ boundary and non-vanishing curvature, then $\widehat{\chi}_K(\xi)$ is equal to

$$C\kappa^{-\frac{1}{2}}\left(\frac{\xi}{|\xi|}\right)\sin\left(2\pi\left(\rho^*(\xi)-\frac{d-1}{8}\right)\right)|\xi|^{-\frac{d+1}{2}}+O(|\xi|^{-\frac{d+3}{2}}),$$

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- From this formula we deduce that if $e^{2\pi i x\cdot a}$ and $e^{2\pi i x\cdot a'}$ are orthogonal in $L^2(K)$, then $\rho^*(a-a')$ is, up to a small error, a shifted integer.
- It turns out that the Erdős Integer Distance Principle still applies in this approximate setting, with the Euclidean norm replaced by a more general (smooth) norm.

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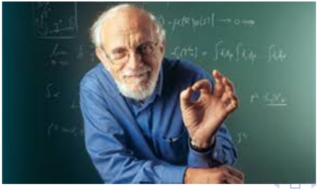

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(Ziegler (2006)) Let $d \geq 2, k \geq 2$*. Suppose* $E \subset \mathbb{R}^d$ *is of positive upper Lebesgue density, and let* E^δ *denote the* δ*-neighborhood of* E*. Let* $V = \{\mathbf{0}, v^1, v^2, \ldots, v^k\} \subset \mathbb{R}^d$*. Then there exists* $r_0 > 0$ *such that, for all* $r > r_0$ *and any* $\delta > 0$*, there exists* $\{x^1, \ldots, x^{k+1}\} \subset E^\delta$ *similar to* $\{\mathbf{0}, v^1, \ldots, v^k\}$ *via scaling* r*.*

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Applying positive density results to frame theory

- Here is the basic idea illustrated in the case of the unit ball in $\mathbb{R}^d$, $d \geq 2$. As we noted above, if $\{e^{2\pi i x \cdot \lambda}\}_{\lambda \in \Lambda}$ is an orthonormal basis for $L^2(B_d)$, then Λ is separated and has density $|B_d|$ by the classical Beurling density theorem.

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- Thicken each point of Λ by a small $\delta > 0$. The resulting set has positive upper (and lower) Lebesgue density, so by the result above due to Furstenberg-Katznelson-Weiss every sufficiently large distance is realized.

- But this cannot be true because as we saw before, the distances between the elements of Λ are zeroes of $J_{\frac{d}{2}}(2\pi \cdot)$, which implies that they are asymtotically close to half integers shifted by $\frac{d-1}{8}$. Consequently, distances between the elements of Λ thickened by δ are $C\delta$ close to half integers shifted by $\frac{d-1}{8}$, so it is impossible to recover every sufficiently large distance.

A stronger formulation

Definition

We say that $\mathcal{E}(\Lambda) = \{e^{2\pi i x \cdot \lambda}\}_{\lambda \in \Lambda}$ is a ϕ-approximate orthogonal basis for $L^2(\Omega)$, Ω a bounded domain in $\mathbb{R}^d$, if $\mathcal{E}(\Lambda)$ is a basis and

$$|\widehat{\chi}_\Omega(\lambda - \lambda')| \leq \phi(|\lambda - \lambda'|),$$

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Theorem

(A. Iosevich and A. Mayeli (2020)) Let ϕ be a any function such that

$$\lim_{t \to \infty} (1+t)^{\frac{d+1}{2}} \phi(t) = 0.$$

Then there does not exist a set Λ such that $L^2(B_d)$ possesses a ϕ-approximate orthogonal basis $\mathcal{E}(\Lambda)$.

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- Lubarskii and Rashkovski (2001) proved that $L^2(K)$ has a Riesz basis of exponentials if K is a symmetric polygon inscribed in the disk.

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- Lubarskii and Rashkovski (2001) proved that $L^2(K)$ has a Riesz basis of exponentials if K is a symmetric polygon inscribed in the disk.
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Exponential Riesz basis for the ball

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- There are no **known** examples of sets E of positive Lebesgue measure such that $L^2(E)$ does not possess a Riesz basis of exponentials.

Kadison-Singer conjecture

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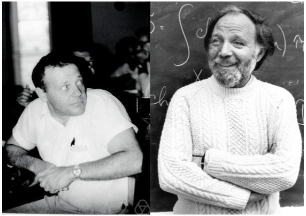
- 

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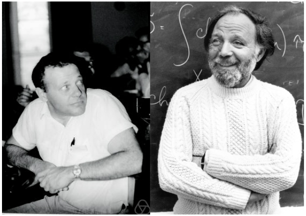
- 

- Consider the separable Hilbert space l^2 and two related C^*-algebras: the algebra B of all continuous linear operators from l^2 to l^2, and the algebra D of all diagonal continuous linear operators from l^2 to l^2.

Kadison-Singer conjecture-continued

- A state on a C^*-algebra A is a continuous linear functional $\varphi : A \to \mathbb{C}$ such that $\varphi(I) = 1$ (where I denotes the algebra's multiplicative identity) and $\varphi(T) \geq 0$ for every $T \geq 0$. Such a state is called **pure** if it is an extremal point in the set of all states on A.

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Kadison-Singer conjecture: alternate formulation

Theorem

(Marcus, Spielman and Srivastava) Let $\epsilon > 0$ *and* $u_1, \ldots, u_m \in \mathbb{C}^n$ *such that* $||u_i||^2 \leq \epsilon$ *for all* $i = 1, 2 \ldots, m$ *and*

$$\sum_{i=1}^{m} |< w, u_i >|^2 = ||w||^2 \quad \forall w \in \mathbb{C}^n.$$

Then there exists a partition of $\{1, 2, \ldots m\}$ *into* S_1, S_2 *such that for* $j = 1, 2,$

$$\sum_{i \in S_j} |< w, u_i >|^2 \leq \frac{(1+\sqrt{2\epsilon})^2}{2} ||w||^2 \quad \forall w \in \mathbb{C}^n.$$

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From Kadison-Singer to universal frame constants

Theorem

(Nitzan-Olevskii-Ulanovskii) There are positive constants c, C such that for every set $S \subset \mathbb{R}^d$ of finite measure there is a discrete set $\Lambda \subset \mathbb{R}^d$ such that $\{e^{2\pi i x \cdot \lambda}\}_{\lambda \in \Lambda}$ is a frame in $L^2(S)$ with frame bounds $c|S|$ and $C|S|$.

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- Let μ_δ denote δ^{-1} times the indicator function of the annulus of radius 1 and width δ. By the Nitzan-Olevskii-Ulanovskii theorem, there exist $C, c > 0$ such that for every $\delta < 0$ there exists a frame $\{e^{2\pi i x \cdot \lambda}\}_{\lambda \in \Lambda_\delta}$ with

$$c||f||^2_{L^2(\mu_\delta)} \leq \sum_{\lambda \in \Lambda_\delta} \left|\widehat{f\mu_\delta}(\lambda)\right|^2 \leq C||f||^2_{L^2(\mu_\delta)}.$$

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- Since $\mu_\delta \to \sigma$, the surface measure on S^{d-1}, it is reasonable to ask whether $L^2(\sigma)$ possesses a frame of exponentials. This question was posed by Nir Lev.

Spheres vs polytopes

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(Iosevich, Lai, Liu and Wyman (2019)) The Hilbert space $L^2(\sigma)$ **does not** *possess a frame of exponentials.*

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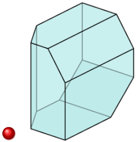

Fourier decay and frames: lower bound

- Our approach to proving that $L^2(\sigma)$ does not possess a frame of exponentials rests on the following results which sets up a rather general framework for these types of problems.

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Theorem

Let μ be a compactly supported Borel measure and suppose that $L^2(\mu)$ possesses a frame of exponentials with the frame spectrum $\Lambda \subset \mathbb{R}^d$. Suppose that there exists constant $C > 0$ and $0 < \gamma \leq d$ such that

$$|\widehat{\mu}(\xi)| \leq C|\xi|^{-\frac{\gamma}{2}}, \qquad \forall \xi \in \mathbb{R}^d.$$

Then

$$\sum_{\lambda \in \Lambda \setminus \{\mathbf{0}\}} \frac{1}{|\lambda|^{\gamma}} = \infty.$$

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Fourier decay and frames: upper bound

Theorem

Let μ be a finite Borel measure that admits a Bessel sequence $E(\Lambda)$ for some countable set $\Lambda \subset \mathbb{R}^d$. Suppose that there exists and $L > 0$ and $\gamma > 0$ such that

$$\sup_{R>0} \inf_{|\lambda|>L} |\lambda|^\gamma \int_{B_R(\lambda)} |\widehat{\mu}(\xi)|^2 d\xi > 0.$$

Then

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- The result about the sphere is obtained by showing that the assumptions of the two theorems above are satisfied if $\mu = \sigma$ and $\gamma = \frac{d-1}{2}$. The resulting contradiction establishes the claim.

Proof of the upper bound

- By assumption, there exists $R > 0$ such that

$$c := \inf_{|\lambda|>L} |\lambda|^{\gamma} \int_{B_R(\lambda)} |\widehat{\mu}(\xi)|^2 d\xi > 0.$$

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- Integrating both sides we obtain

$$\sum_{\lambda\in\Lambda, |\lambda|>L} \int_{B_R(-\lambda)} |\widehat{\mu}(\xi)|^2 d\xi$$

Proof of the upper bound (conclusion)

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- which shows that

$$\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^\gamma} < \infty.$$

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- Fixing $R > 1$, for all $|\lambda| > 2R$ and $|\xi| \leq R$, we have

$$|\lambda + \xi| > \frac{|\lambda|}{2}.$$

Proof of the lower bound (continued)

- It follows that

$$\sum_{|\lambda|>2R} |\widehat{\mu}(\lambda+\xi)|^2 \lesssim \sum_{|\lambda|>2R} |\lambda+\xi|^{-\gamma} \lesssim \sum_{|\lambda|>2R} |\lambda|^{-\gamma}.$$

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- Therefore for R large enough and all $|\xi| \le R$,

$$\frac{c}{2} \le \sum_{|\lambda|\le 2R} |\widehat{\mu}(\lambda+\xi)|^2.$$

Proof of the lower bound (continued some more)

- Integrating this inequality over the ball of radius R centered at the origin, we obtain

$$R^d \lesssim \sum_{|\lambda|\leq 2R} \int_{|\xi|\leq R} |\widehat{\mu}(\lambda+\xi)|^2 d\xi = \sum_{|\lambda|\leq 2R} \int_{B_R(-\lambda)} |\widehat{\mu}(\xi)|^2 d\xi$$

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- Applying the Fourier decay condition, we obtain

$$\int_{B_{3R}(\mathbf{0})} |\widehat{\mu}(\xi)|^2 d\xi \lesssim \int_1^{3R} r^{-\gamma} r^{d-1} dr \lesssim \begin{cases} R^{d-\gamma} & \text{if } \gamma < d \\ \log R & \text{if } \gamma = d \end{cases}$$

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$$\#\{\Lambda \cap B_{2R}(\mathbf{0})\} \geq R^{\gamma} \text{ if } \gamma < d,$$

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- The desired contradiction follows.