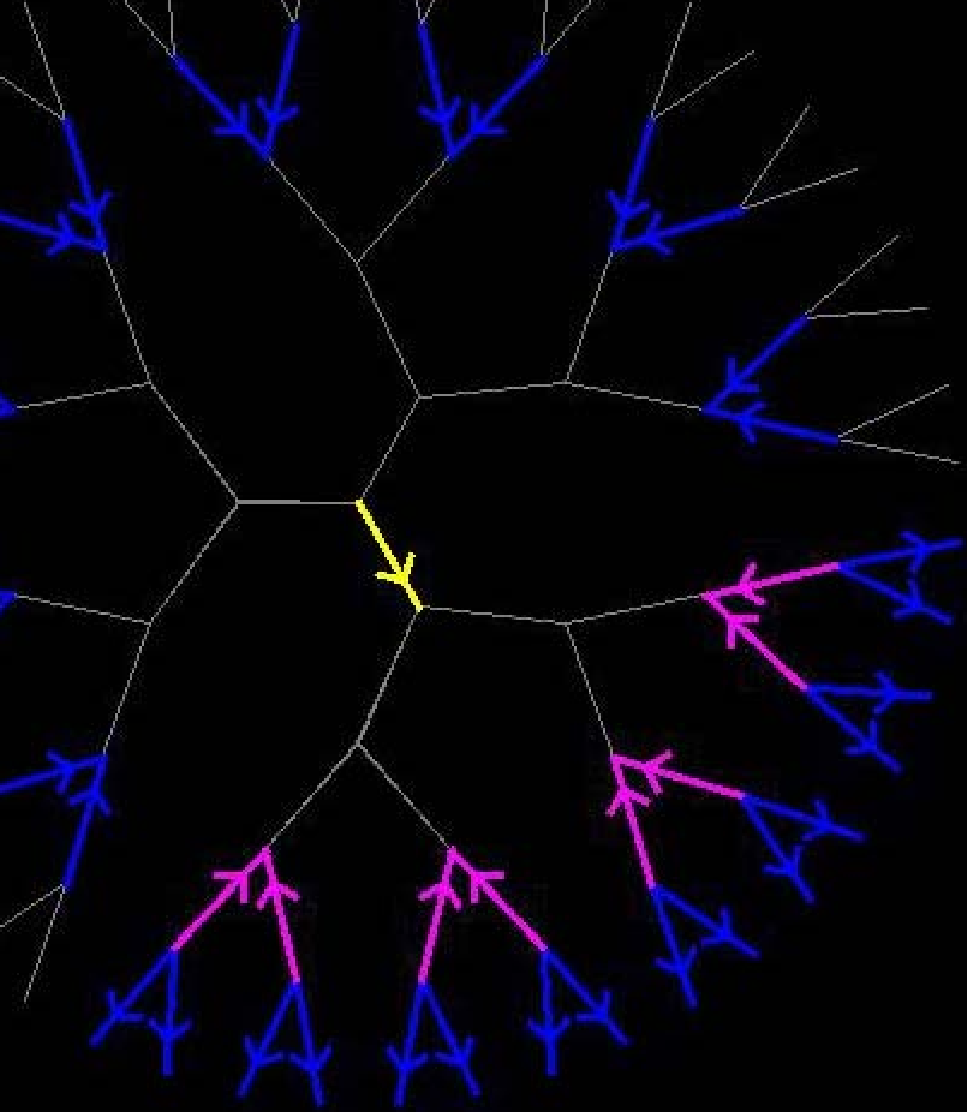

Visual Mathematics

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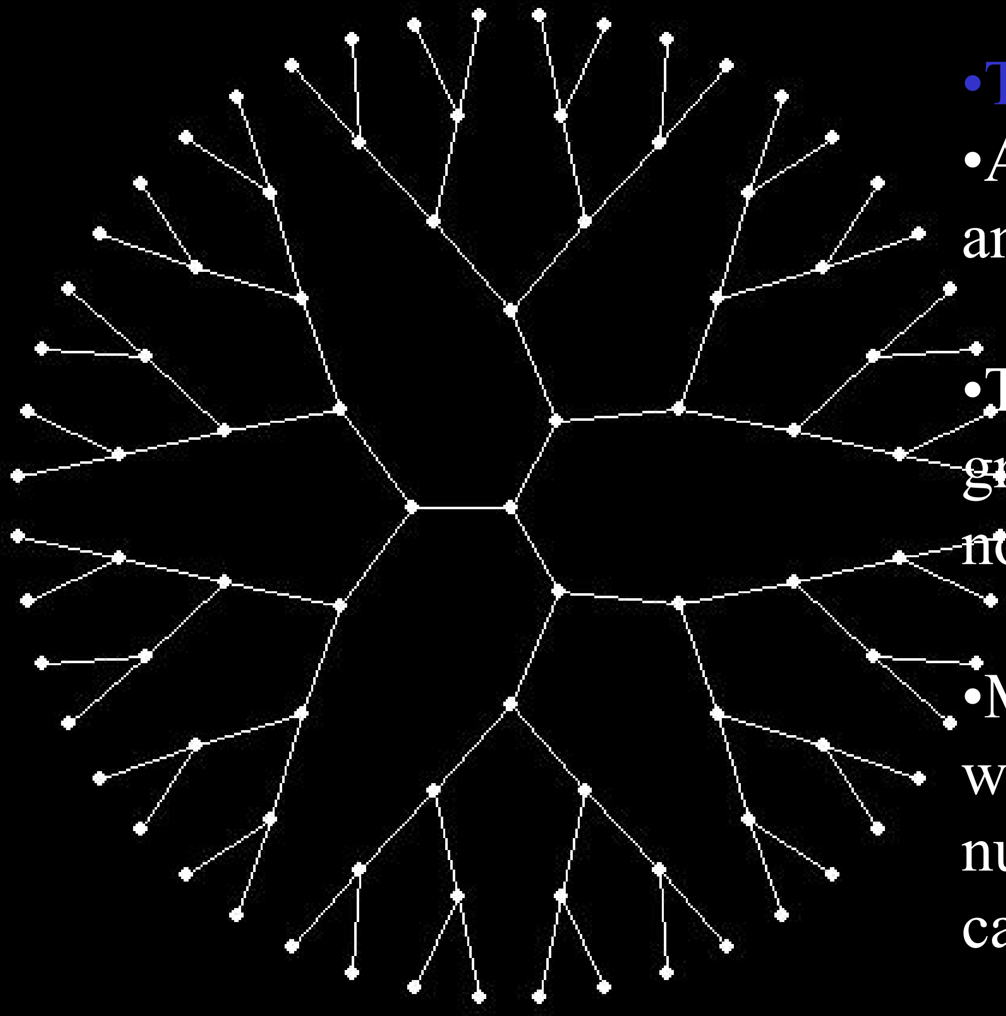


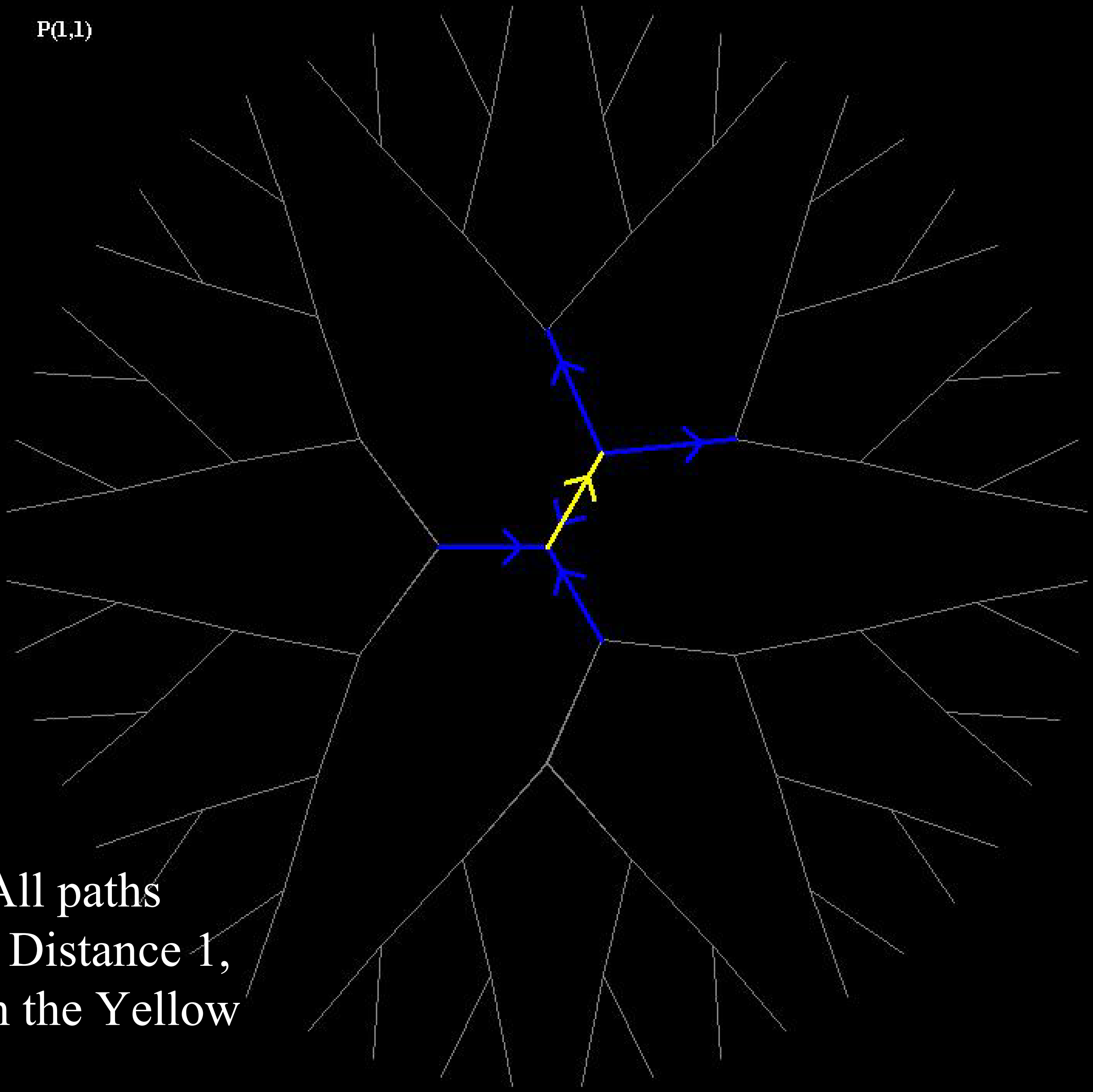
- The following picture is a Graph
- A graph is any group of vertices and edges between them.
- This graph is a special kind of graph called a tree because it has no loops.
- More than just a tree, it is a tree in which every vertex has the same number of branches, sometimes we call this a regular tree.

Distance in the Graph

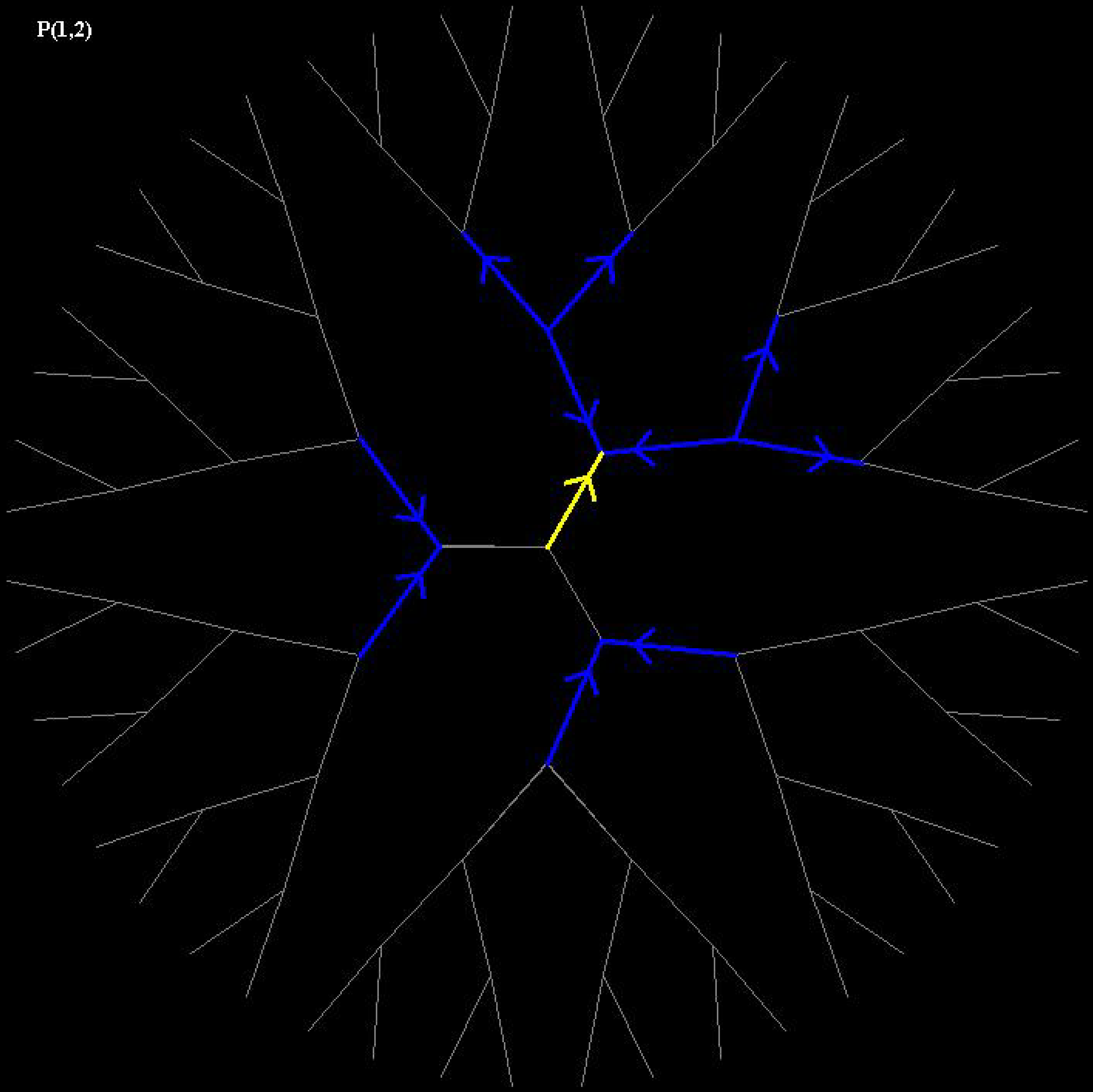
Notice we can connect vertices with lines drawn on the graph. These lines we call paths. If we count the number of lines in a path we know the distance between the two points.

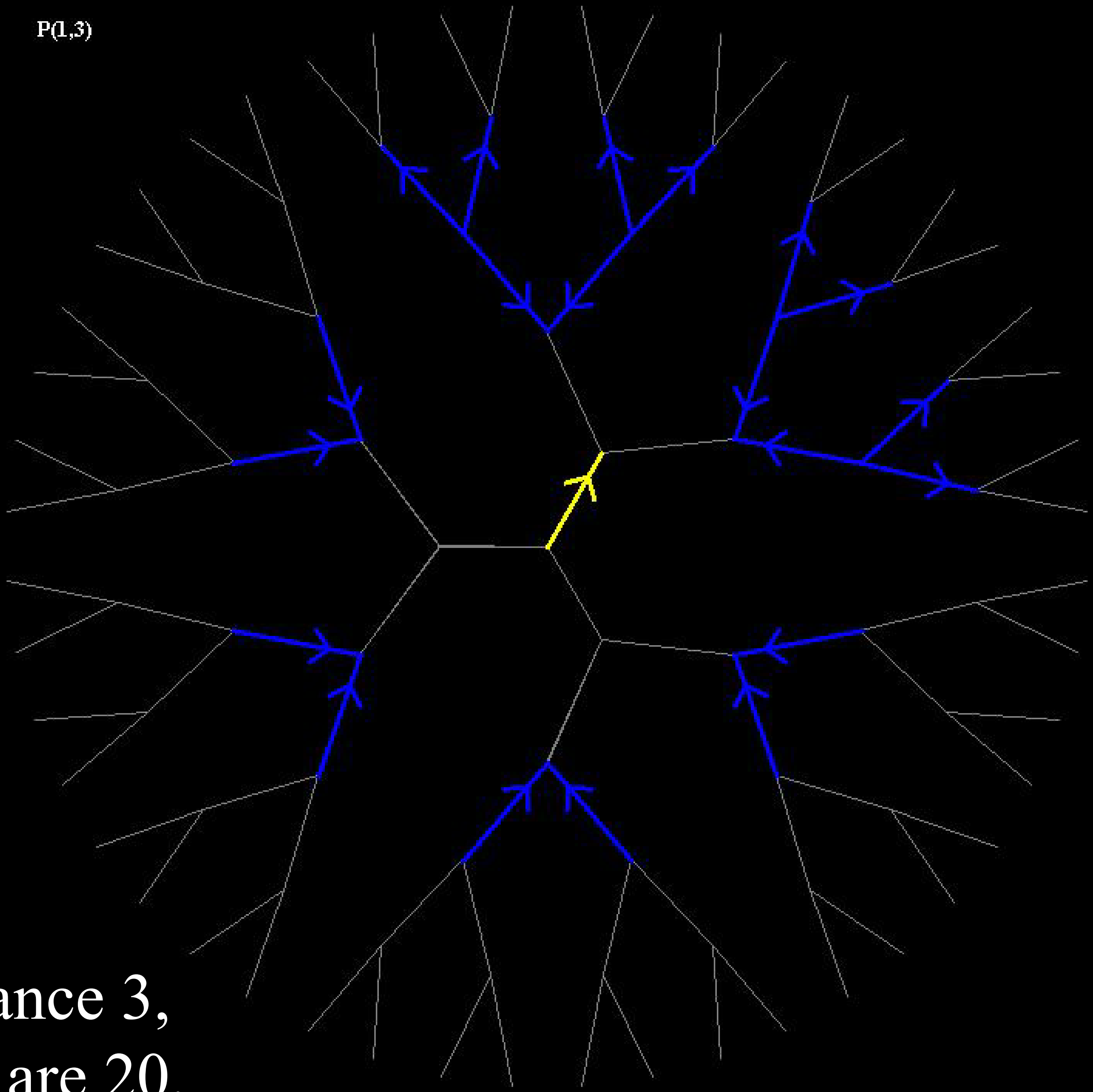
Notice trees are special because there is only one path between any two points.

For our case we will put arrows on the lines to know what direction we travel in.

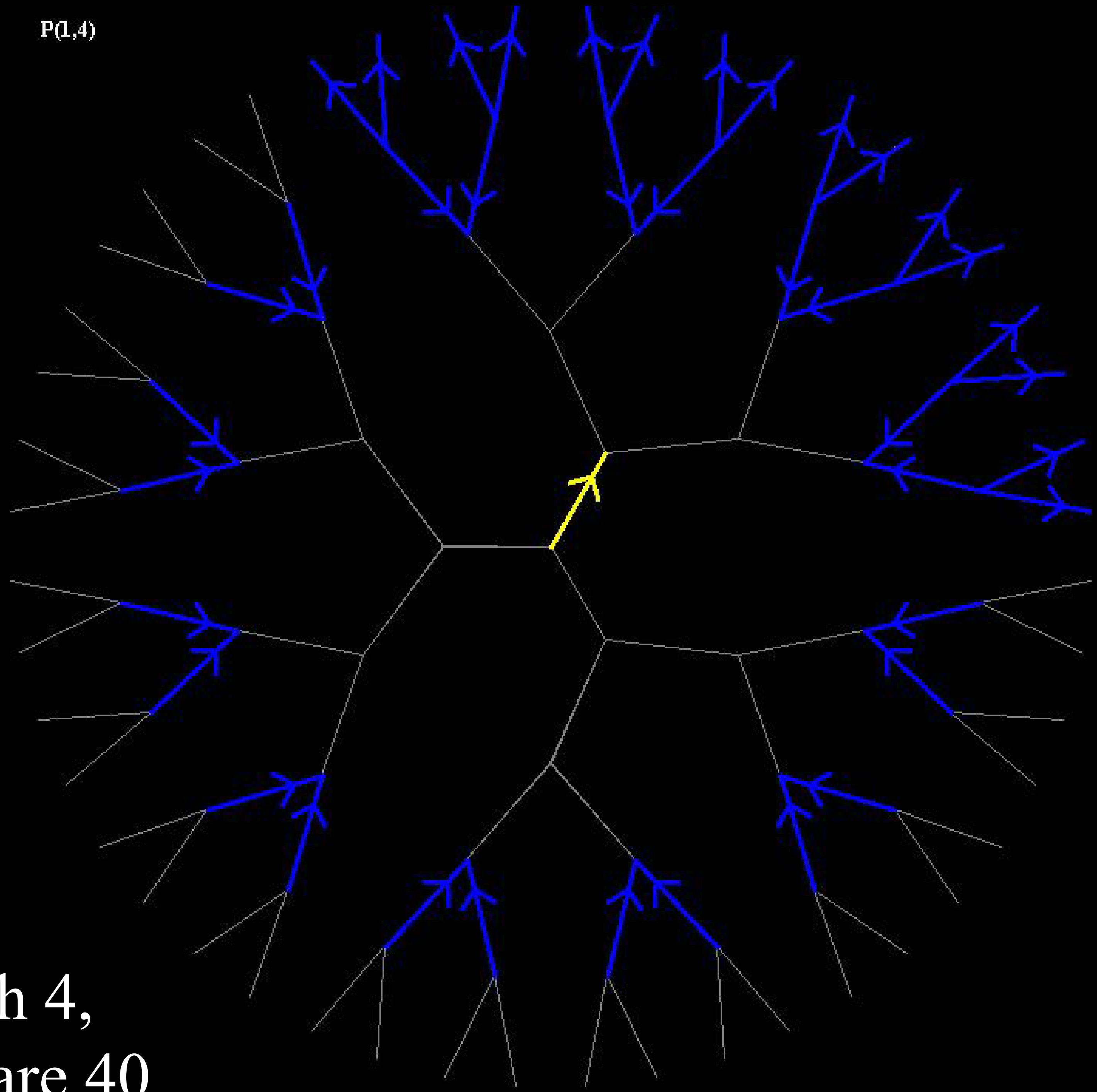


All paths
At a Distance 1,
From the Yellow

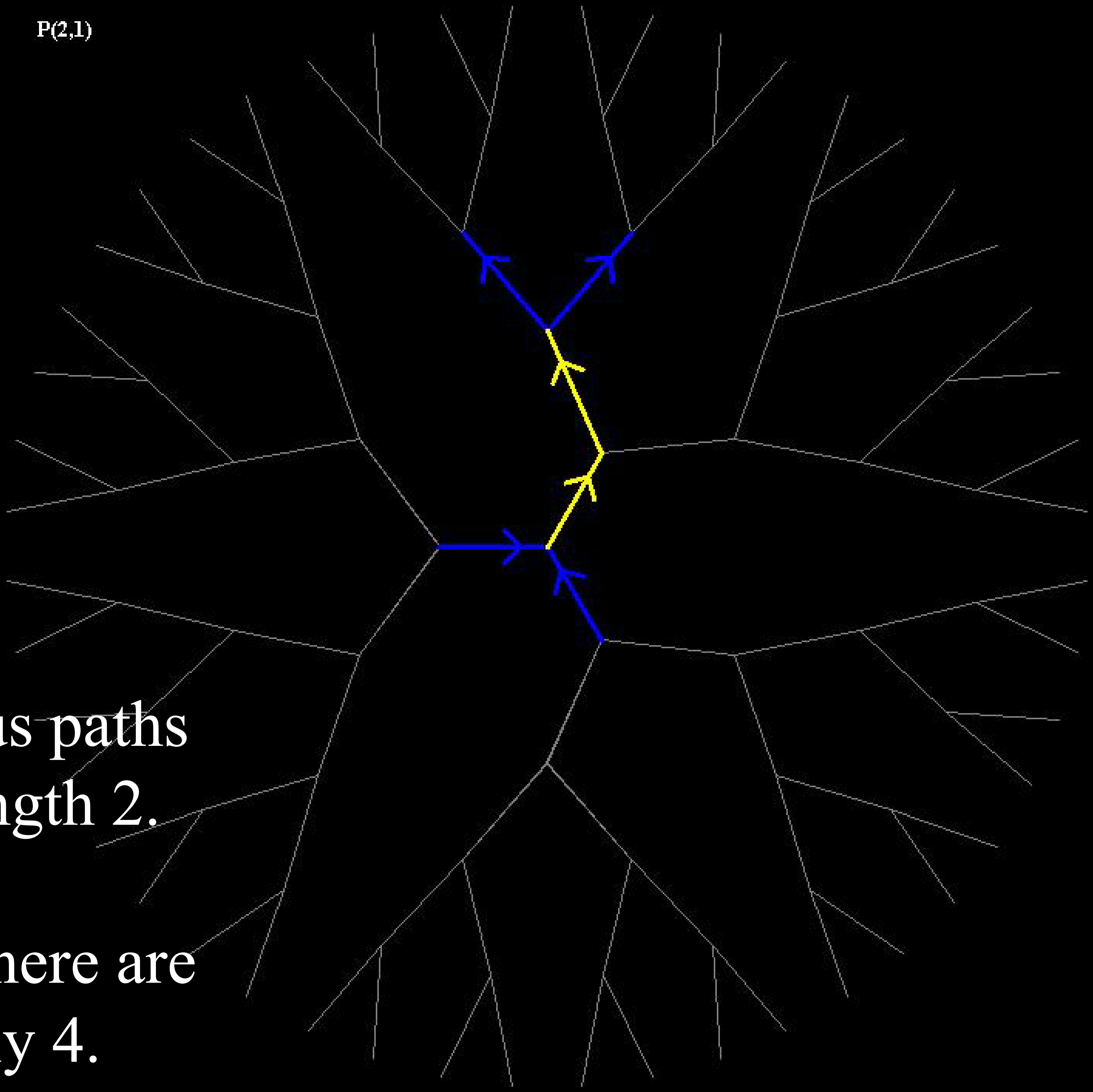

P(1,2)



Distance 3,
there are 20.

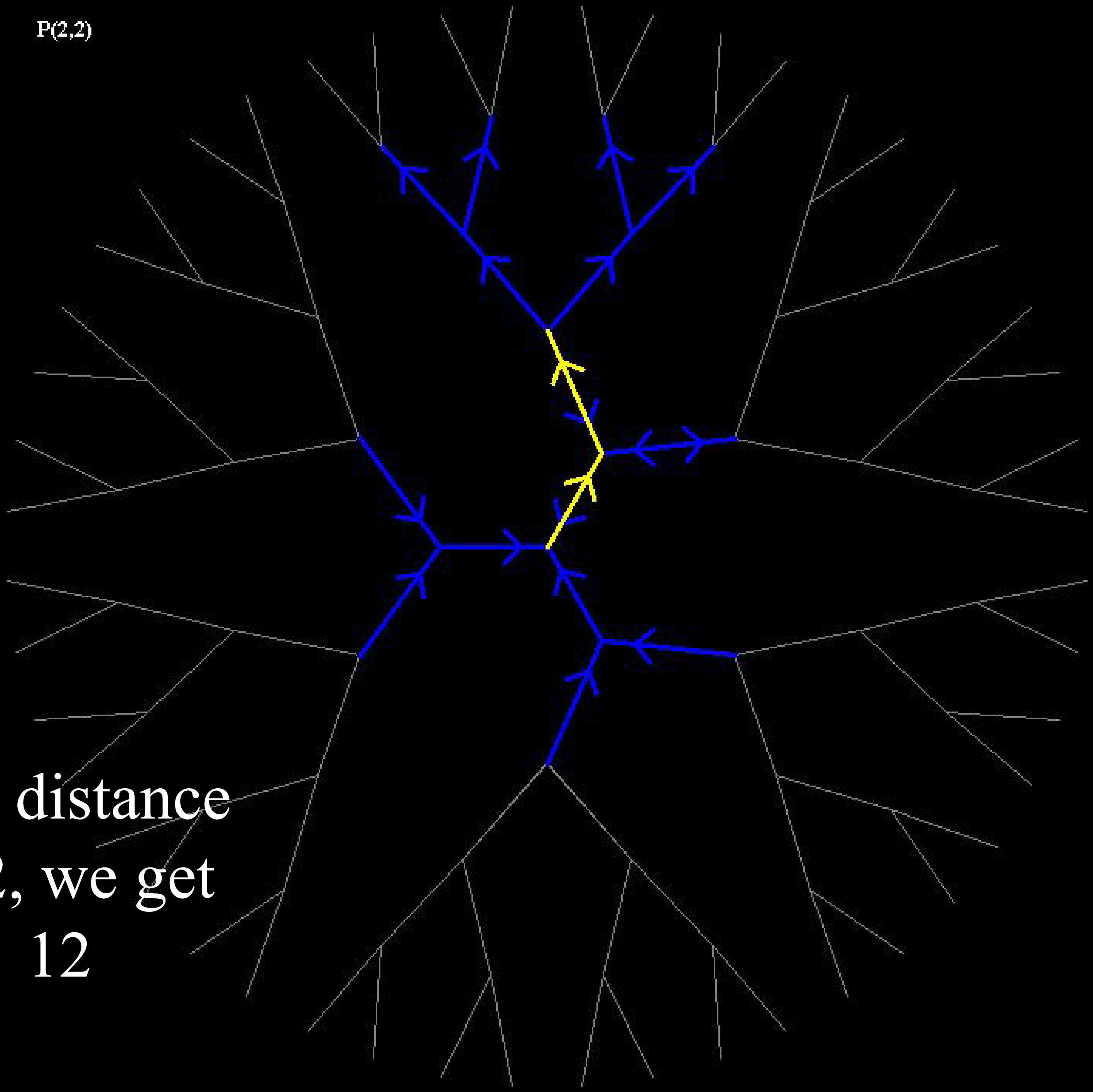


With 4,
there are 40

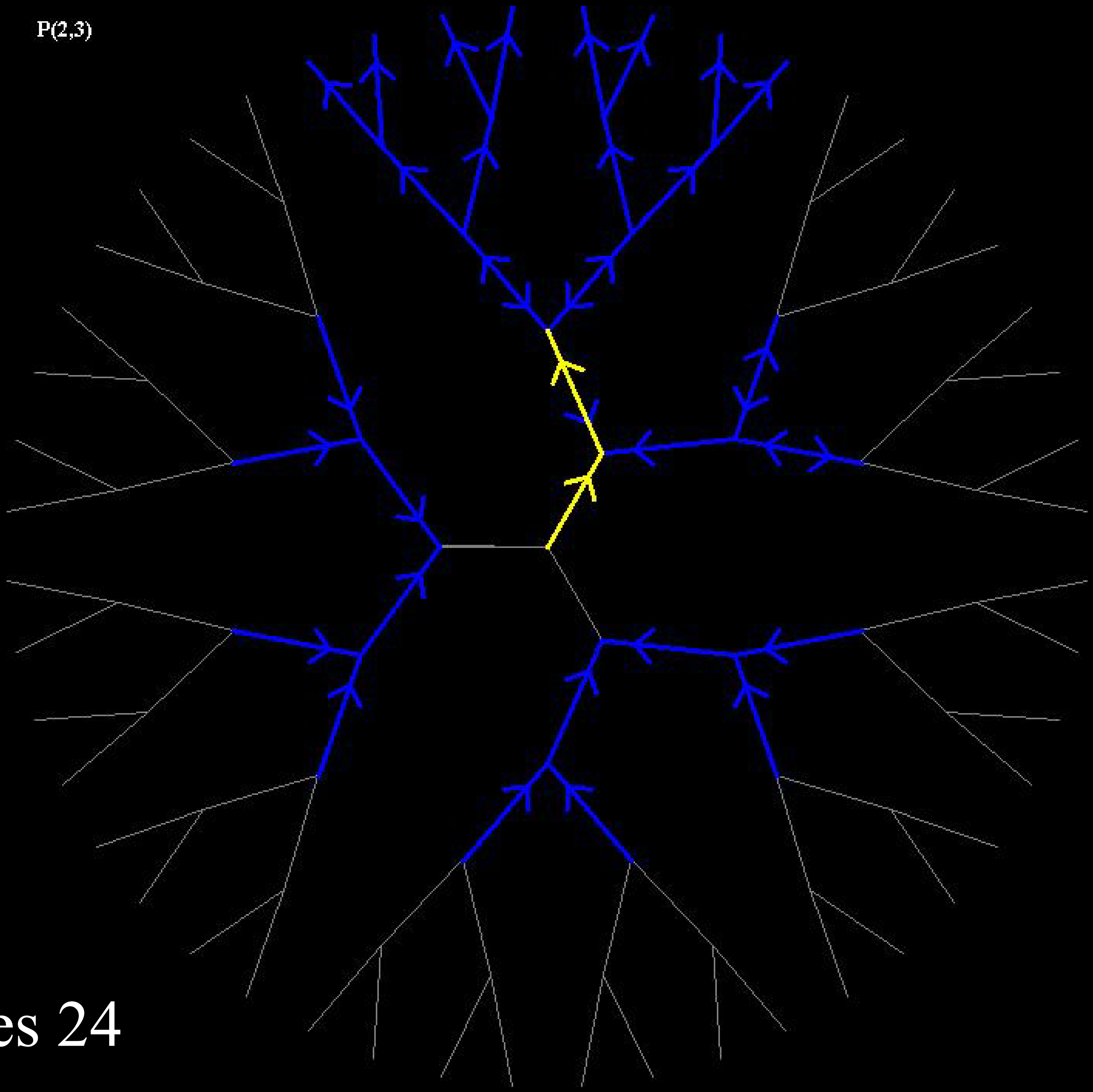


Now us paths of length 2.

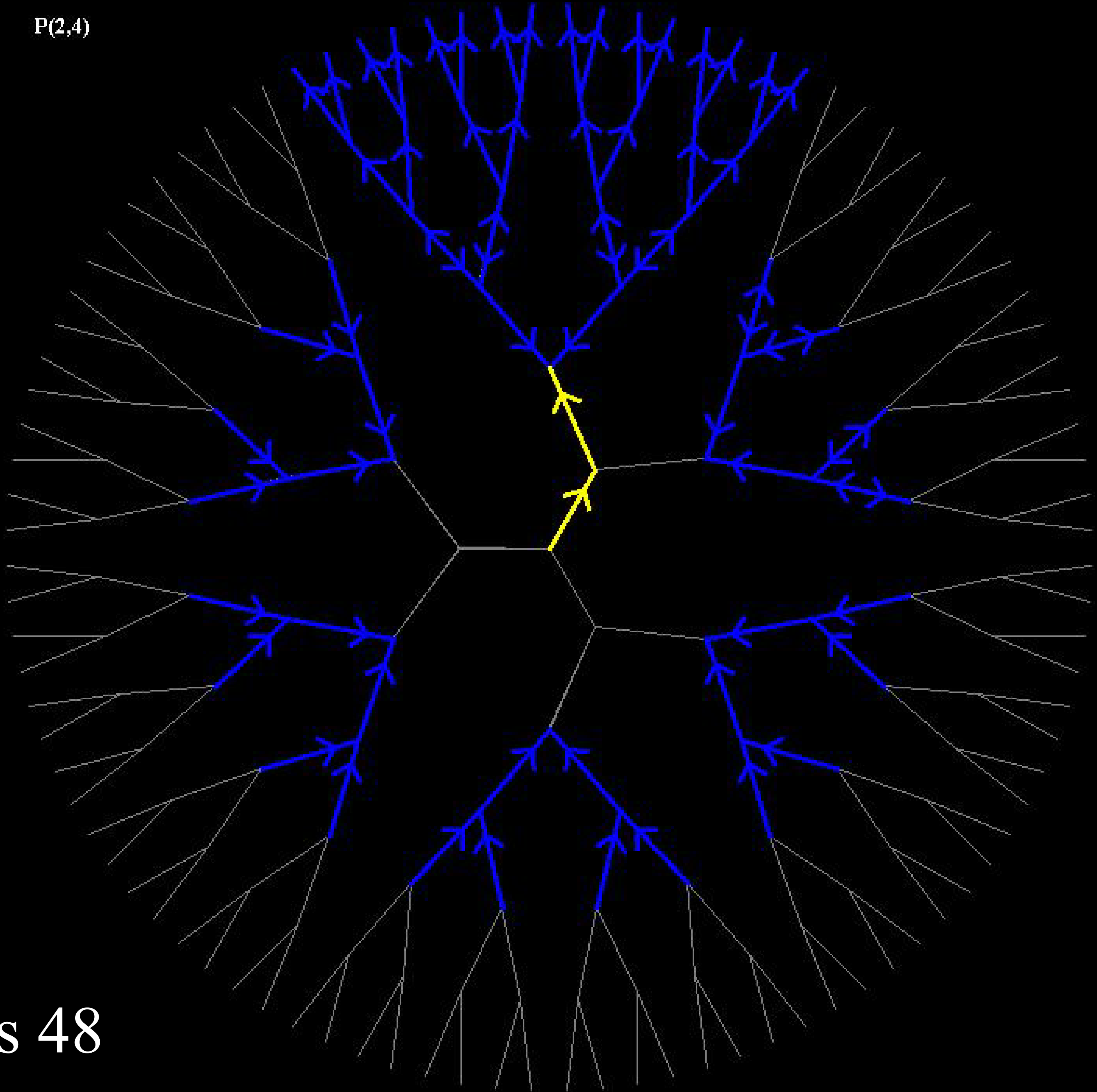
Here there are only 4.



At a distance of 2, we get 12



3, gives 24



4, is 48

The Problem

- We saw the numbers seem to grow by

5, 10, 20, 40, etc. … and
4, 12, 24, 48, etc, …

what happens if we change the tree?

- How can we write the formula and be sure it works?

The formulas.

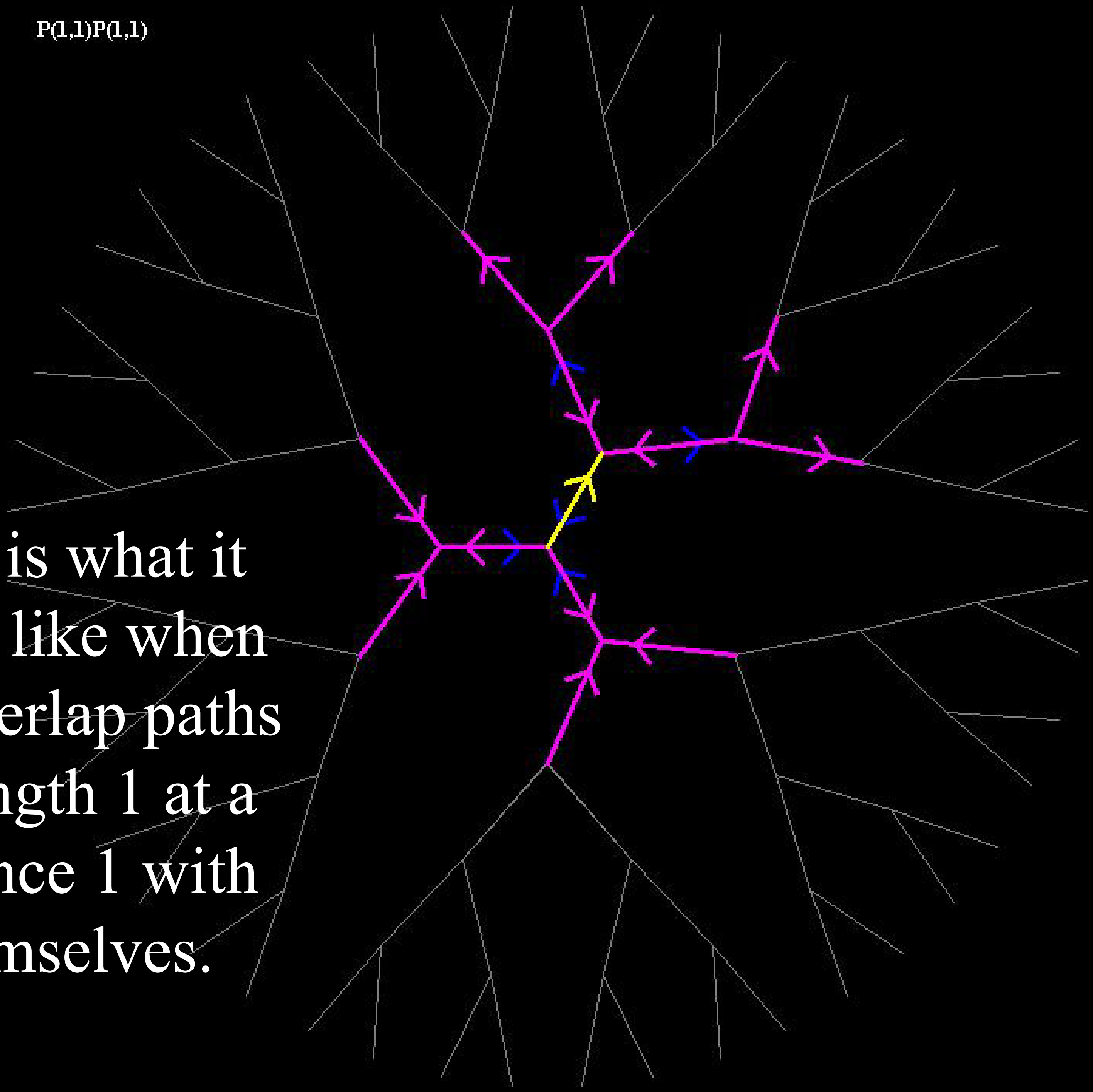
- Our guess is that the numbers grow as follows:

$(p+1)p^{k-1}$ for length 1.

$(2p+1)p^{k-1}$ for length 2, well see why.

What is the Picture?

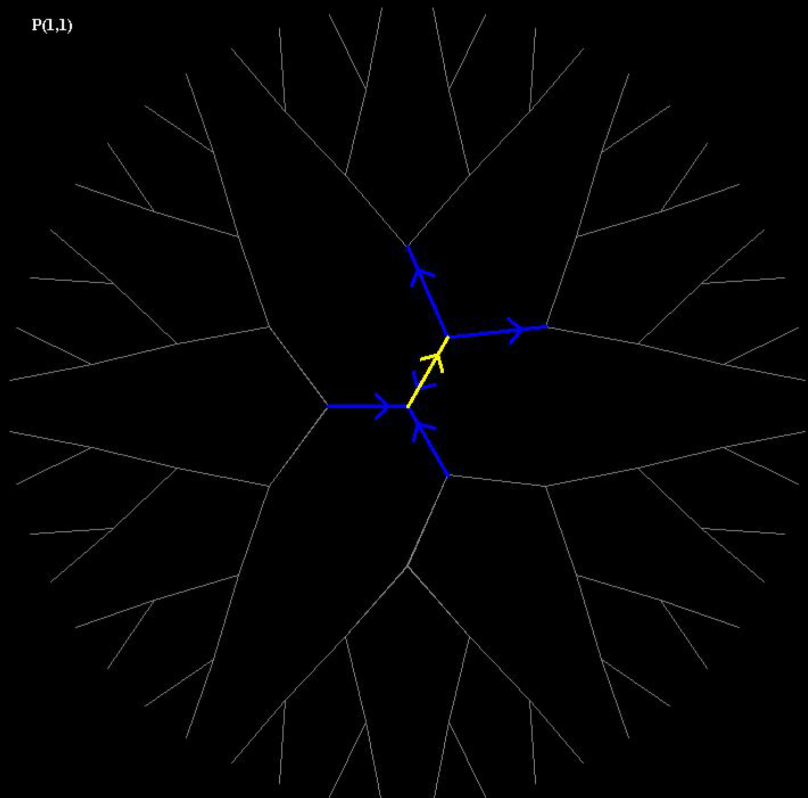
- If we look at how these lines grow from the previous distance maybe we can see how to create the formula.
- We define a covering (or product) of two pictures by simply overlapping one with another. We may have to twist some lines but that is allowed.

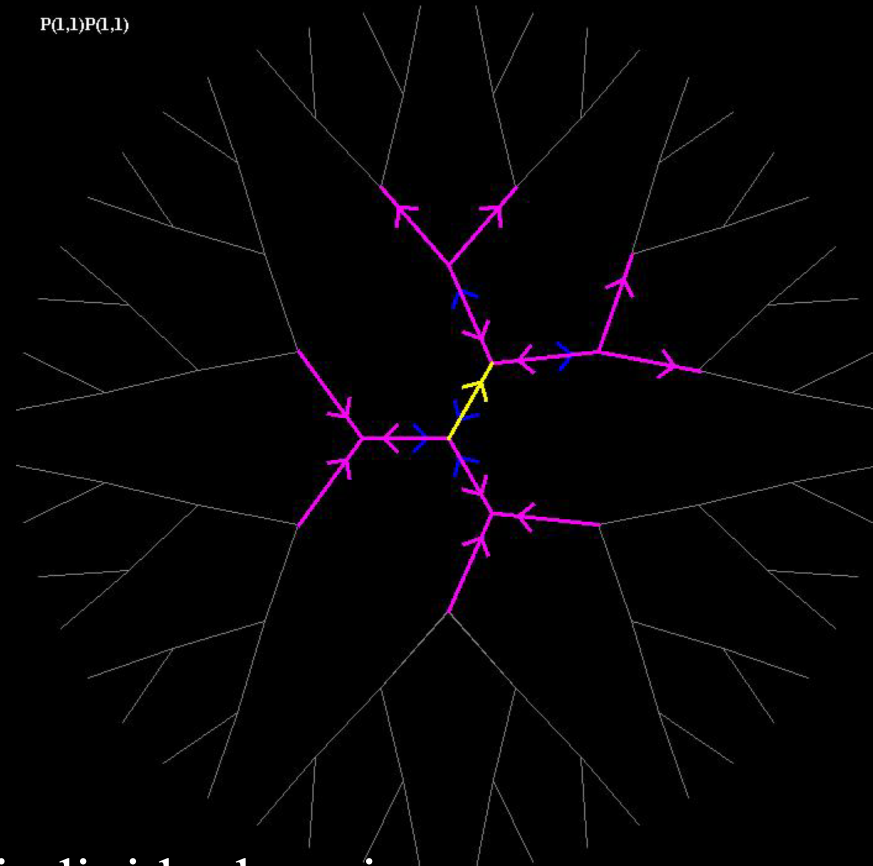


This is what it looks like when we overlap paths of length 1 at a distance 1 with themselves.

Here is what we over lap.

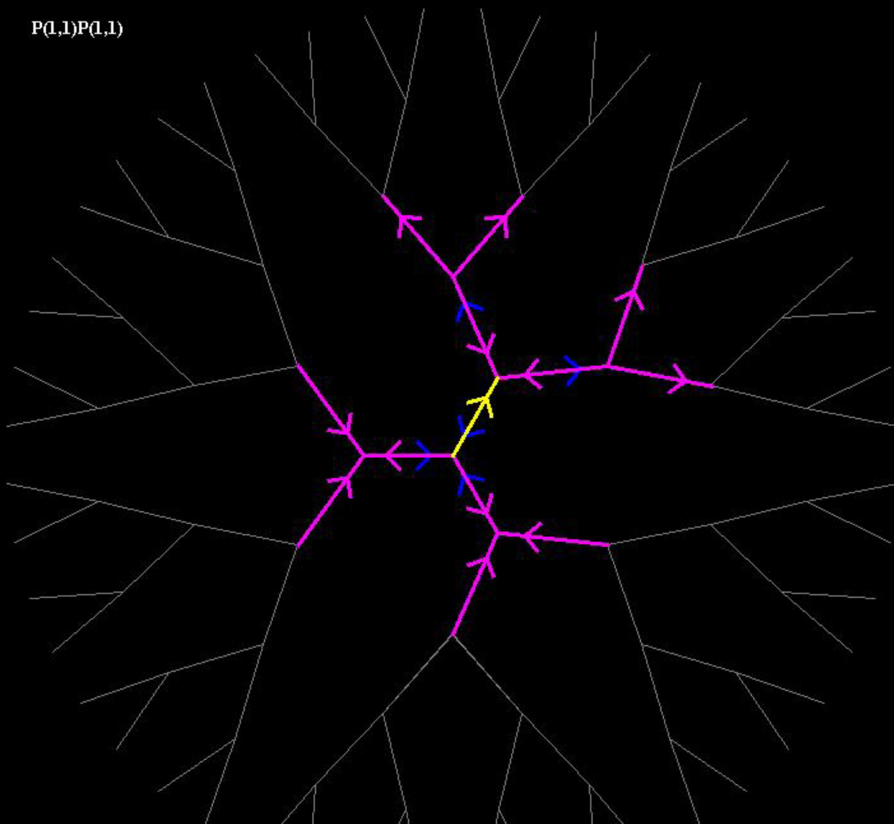
Here is what it looks like
When we finish.

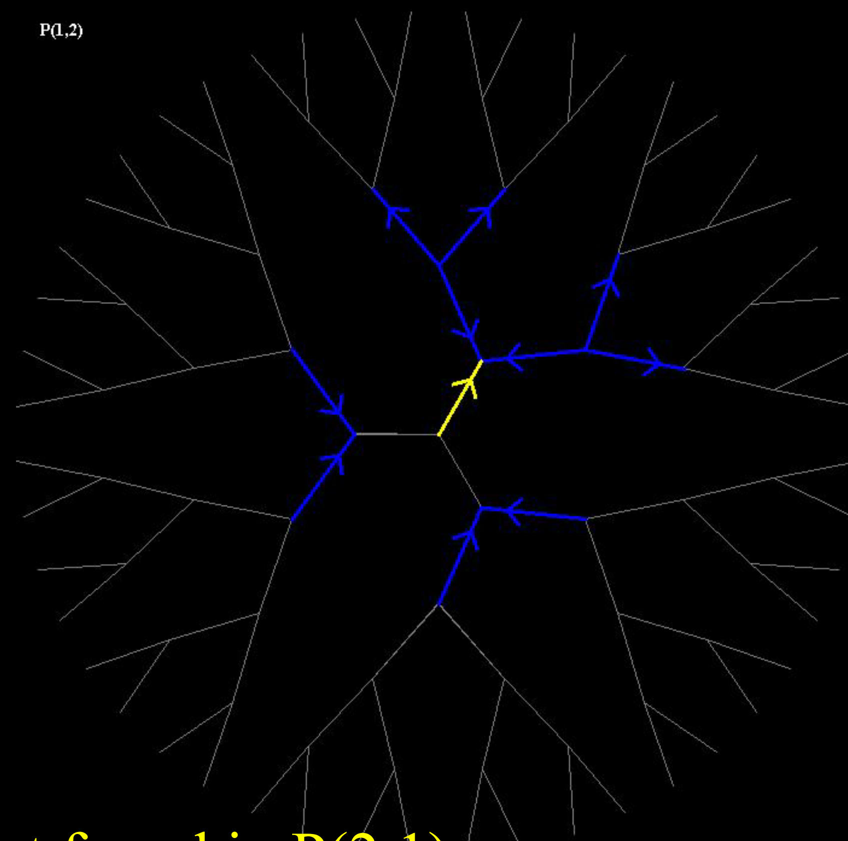




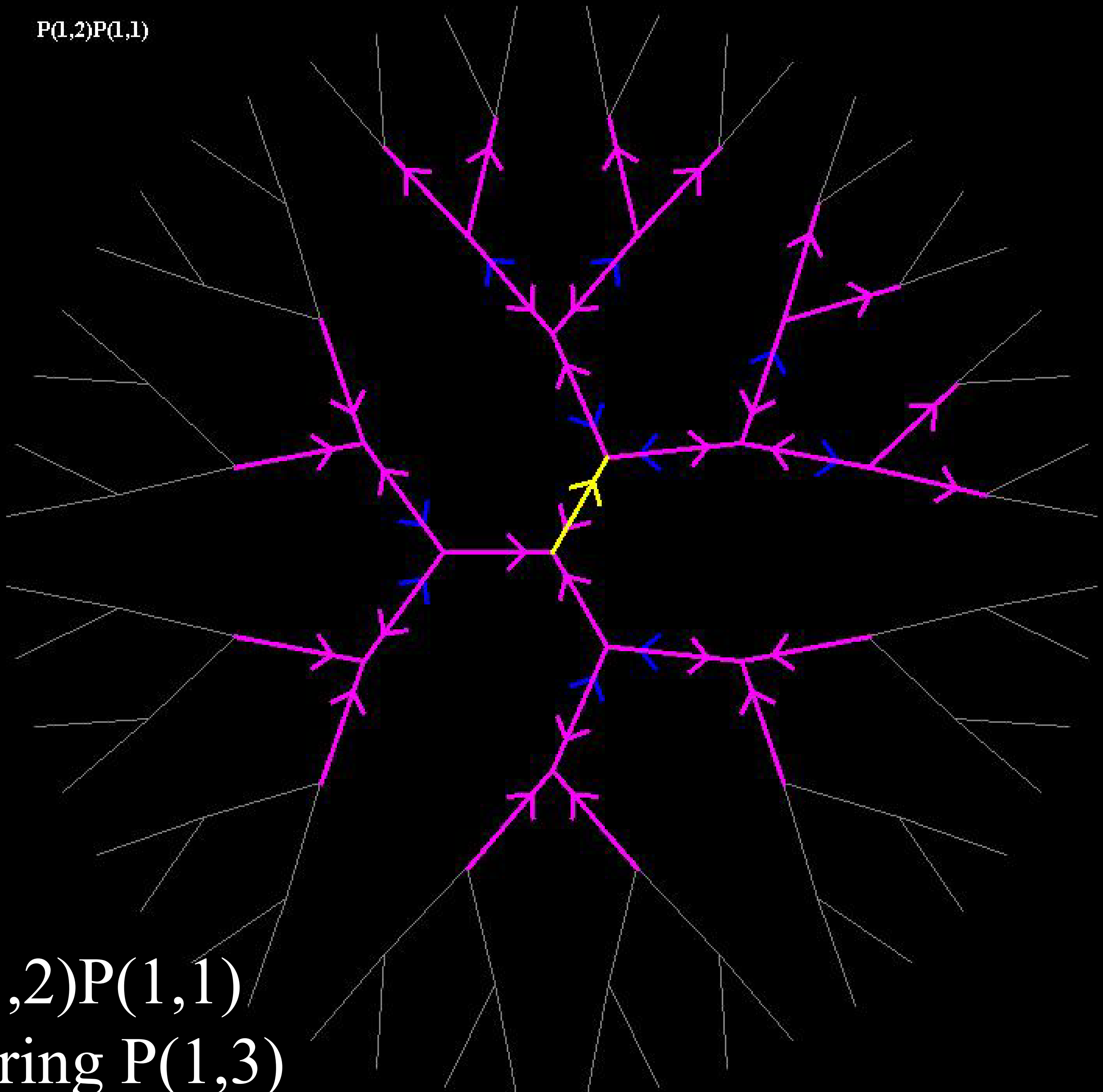
Can you see the 5 individual copies
Of the cover?

Compare the cover to what we get in the Next step:

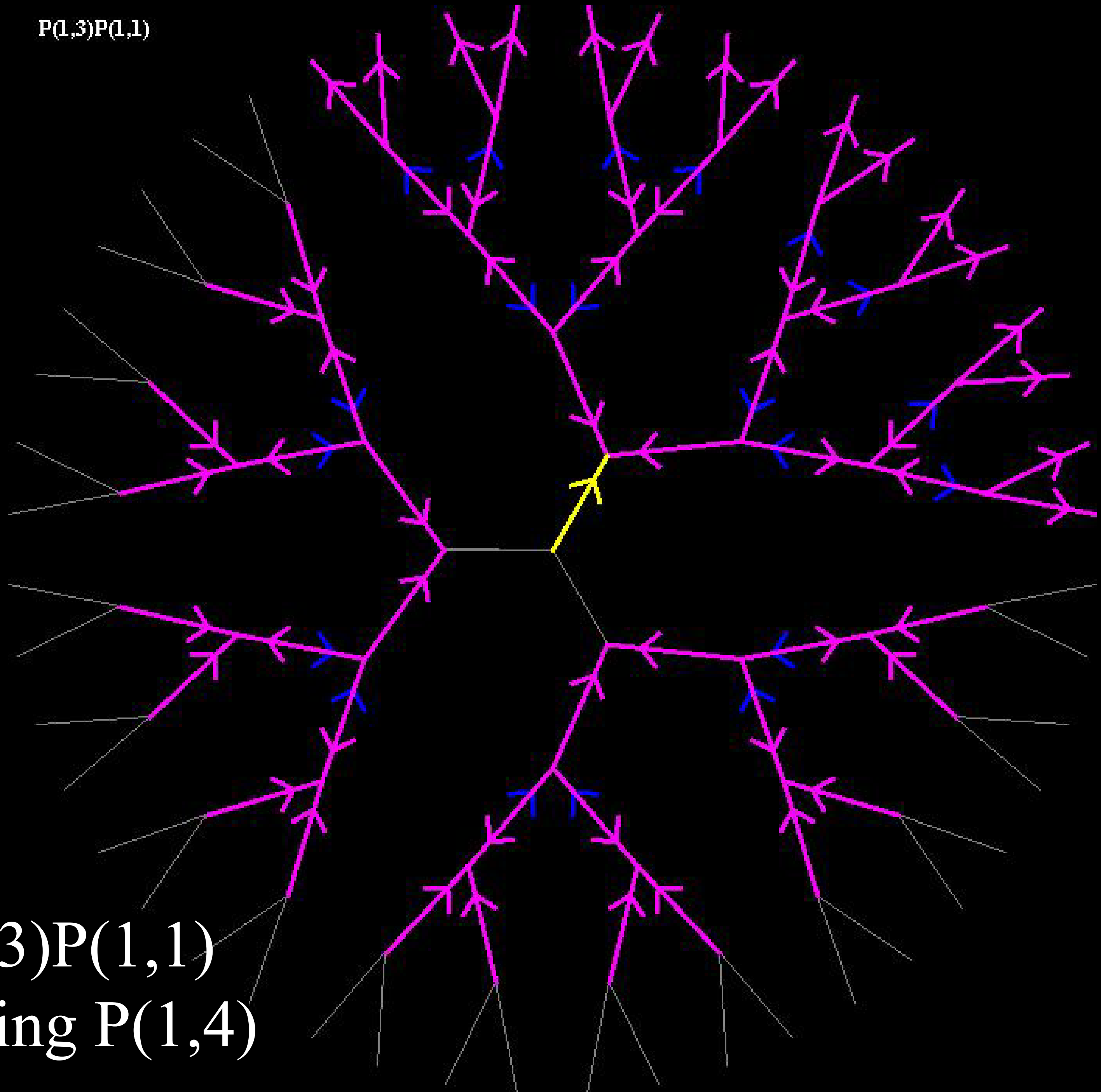




However there are new paths not found in P(2,1).



P(1,2)P(1,1)
covering P(1,3)

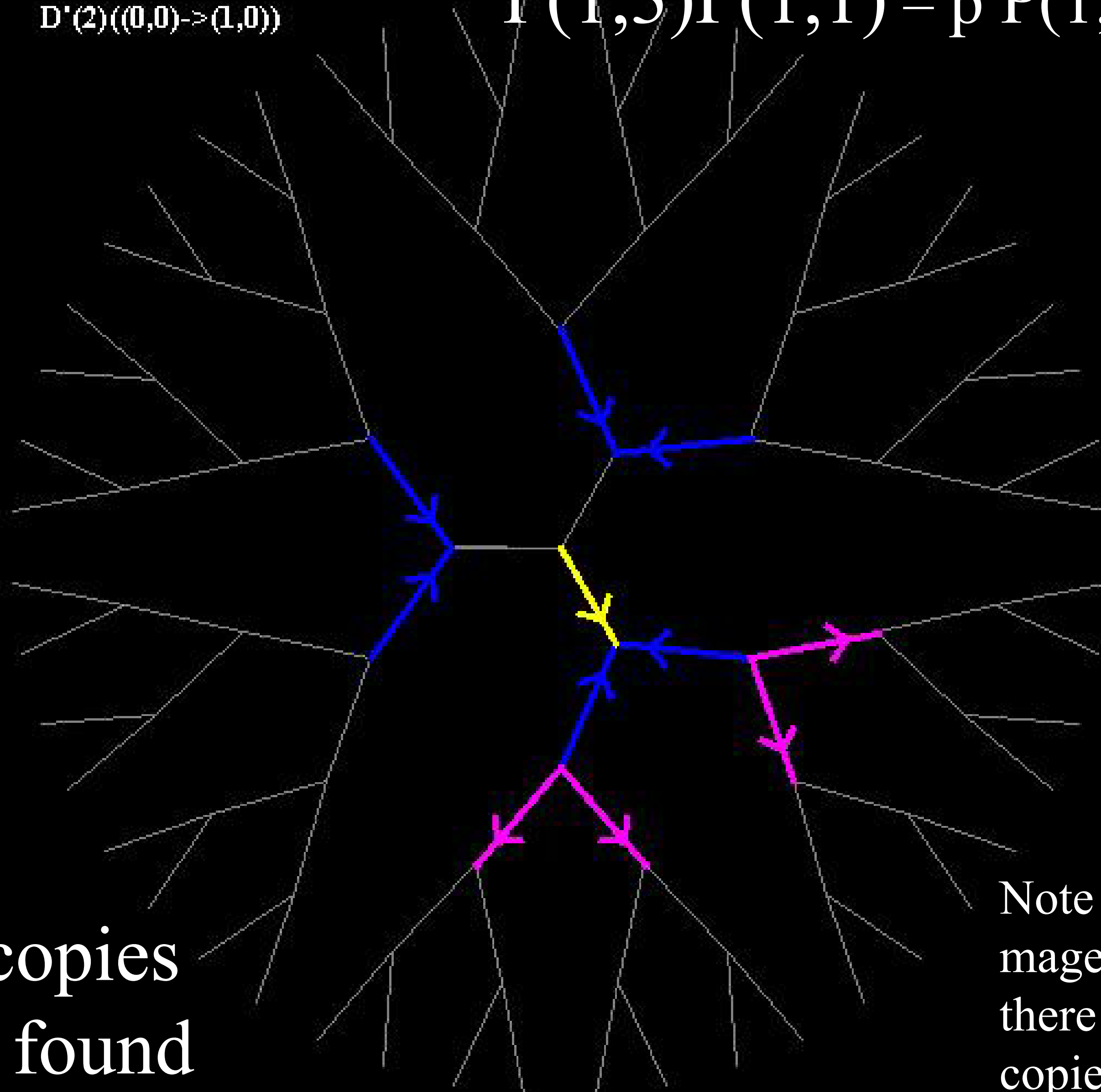


P(1,3)P(1,1)
covering P(1,4)

Decomposition

- Covering things gives us to many new arrows, but at least it gives us enough. Let’s see if we can figure out how to count them.

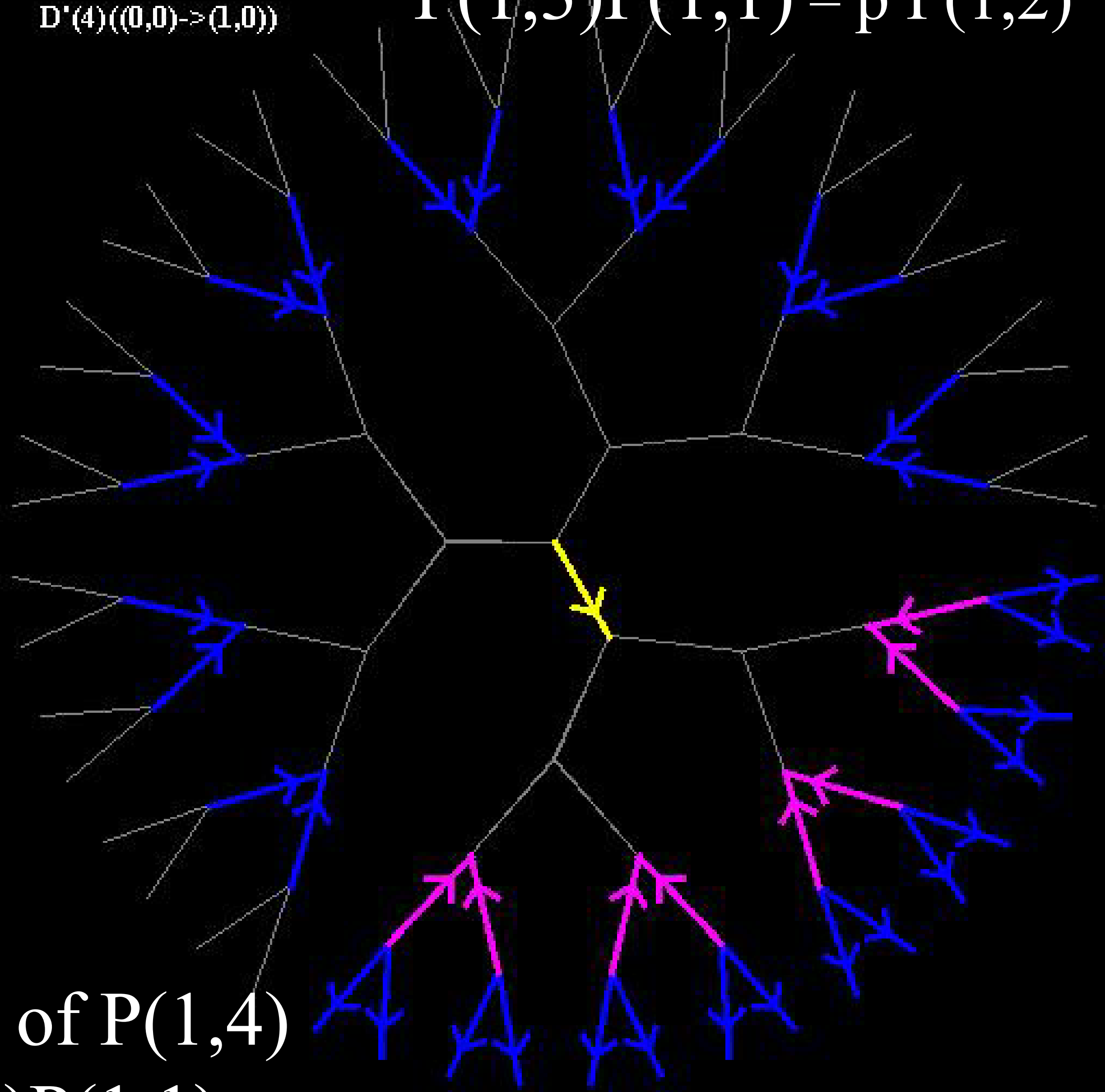
P(1,3)P(1,1) = p P(1,2) + ?



p many copies of P(1,2) found in P(1,3)P(1,1)

Note the color magenta means there are more copies of these paths than the blue.

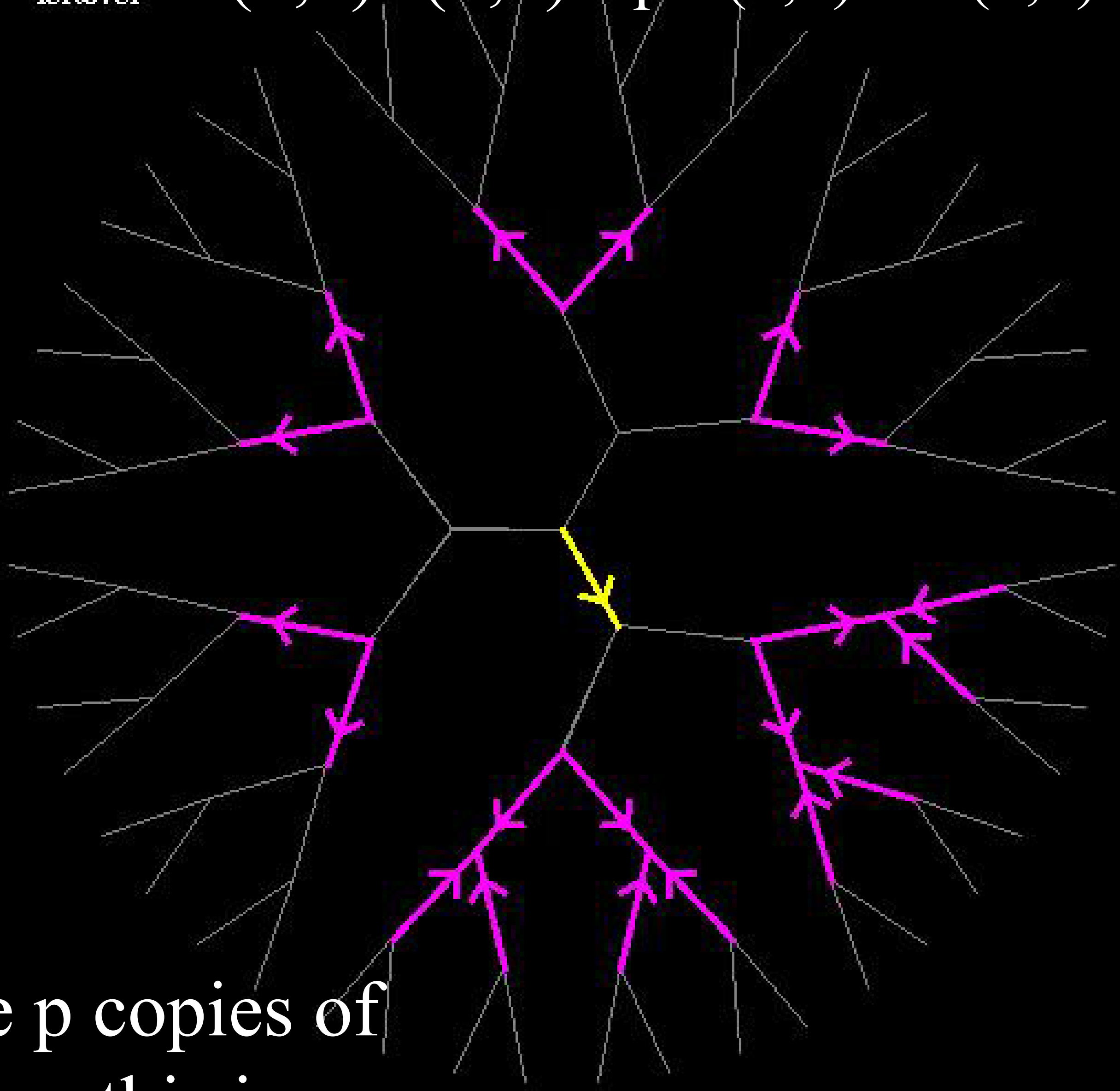
$P(1,3)P(1,1) = p\,P(1,2) + P(1,4) + ?$



The copy of P(1,4) in P(1,3)P(1,1).

leftover

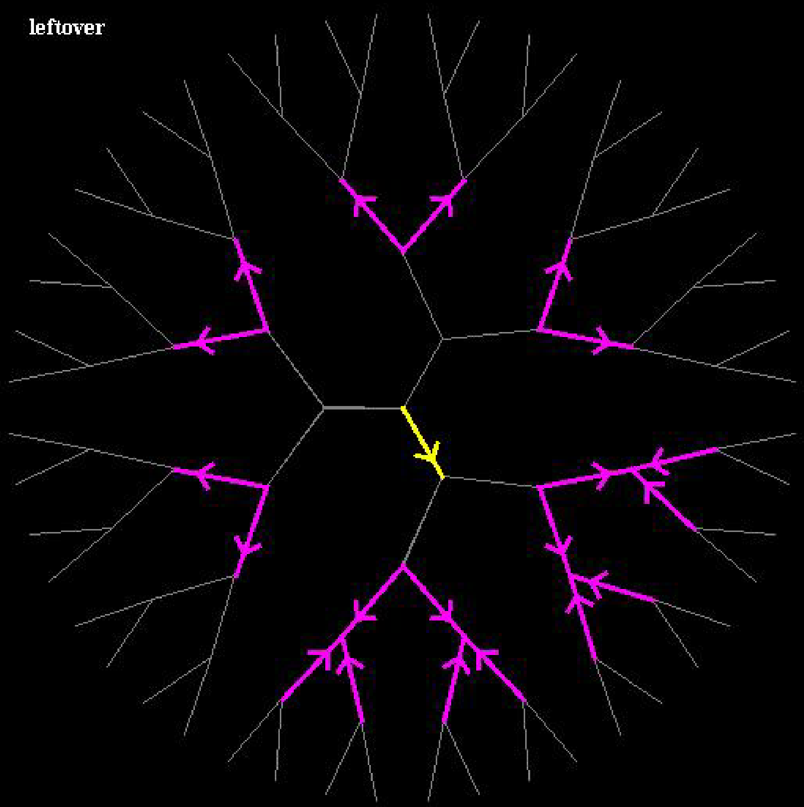
$$P(1,3)P(1,1) = p\,P(1,2) + P(1,4) + p\,S$$

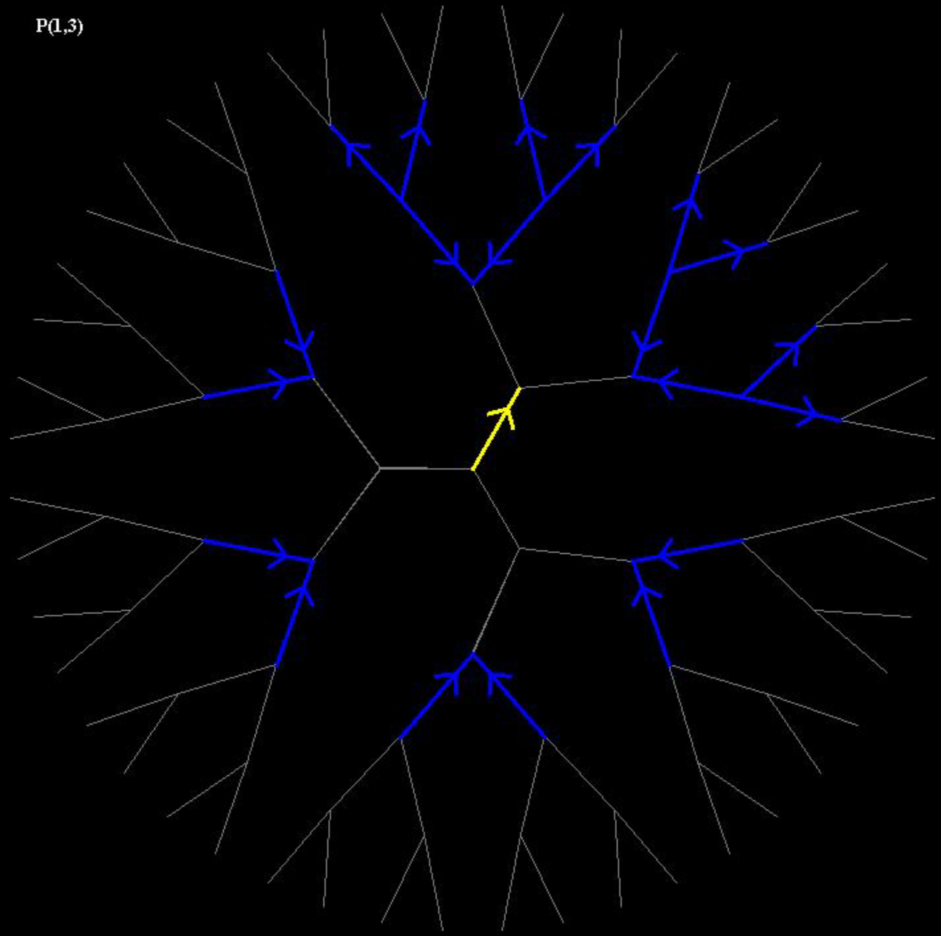


There are p copies of whatever this is.

But hold on, this looks similar
To One we already know about.

Only the arrows
Are all in the
Wrong direction!





A reverse orientation copy of P(1,3)!

Permutations

- To deal with this wrong direction all we have to do is flip all the arrows.
- To do this we use a well-known tool called a permutation.

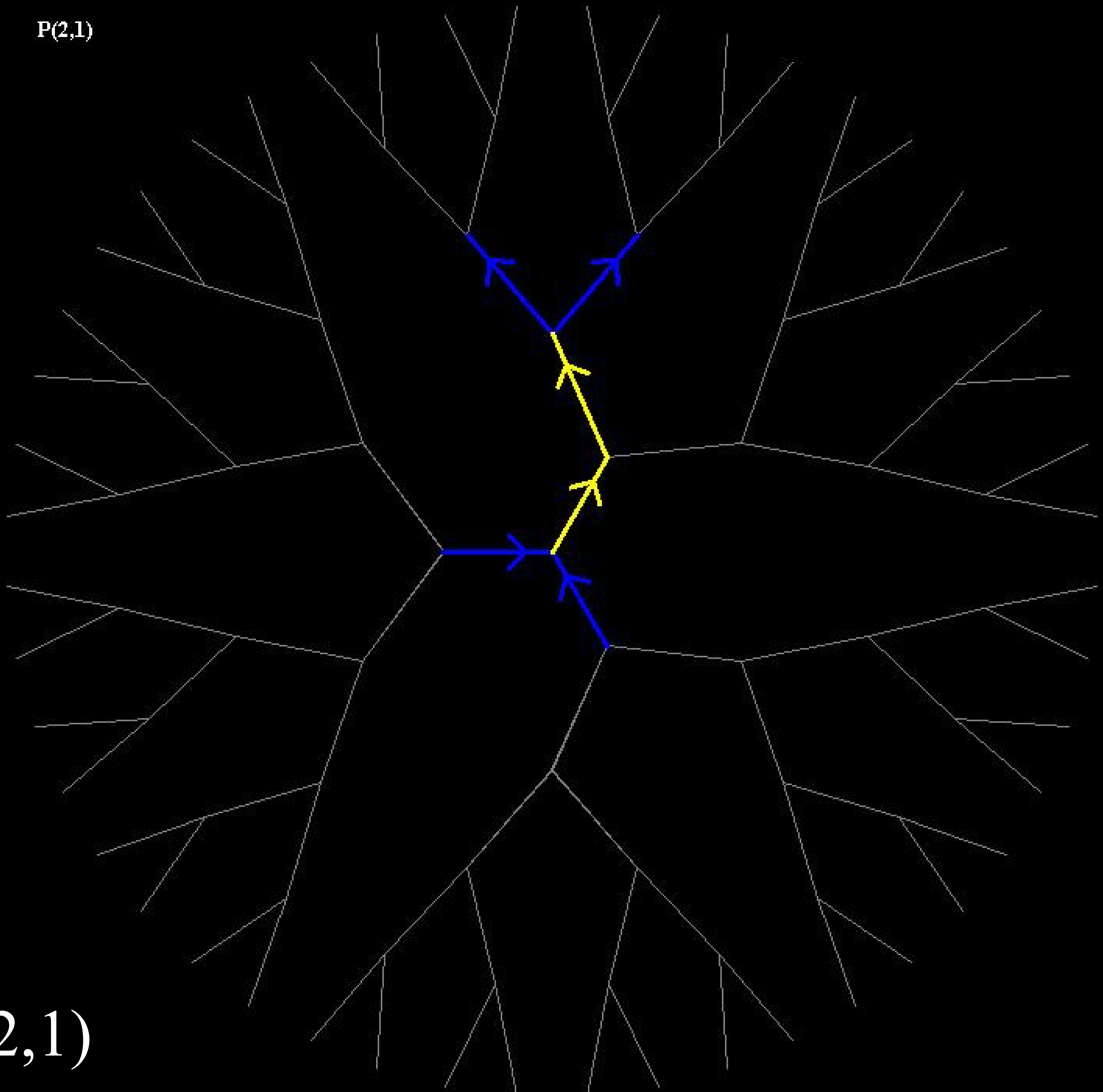
The Recursive Form of length 1.

- Let F be a permutation matrix that flips orientation. (Here is where you can catch my lie.)

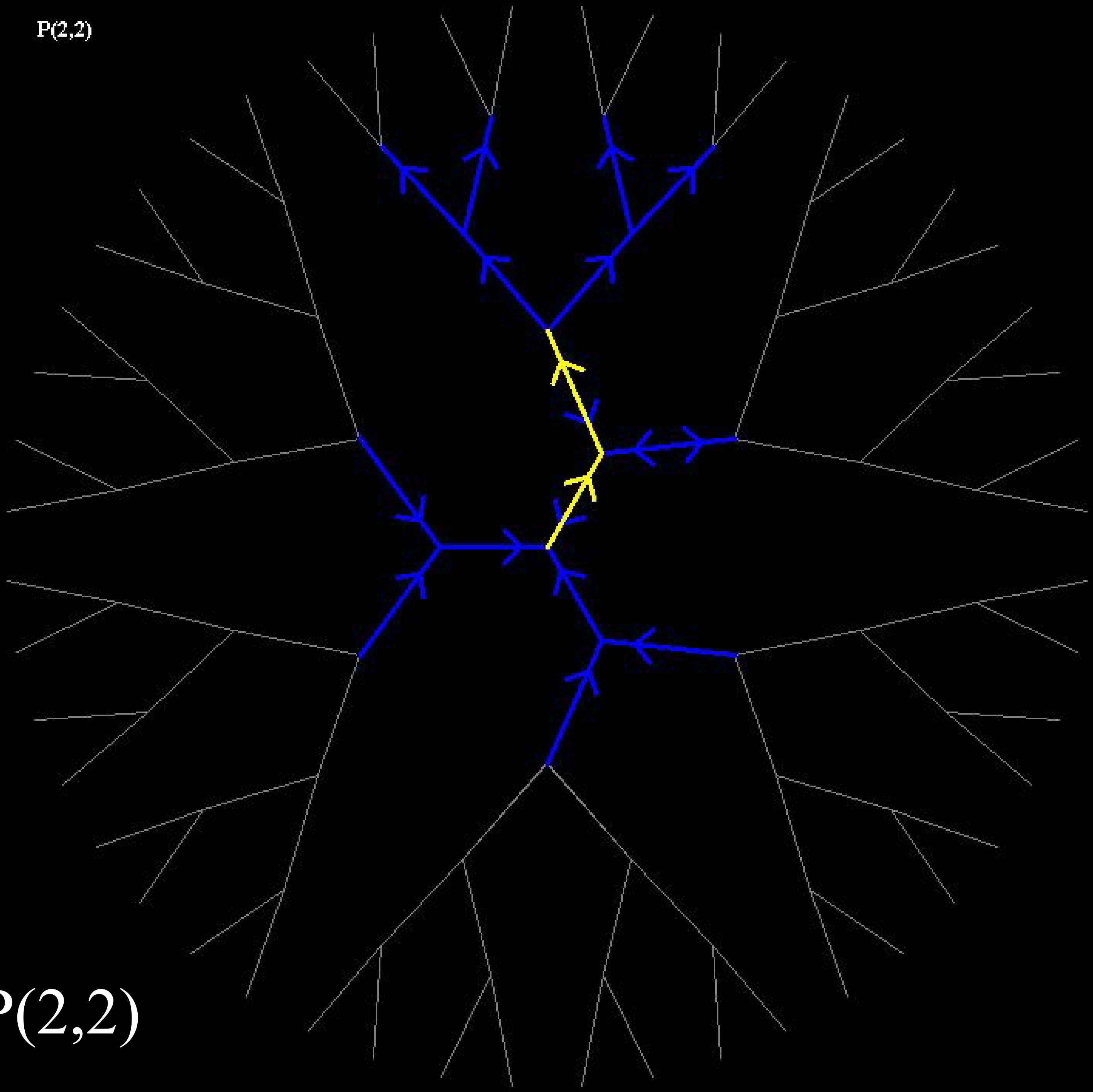
- D(n)D(1) = D(n+1) + p D(n)F + p D(n-1)

- For all n>2.

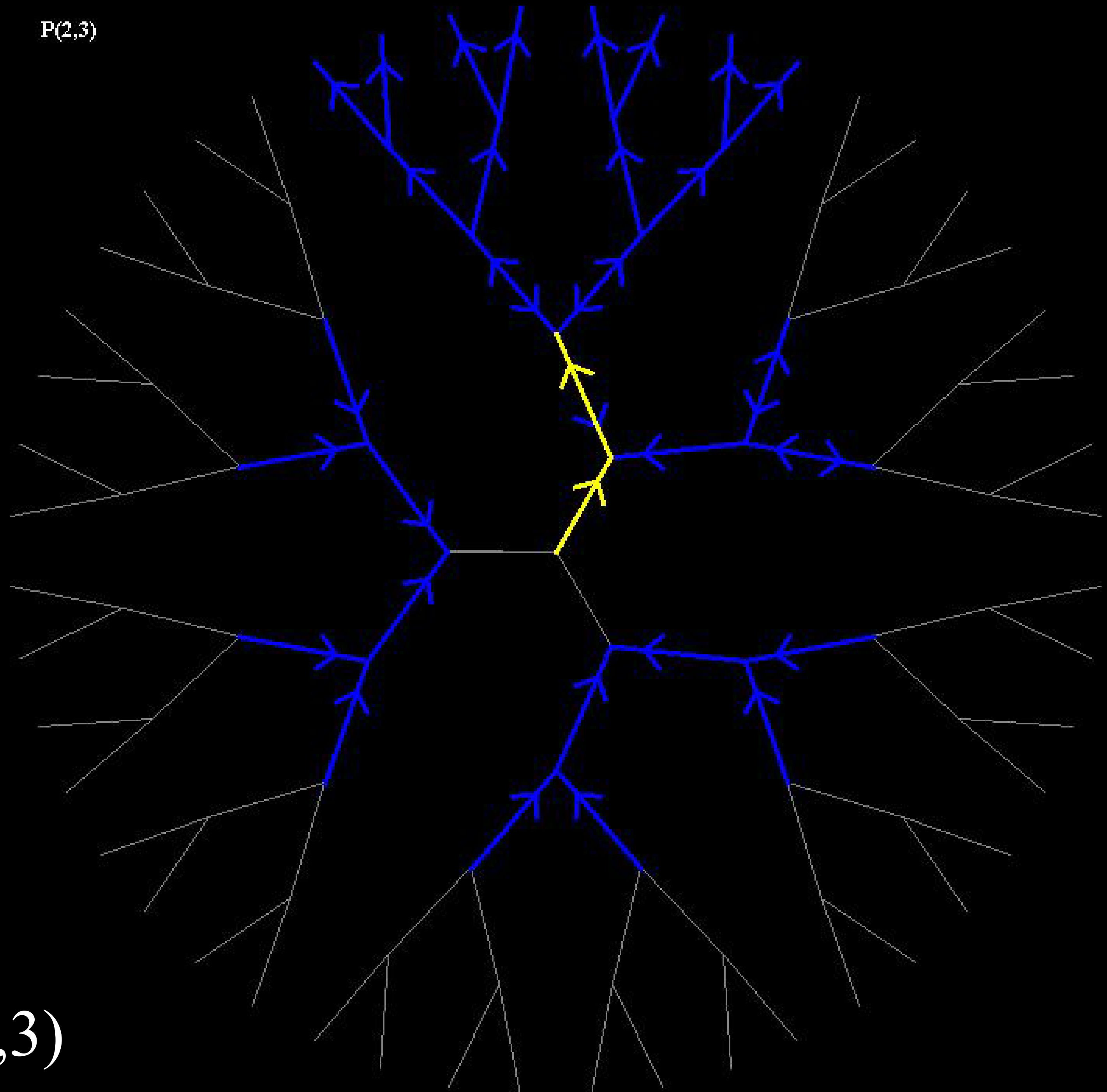
    » Now to Length 2….



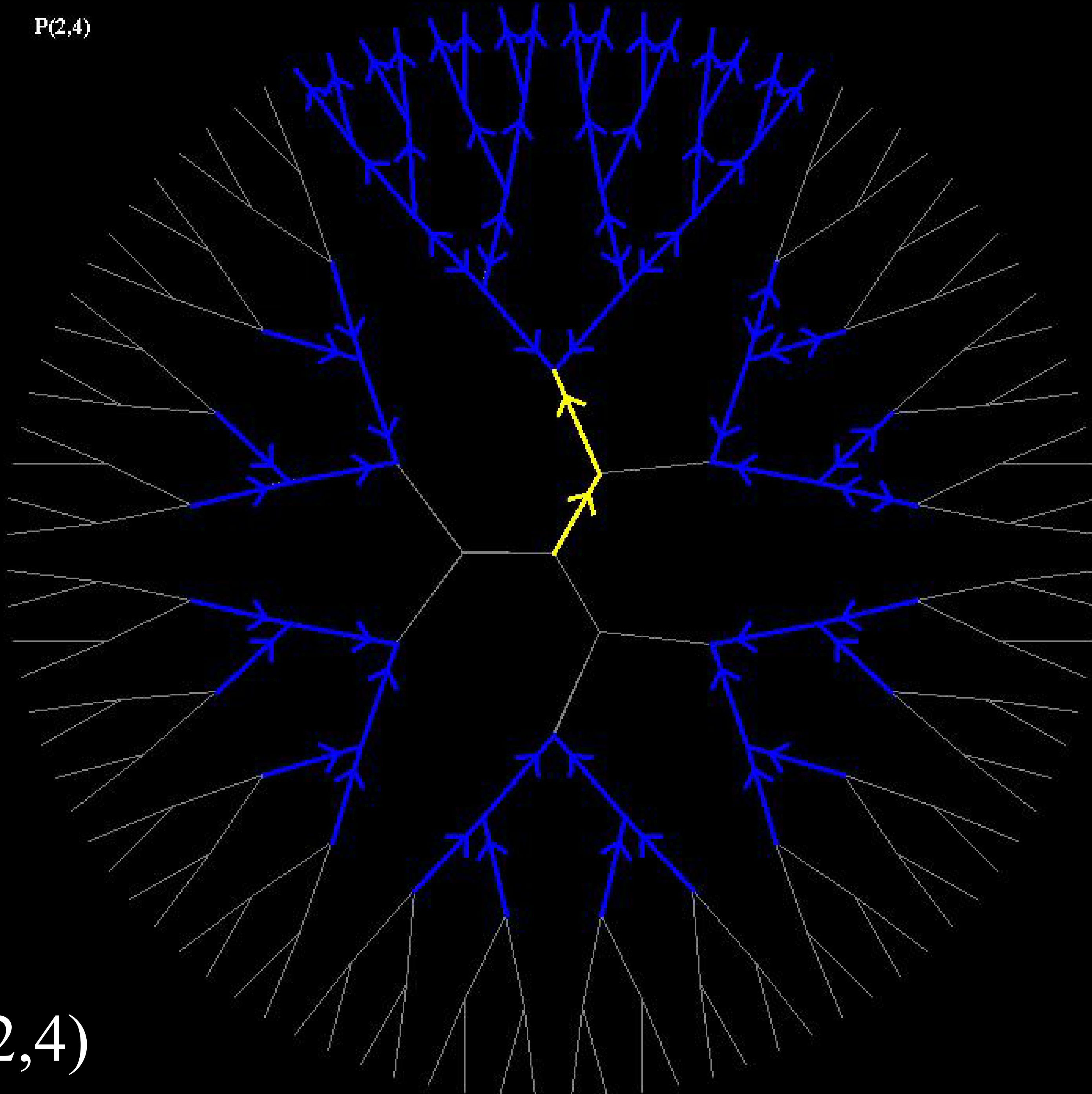
P(2,1)



P(2,2)



P(2,3)



P(2,4)

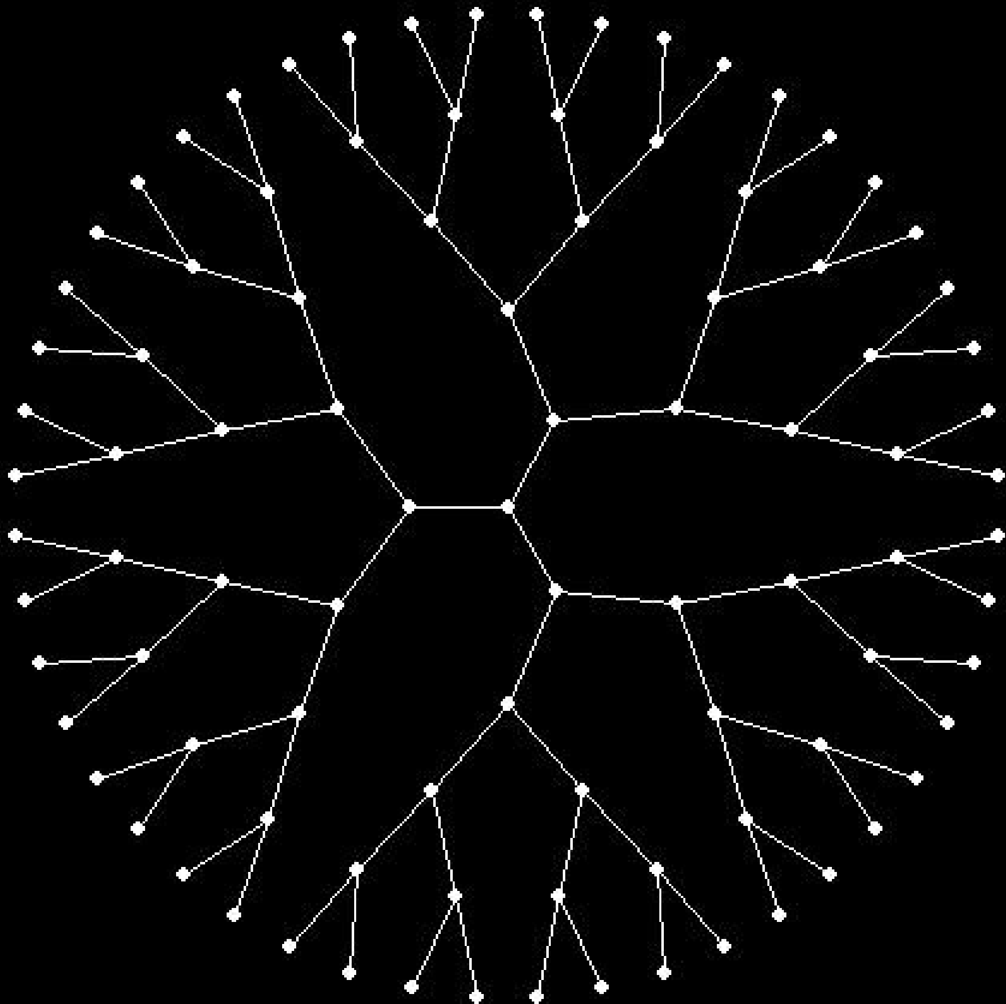
K-Distance Graphs

The symmetry in length one cases comes from the fact that each path is contained in exactly one edge.

To "clean up" the longer length k paths we redraw them inside the k-distance graph of the regular tree.

- Vertices: same as in the tree.
- Edges now become lines between vertices that are 2 points away from each other.

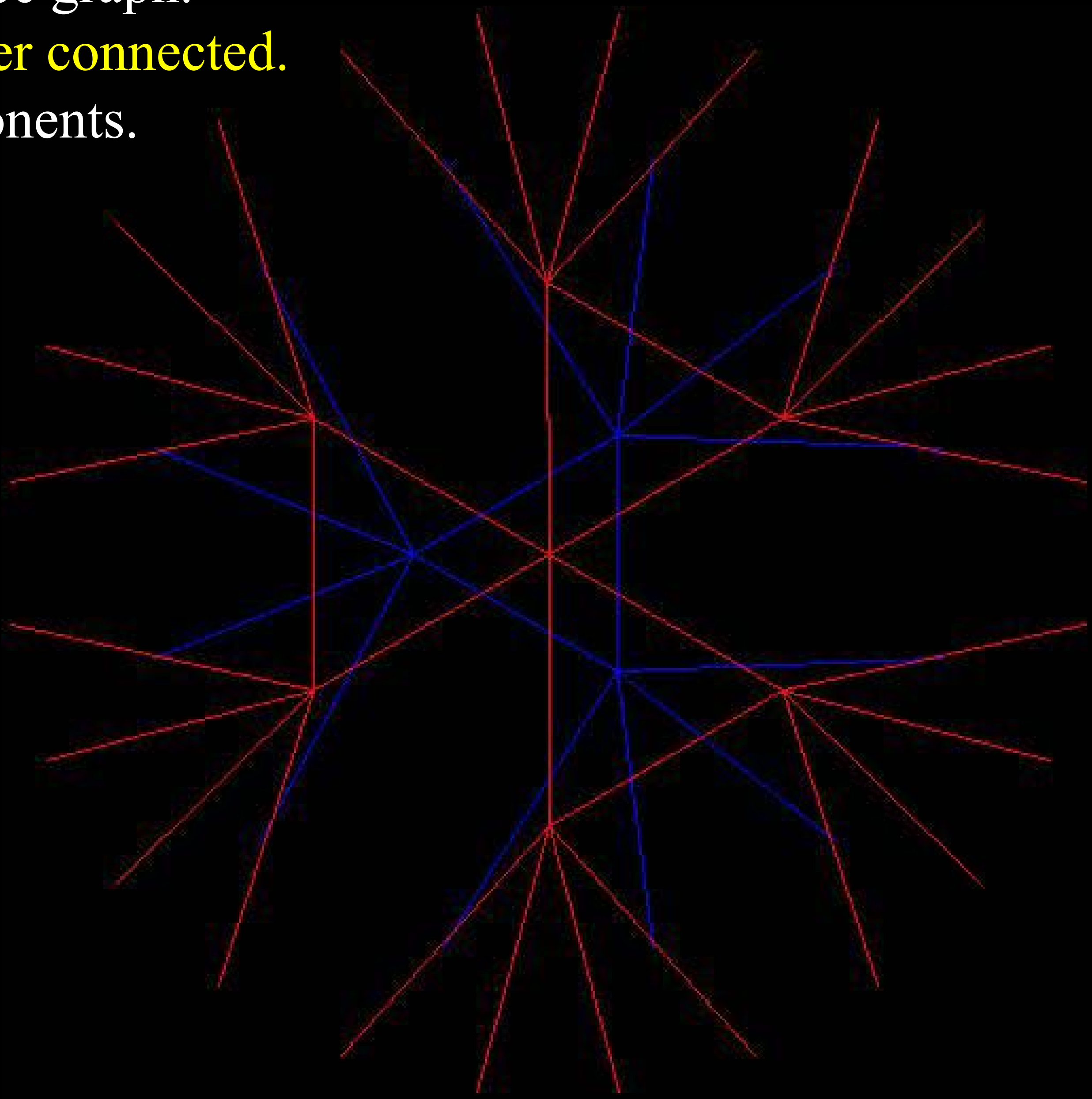
1-distance graph, same as the tree.



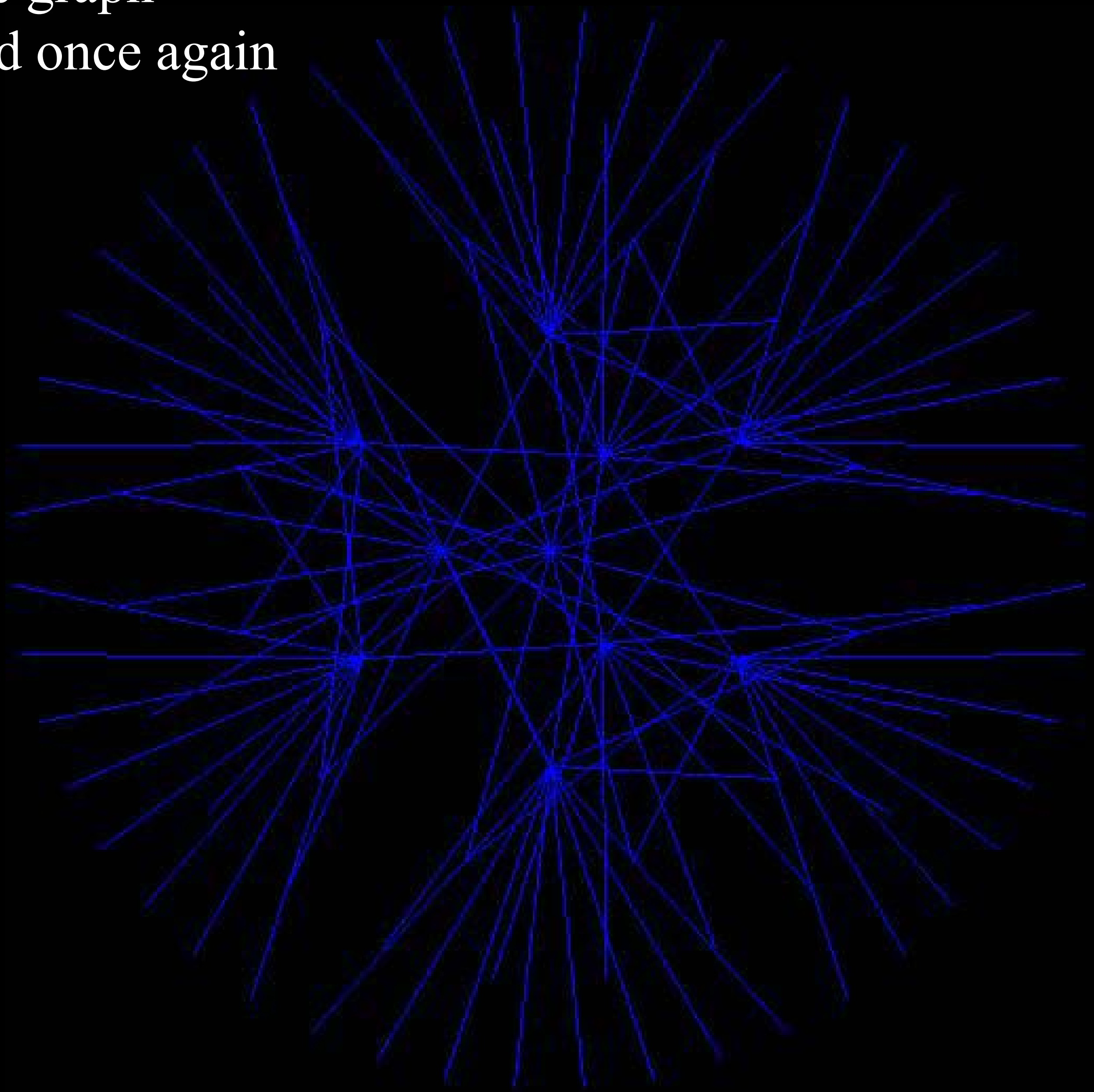
2-distance graph.

No longer connected.

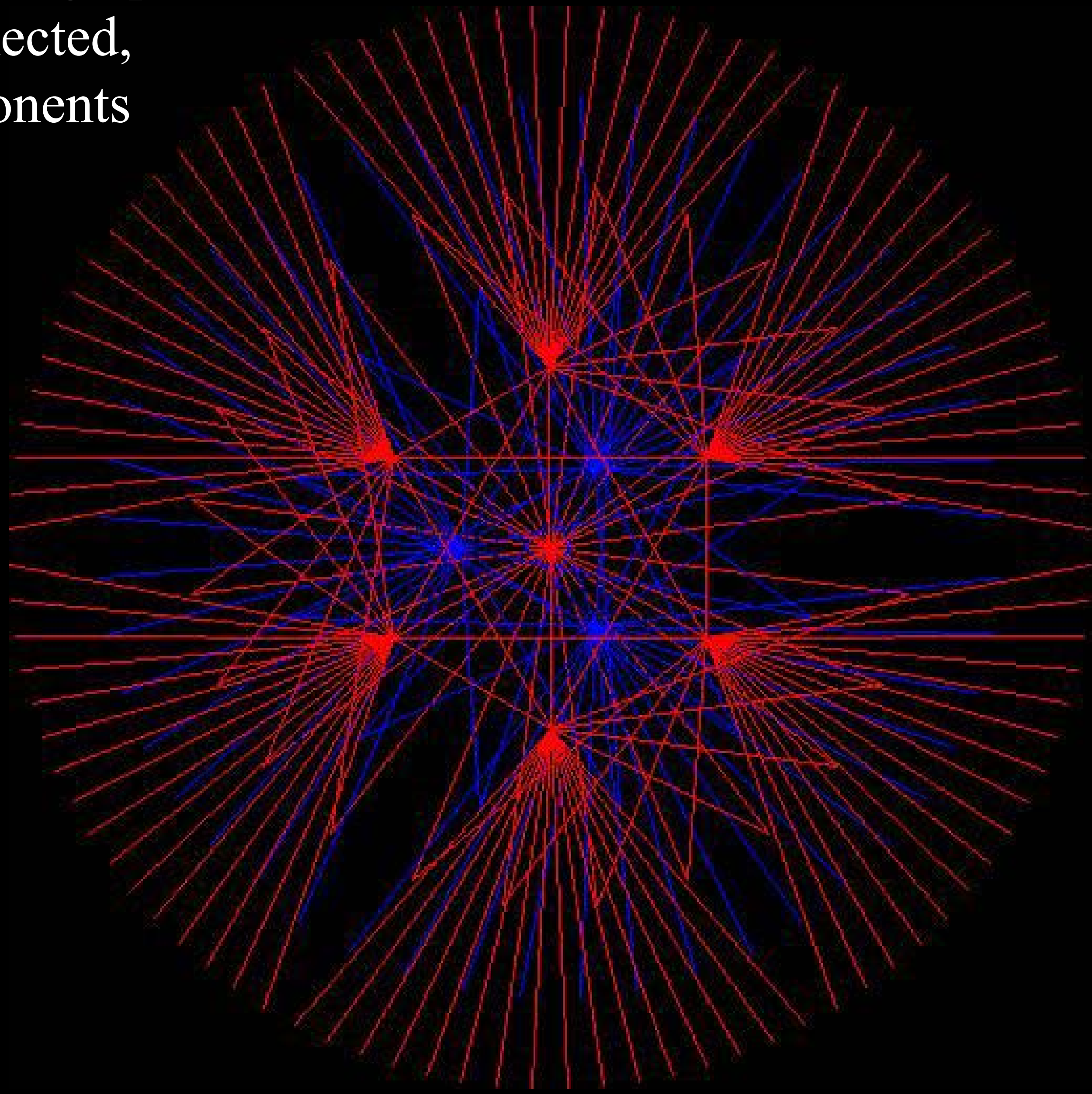
2-components.



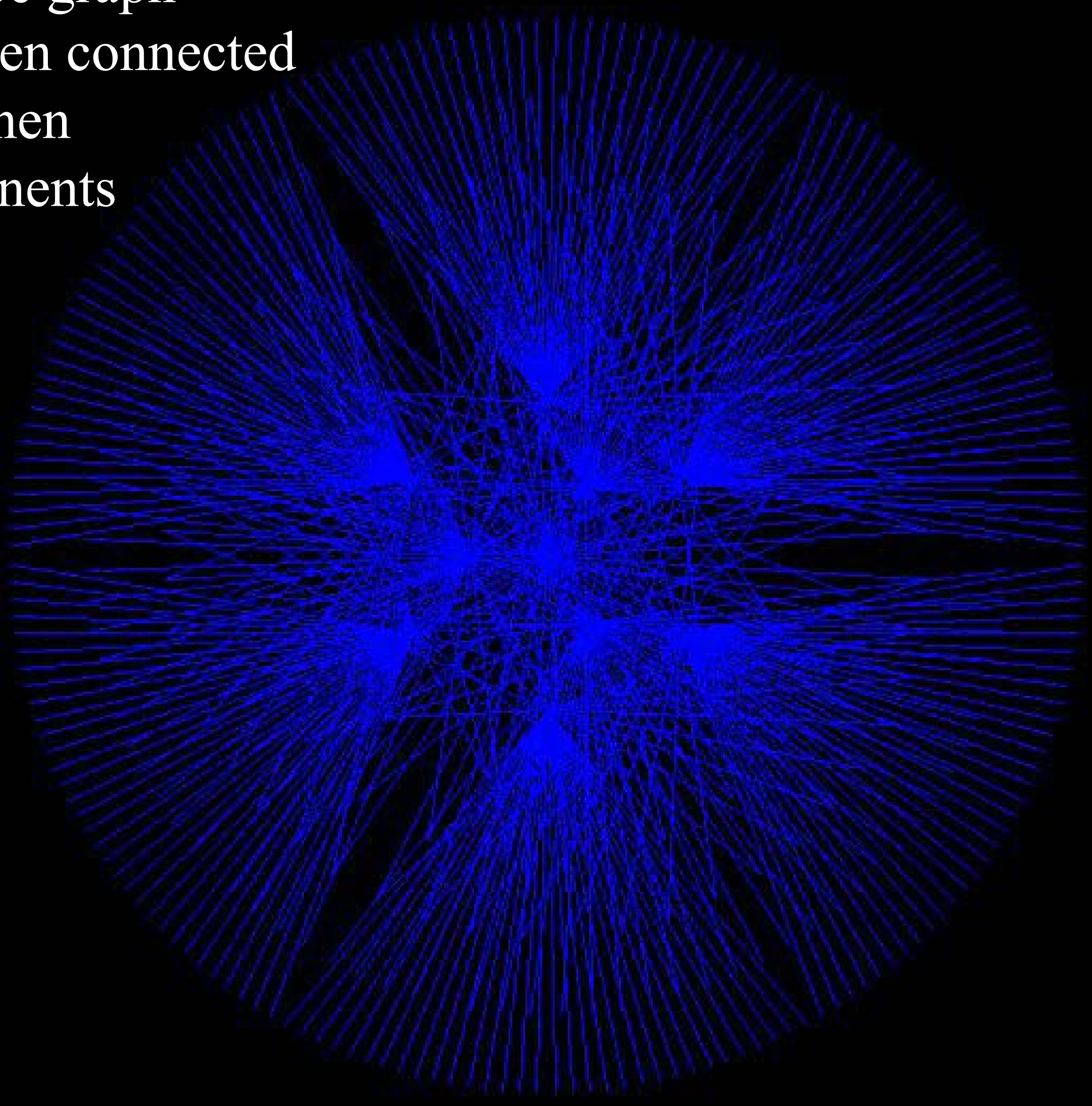
3-distance graph
Connected once again

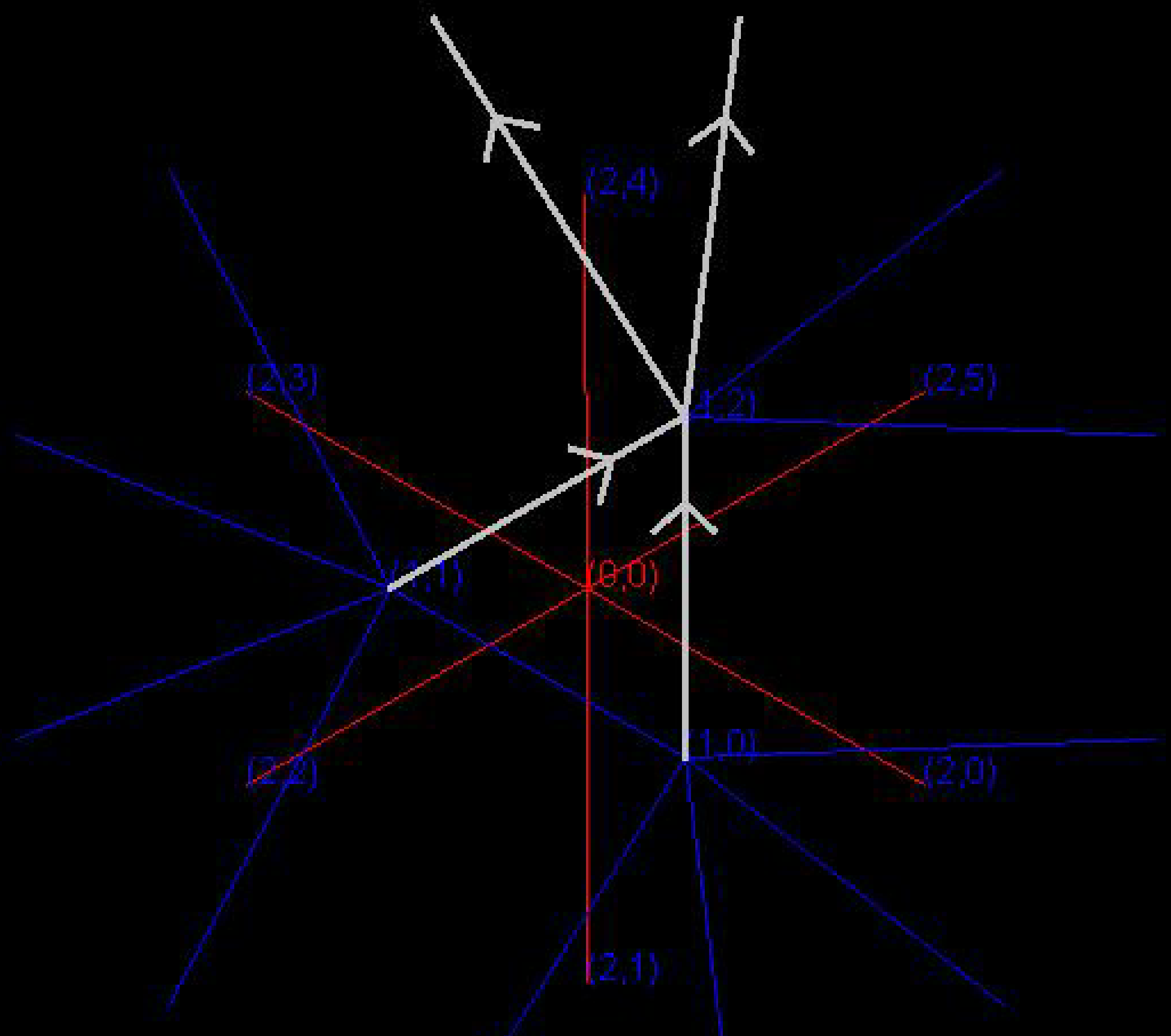


4-distance graph
Disconnected,
2-components

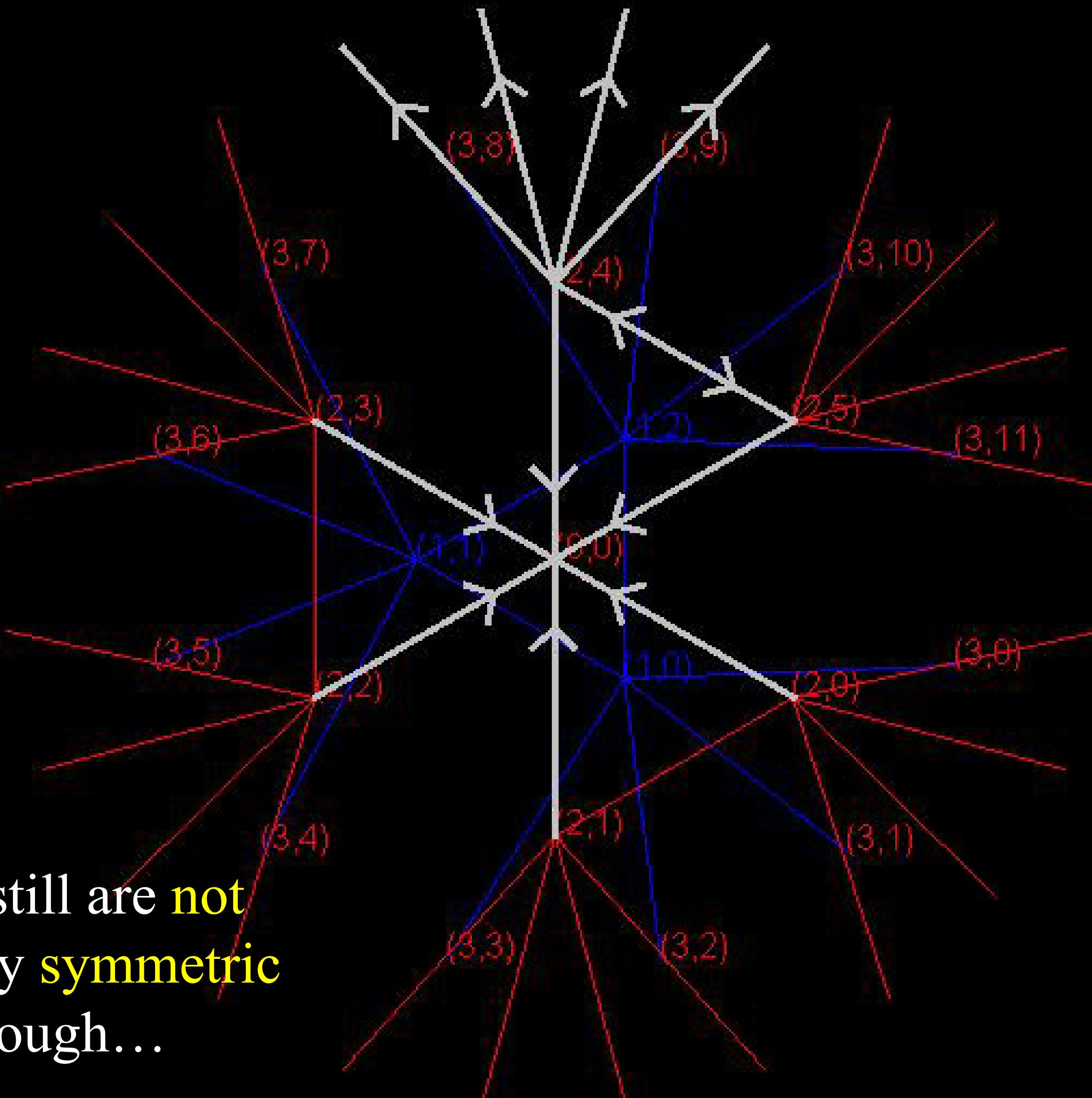


5-distance graph
Odd k then connected
Even k then
2 components

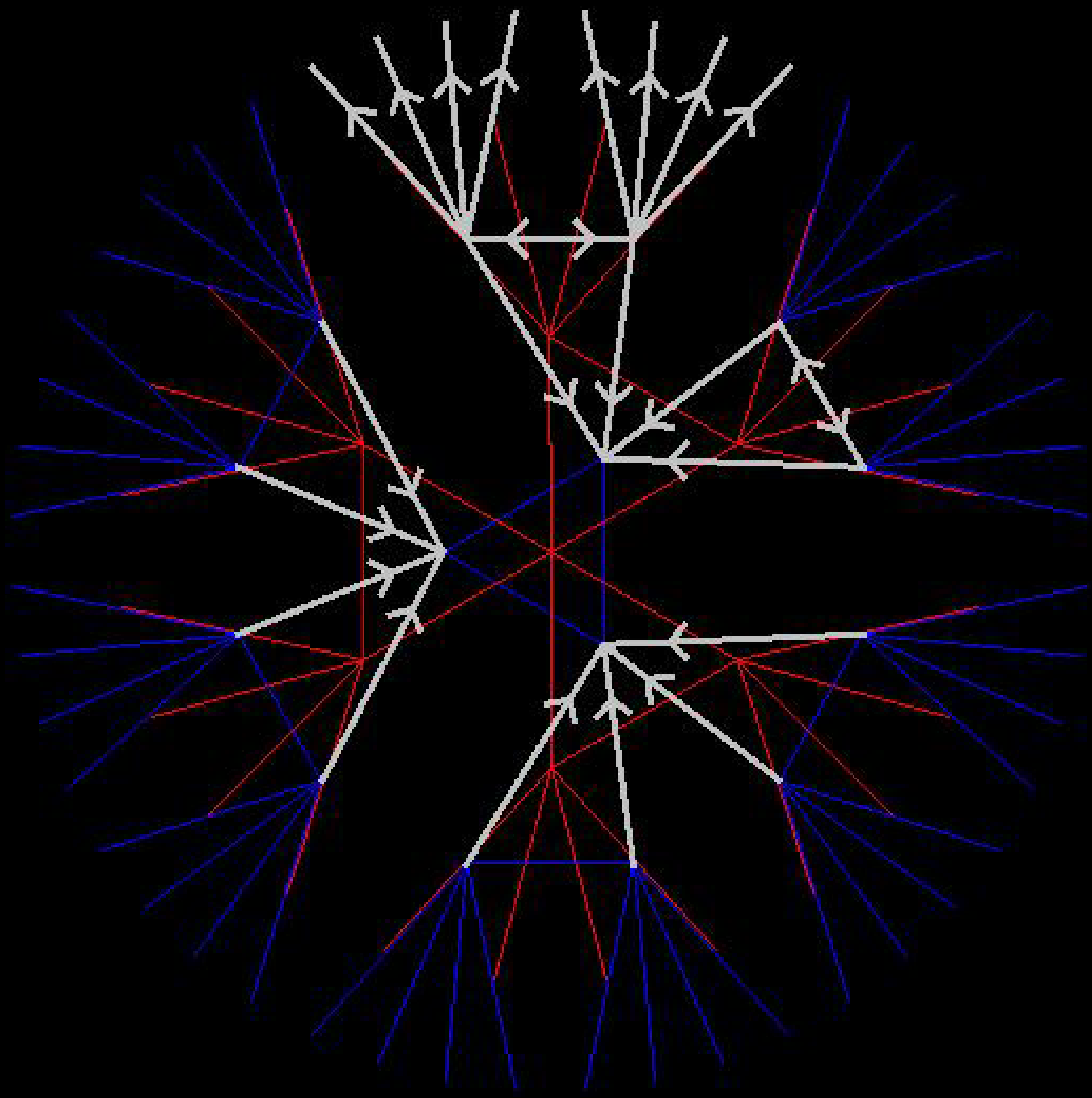


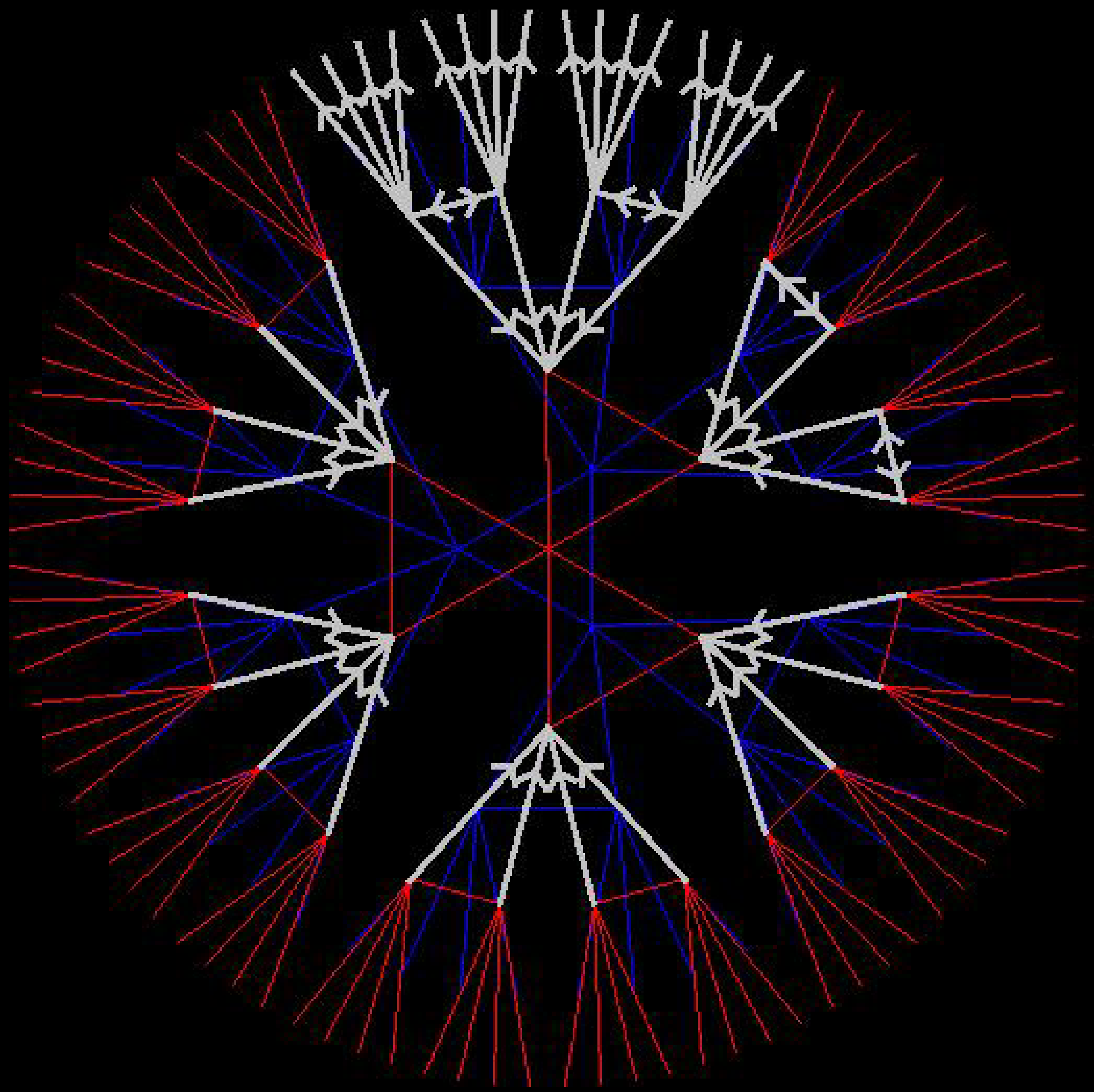


Notice our paths become
Straight again!



They still are not
Perfectly symmetric
Though…





Null-Orbit Count Matrix - M(p) - for p=2.

LENGTH

DISTANCE

1	1	1	1	1
3	5	4	4	4
6	10	12	10	10
12	20	24	28	24
24	40	48	56	64

Null-Orbit Count Matrix - M(p) - for p=3.

LENGTH

DISTANCE

1	1	1	1	1
4	7	6	6	6
12	21	27	24	24
36	63	81	99	90
108	189	243	297	351

Count Matrix - M(p) - for any p.

LENGTH

DISTANCE

1	1	1	1	1	1
$p+1$	$M_{1,1} = 2p+1$	$2p$	…	…	…
$(p+1)p$	$(2p+1)p$	$M_{2,2}$	$M_{2,2}-p$	…	…
:	:	:	:	…	…
:	:	:	:	$M_{i,i}$	$M_{i,i}-p^{i-1}$
$(p+1)p^k$	$(2p+1)p^k$	$M_{2,2}\, p^k$	…	$M_{i,i}\, p^{k-i}$	:

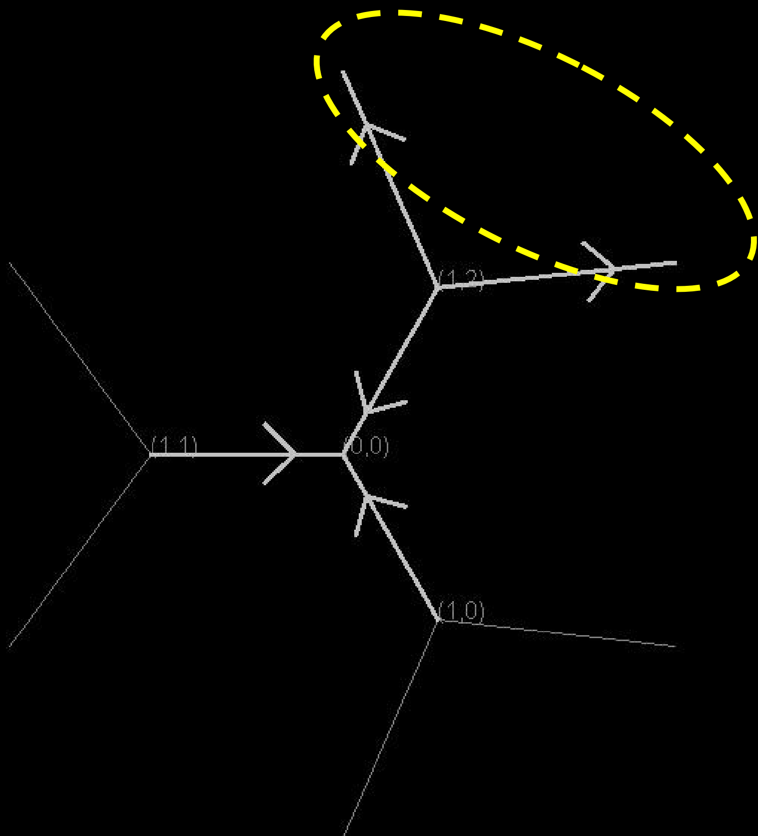
Bipartite Reorganization

Prop: *When i is odd, the graph P(i,j) is bipartite.*

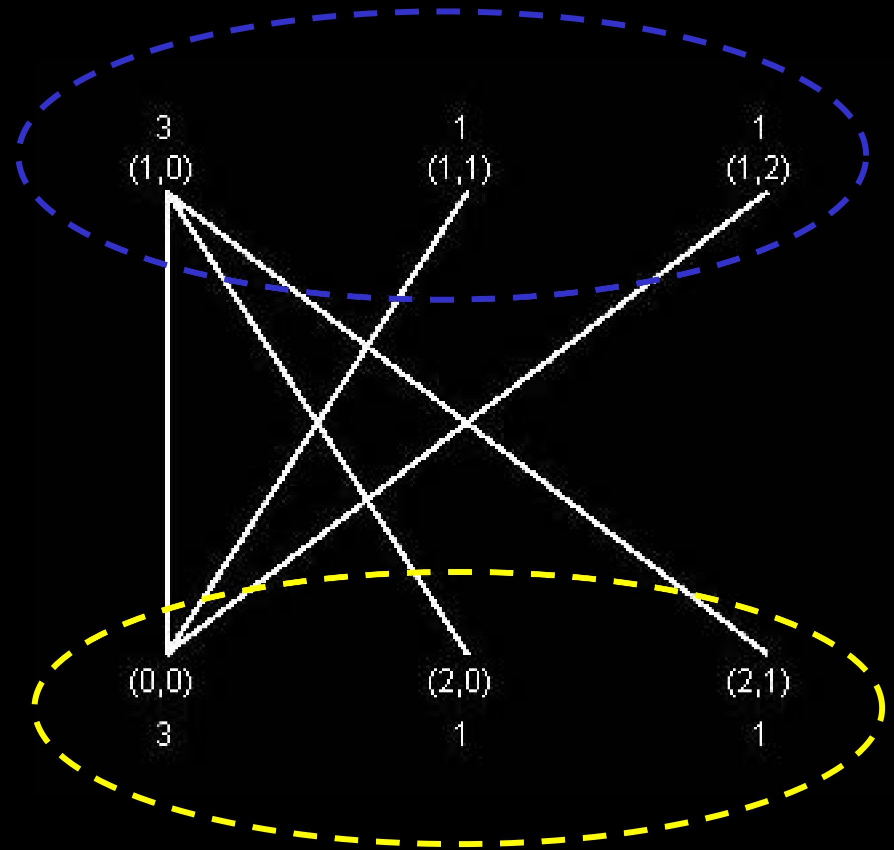
The *X* set is the set of all start vertices and *Y* the set of all end vertices of paths in *P(i,j)*.

Even length paths may in some cases end at the start of another path. Odd length paths may cross such points but will never end there.

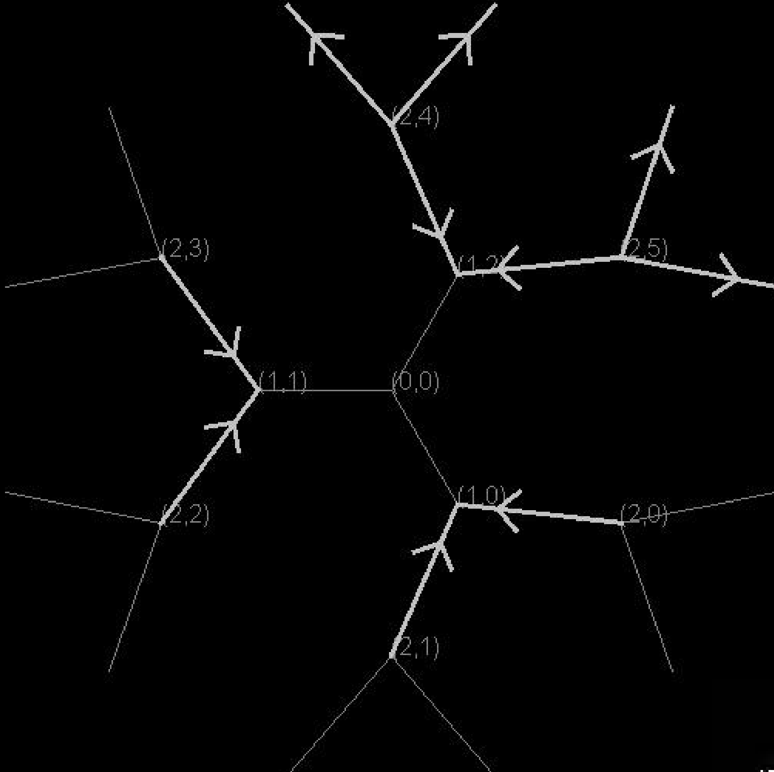
We can fold the graph up to
See this with two rows.



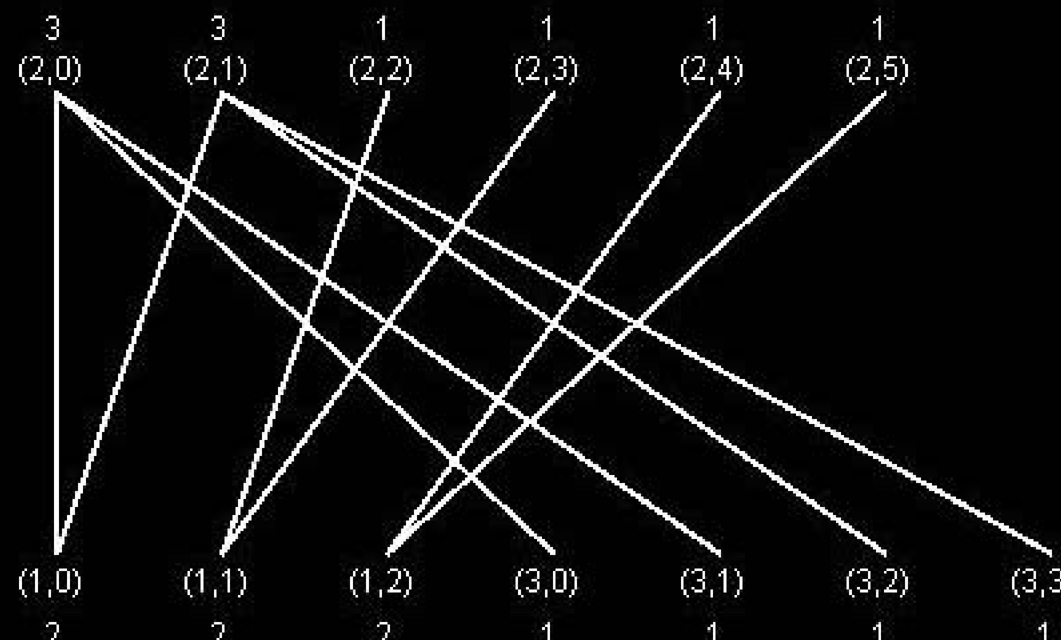
Source



Sink



We notice now the degree sequence of the sink set has changed to three 2's instead of two 3's. The graph is no longer symmetric.



Stability - Reducing the Search Space

Theorem: *If $j > i+1$ then $P(i,j)$ decomposes into q copies of $P(i,j-1)$. $P(i,i+1)$ is called stable and $P(i,j)$ super-stable.*

Corollary: *The degree sequence $S(i,j)$ of $P(i,j)$ is the disjoint union of q copies of the $S(i, j-1)$.*

Corollary: *A formula for $P(i,i+2)$ is a formula for $P(i,j)$, $j > i+2$. That is rewrite the formula by substitution.*

Unsolved Problem

- For paths of length 2 or greater, there is no formula yet derived.
- What is known:
  - No direct formula can exist.
  - No formula using flips can exist.
  - No formula will ever describe the entire column since length 2 is the first column that is not completely stable.

Other Approaches

- Using alternate permutation matrices at different locations allows for use of multiple central paths(even different paths) that can be used to increase the cardinality at any path to the desired amount.