TREES, HECKE OPERATORS, AND QUADRATIC FORMS

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ABSTRACT. We study paths on a tree associated to a definite rational quaternion algebra $\mathbf{H}$, and show that the class number of an order of level p^n in $\mathbf{H}$ is equal to the number of orbits of a certain group action on the set of paths of legnth n on the tree. We establish recursion formulas among the orbits of the paths, which allow us to prove relations for the Brandt matrices $B(p^k)$. Consequently, we obtain relations for the Hecke operators $T(p^k)$ for primes dividing the level. As an application, we compute the representation numbers of the norm form of the algebra restricted to the order. This generalizes results of Vignéras and Pays who studied maximal orders in quaternion algebras.

0. Introduction

Let $\mathbf{H}$ be a rational quaternion algebra, Γ a maximal order in $\mathbf{H}$, and p a prime which splits in $\mathbf{H}$, so $\mathbf{H}_p = \mathbf{H} \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong M_2(\mathbf{Q}_p)$. The set of all maximal orders in $\mathbf{H}_p$ can be made into a graph by defining a distance between the maximal orders, and placing an edge between any two orders at distance one; this graph is a $(p+1)$-regular tree. Now consider maximal orders Γ' in the global algebra $\mathbf{H}$ with the property that $\Gamma'_q = \Gamma_q$ for all $q \neq p$, and define the distance between two of these orders to be the distance between their corresponding localizations at the prime p. The unit group of $\Gamma^{(p)} = \Gamma \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{p}]$ acts on these orders, and the class number of Γ is equal to the number of orbits of $\Gamma^{(p)\times}$ under this action.

Now let Λ be an order of level p^n in $\mathbf{H}$, and as before set $\Lambda^{(p)} = \Lambda \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{p}]$. We show there is a group action of $\Lambda^{(p)\times}$ on the set of paths of length n on the tree, and that the class number of Λ is equal to the number of orbits of $\Lambda^{(p)\times}$ under this action. This result allows us to define certain matrices which are very close to the classical Brandt matrices $B(p^k)$, for primes dividing the level of Λ. Using the structure of the tree, we see that when $n = 1$ we can very easily derive recursion formulas for these matrices. As an application, we show this allows us to recursively compute the representation numbers of certain quadratic forms over $\mathbf{Q}$.

1. Preliminaries

Let $\mathbf{H}$ be a rational quaternion algebra; that is, a central simple algebra of dimension 4 over $\mathbf{Q}$. For a prime $p \in \mathbf{Z}$ (including infinity), we denote by $\mathbf{H}_p$ the quaternion algebra $\mathbf{H} \otimes_{\mathbf{Q}} \mathbf{Q}_p$ over $\mathbf{Q}_p$. The prime p ramifies in $\mathbf{H}$ if $\mathbf{H}_p$ is a division algebra; p splits in $\mathbf{H}$ if $\mathbf{H}_p \cong M_2(\mathbf{Q}_p)$. We shall be considering only definite quaternion algebras; so $\mathbf{H}$ ramifies at the infinite prime.

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A subset $\Lambda \subset \mathbf{H}$ is called an $\mathbf{Z}$-order if Λ is a ring containing $\mathbf{Z}$ as a subring, is finitely generated as a $\mathbf{Z}$-module and $\Lambda \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{H}$. We shall be interested in the following type of $\mathbf{Z}$-orders.

Definition 1. *Let N be a positive integer prime to the discriminant $D = D(\mathbf{H}/\mathbf{Q})$ of $\mathbf{H}$. An order Λ has level N if for each finite prime q, $\Lambda_q = \Lambda \otimes_{\mathbf{Z}} \mathbf{Z}_q$ satisfies:*

(1) *Λ_q is the unique maximal order of $\mathbf{H}_q$ when $q|D$*
(2) $\Lambda_q \cong \begin{pmatrix} \mathbf{Z}_q & \mathbf{Z}_q \\ q^{ord_q(N)}\mathbf{Z}_q & \mathbf{Z}_q \end{pmatrix}$, *otherwise.*

If Λ_q satisfies the second condition listed above, we say Λ_q is an Eichler order of level $q^{\mathrm{ord}_q(N)}$ in $\mathbf{H}_q$.

2. The Tree and Class Numbers

We identify $\mathbf{H}_p$ with $M_2(\mathbf{Q}_p)$ when $p \nmid D$, so the maximal orders of $\mathbf{H}_p$ are the conjugates of $M_2(\mathbf{Z}_p)$. Defining the distance between two maximal orders by

$$d(x_p^{-1}M_2(\mathbf{Z}_p)x_p, y_p^{-1}M_2(\mathbf{Z}_p)y_p) = \mathrm{ord}_p(\det(x_p y_p^{-1})) - 2 \cdot \min\{\mathrm{ord}_p(x_p y_p^{-1})_{ij}\},$$

one obtains the following:

Theorem 1. ([6], p. 40) *The graph T whose vertices are the maximal orders of $\mathbf{H}_p$ and whose edges are the pairs of vertices at distance 1 is a $(p+1)$-regular tree.*

The following gives a useful characterization of the Eichler orders in $\mathbf{H}_p$ (see [6], Lemma 2.4):

Proposition 1. *Let Λ_p be an order in $\mathbf{H}_p$. The following are equivalent:*

(1) $\Lambda_p = x_p^{-1}\begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p^n\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix} x_p$ *for some $x_p \in \mathbf{H}_p^{\times}, n \in \mathbf{Z}^{+}$.*
(2) *There exists a unique pair of maximal orders Γ_p^0 and Γ_p^n of $\mathbf{H}_p$ such that $\Lambda_p = \Gamma_p^0 \cap \Gamma_p^n$.*
(3) *Λ_p is the intersection of two maximal orders at distance n in T.*

As noted in [4], $t_p \in \mathbf{H}_p^{\times}$ normalizes $\Lambda_p = x_p^{-1}\begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p^n\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix} x_p = \Gamma_p^0 \cap \Gamma_p^n$ if and only if conjugation by t_p permutes the two maximal orders Γ_p^0 and Γ_p^n. This implies that $t_p \in \Lambda_p^{\times}\mathbf{Q}_p^{\times}$ or $t_p \in \Lambda_p^{\times}\mathbf{Q}_p^{\times}s_p$, where $s_p = x_p^{-1}\begin{pmatrix} 0 & 1 \\ p^n & 0 \end{pmatrix} x_p$. Thus, the normalizer of Λ_p is given by $N(\Lambda_p) = \Lambda_p^{\times}\mathbf{Q}_p^{\times} \cup \Lambda_p^{\times}\mathbf{Q}_p^{\times}s_p$.

We now shift our attention to the global setting. Fix an order Λ of level p^n in $\mathbf{H}$, and observe that the set

$$X_\Lambda = \{\text{maximal } \Gamma \subset \mathbf{H} \,|\, \Gamma_q = \Lambda_q \text{ for all primes } q \neq p\}$$

is in one-to-one correspondence with the vertices of T by the local-global correspondence for orders. We define the distance between two global maximal orders to be the distance between their corresponding localizations at the prime p. To make X_Λ into a graph, we put an edge between two orders distance 1 apart.

By a path of length n in X_Λ we mean a sequence of vertices $(\Gamma^0, \Gamma^1, \cdots, \Gamma^n)$ such that $d(\Gamma^i, \Gamma^{i+1}) = 1$ for $i = 0, \cdots, n-1$. If the sequence is such that $\Gamma^i \neq \Gamma^{i+2}$ for $i = 0, \cdots, n-2$ then the path is called a path of length n; i.e., a path of length n without backtracking. Since X_Λ is a tree, it is clear that any path is uniquely determined by Γ^0 and Γ^n, so we write (Γ^0, Γ^n) for the path from Γ^0 to Γ^n. We will say that the path $g = (\Gamma^0, \Gamma^n)$ has the opposite orientation to $\overline{g} = (\Gamma^n, \Gamma^0)$. Denote by $P_n(X_\Lambda)$ the set of paths in X_Λ of length n.

Let $J_{\mathbf{H}}$ denote the ideles of the quaternion algebra $\mathbf{H}$; so if Λ is an order of level p^n,

$$J_{\mathbf{H}} = \left\{ \tilde{x} = (x_q) \in \prod_q \mathbf{H}_q^\times \,\middle|\, x_q \in \Lambda_q^\times \text{ for almost all } q \right\},$$

where the product is over all finite and infinite primes of $\mathbf{Q}$. A left Λ-ideal is an $\mathbf{Z}$-lattice I on $\mathbf{H}$ so that $I_q = \Lambda_q x_q$ for all finite q with $x_q \in \mathbf{H}_q^\times$. Further, $x_q \in \Lambda_q^\times$ for almost all q (since $I_q = \Lambda_q$ for almost all q), so there exists an $\tilde{x} \in J_{\mathbf{H}}$ whose components are equal to x_q for all finite q. Hence we may write $I = \Lambda\tilde{x}$, where $\tilde{x} \in J_{\mathbf{H}}$ is an appropriate idele. The left order of I is given by $\{x \in \mathbf{H} \,|\, xI \subset I\}$, and the right order of I is $\{x \in \mathbf{H} \,|\, Ix \subset I\}$. I is said to be normal if its left (hence right) order is maximal.

Two left Λ-ideals I and J are in the same ideal class if $I = Jx$ for some $x \in \mathbf{H}^\times$. If

$$U(\Lambda) = \{\tilde{x} = x_q \in J_{\mathbf{H}} \,|\, x_q \in \Lambda_q^\times \text{ for all finite primes } q\},$$

then the left ideal classes of Λ are in one-to-one correspondence with the cosets $U(\Lambda)\backslash J_{\mathbf{H}}/\mathbf{H}^\times$. The number of ideal classes is finite and is called the class number of Λ. The cardinality of the classes of two-sided ideals of Λ is called the class number of two-sided ideals. If $\{I_1, I_2, \ldots, I_h\}$ is a complete set of left Λ-ideal class representatives with corresponding right orders $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_h\}$, then the number of mutually non-isomorphic orders in this set is called the type number of Λ. We shall need the following

Lemma 1. (Lemma 5.6 of [6]**)** *Let $\{\Lambda = \Lambda_1, \Lambda_2, \ldots, \Lambda_t\}$ give a complete set of the types of orders of level p^n in $\mathbf{H}$. If h_i is the class number of two-sided Λ_i-ideals, then $h = \sum_{i=1}^t h_i$.*

Now set $\Lambda^{(p)}$ equal to the $\mathbf{Z}^{(p)}$-order $\Lambda \otimes_{\mathbf{Z}} \mathbf{Z}^{(p)}$, where

$$\mathbf{Z}^{(p)} = \{x \in \mathbf{Q} \,|\, x \in \mathbf{Z}_q \text{ for all finite primes } q \neq p\} = \bigcap_{q \neq p} \mathbf{Z}_{(q)}.$$

Note that $\Lambda_q^{(p)} = \Lambda_q$ when $q \neq p$, hence $a \in \Lambda^{(p)\times}$ implies that $a \in \Lambda_q^\times$ for all $q \neq p$. Let $G = (\Gamma^0, \Gamma^n)$ be in $P_n(X_\Lambda)$ and $a \in \Lambda^{(p)\times}$. By the local-global correspondence for orders, the pair $(a^{-1}\Gamma^0 a, a^{-1}\Gamma^n a)$ is uniquely determined by the set of local orders $\big\{(a^{-1}\Gamma_q^0 a\,,\; a^{-1}\Gamma_q^n a)\big\}$. When $q \neq p$, $(a^{-1}\Gamma_q^0 a\,,\; a^{-1}\Gamma_q^n a) = (a^{-1}\Lambda_q a\,,\; a^{-1}\Lambda_q a) = (\Lambda_q\,,\; \Lambda_q)$ since $a \in \Lambda_q^\times$. Hence,

$$\big\{(a^{-1}\Gamma_q^0 a\,,\; a^{-1}\Gamma_q^n a)\big\} \leftrightarrow \left\{ \begin{array}{cc} (\Lambda_q\,,\; \Lambda_q) & \text{if } q \neq p \\ (a^{-1}\Gamma_p^0 a\,,\; a^{-1}\Gamma_p^n a) & \text{if } q = p. \end{array} \right\}.$$

As the local orders Γ_p^0 and Γ_p^n are at distance n in T, it is easy to see that the orders $a^{-1}\Gamma_p^0 a$ and $a^{-1}\Gamma_p^n a$ are also at distance n as well. Hence $(a^{-1}\Gamma^0 a, a^{-1}\Gamma^n a)$ defines an element of $P_n(X_\Lambda)$, and we see there is a natural action of $\Lambda^{(p)\times}$ on the set $P_n(X_\Lambda)$.

Recall that the quaternion algebra $\mathbf{H}/\mathbf{Q}$ will satisfy the Eichler condition relative to a Dedekind domain R with quotient field $\mathbf{Q}$ if at least one prime of $\mathbf{Q}$ that does not arise from a prime ideal of R splits in $\mathbf{H}$. We now recall a special case of Eichler's norm theorem.

Theorem 2. ([5], §34, Theorem 34.9) *Let R be a Dedekind domain with quotient field $\mathbf{Q}$, and let $\mathbf{H}$ be a quaternion algebra which satisfies the Eichler condition relative to R. If I is a normal ideal of $\mathbf{H}$, then I is principal if and only if its reduced norm is principal.*

We shall apply this theorem to the order $\Lambda^{(p)}$. Note that

$$\Lambda^{(p)} = \bigcap_{q\neq p}(\Lambda_q \cap \mathbf{H})$$

is a maximal $\mathbf{Z}^{(p)}$-order since it localizes to a maximal order for every prime in $\mathbf{Z}^{(p)}$, so every left $\Lambda^{(p)}$-ideal is normal. Since the norm of any left $\Lambda^{(p)}$-ideal I is a principal ideal of $\mathbf{Z}^{(p)}$, by Eichler's theorem, I must then be principal.

We can now prove the following generalization of Vignéras ([6], p. 147):

Theorem 3. *The number of orbits in $P_n(X_\Lambda)$ under the action of $\Lambda^{(p)\times}$ is equal to the class number of Λ.*

Proof. We shall establish a bijection between the following three sets:

$$U(\Lambda)\backslash J_{\mathbf{H}}/\mathbf{H}^\times \longleftrightarrow \Lambda_p^\times \mathbf{Q}_p^\times \backslash \mathbf{H}_p^\times/\Lambda^{(p)\times} \longleftrightarrow P_n(X_\Lambda)/\Lambda^{(p)\times}.$$

As noted above, $\Lambda^{(p)}$ has class number one, so that $|U(\Lambda^{(p)})\backslash J_{\mathbf{H}}/\mathbf{H}^\times| = 1$. We then can write $J_{\mathbf{H}}$ as

$$J_{\mathbf{H}} = U(\Lambda^{(p)})\mathbf{H}^\times = (\prod_q \Lambda_q^{(p)\times}) \cdot \mathbf{H}^\times = (\mathbf{H}_\infty^\times \mathbf{H}_p^\times \prod_{q\neq p} \Lambda_q^\times) \cdot \mathbf{H}^\times.$$

Accordingly, if $\tilde{h} \in J_{\mathbf{H}}$, we write $\tilde{h}$ as $(h_\infty, h_p, h_q) \cdot h$, where $h_\infty \in \mathbf{H}_\infty^\times$, $h_p \in \mathbf{H}_p^\times$, and $h_q \in \prod_{q\neq p} \Lambda_q^\times$. We first establish a bijection between $U(\Lambda)\backslash J_{\mathbf{H}}/\mathbf{H}^\times$ and $\Lambda_p^\times \mathbf{Q}_p^\times \backslash \mathbf{H}_p^\times/\Lambda^{(p)\times}$ by mapping $\Lambda_p^\times \mathbf{Q}_p^\times h_p \Lambda^{(p)\times}$ to $U(\Lambda)(1, h_p, 1)\mathbf{H}^\times$. With the breakdown of $\tilde{h} \in J_{\mathbf{H}}$ just noted, we at once that the map is surjective.

To show that the map is well-defined, we need to show that $U(\Lambda)(1, r_p k_p h_p r, 1)\mathbf{H}^\times = U(\Lambda)(1, h_p, 1)\mathbf{H}^\times$, where $r_p \in \Lambda_p^\times$, $k_p \in \mathbf{Q}_p^\times$ and $r \in \Lambda^{(p)\times}$. Using the fact that $J_{\mathbf{Q}} = U(\mathbf{Z})\mathbf{Q}^\times$, and that $\mathbf{Q}^\times$ is in the center of $\mathbf{H}^\times$ and $\mathbf{H}_q^\times$ for all q, we have $U(\Lambda)(1, r_p k_p h_p r, 1)\mathbf{H}^\times = U(\Lambda)(1, h_p r, 1)\mathbf{H}^\times$. Next, since $r \in \Lambda^{(p)\times} \subset \mathbf{H}^\times$, it must be that $U(\Lambda)(1, h_p r, 1)\mathbf{H}^\times = U(\Lambda)(r^{-1}, h_p, r^{-1})\mathbf{H}^\times$. But since $\Lambda^{(p)\times} \subset \Lambda_q^\times$ when $q \neq p$, we see that $U(\Lambda)(r^{-1}, h_p, r^{-1})\mathbf{H}^\times = U(\Lambda)(1, h_p, 1)\mathbf{H}^\times$, and the map is well-defined.

Finally, suppose that $U(\Lambda)(1, h_p, 1)\mathbf{H}^\times = U(\Lambda)(1, l_p, 1)\mathbf{H}^\times$, so $(1, l_p, 1) = (r_\infty h, r_p h_p h, r_q h)$, for some $(r_\infty, r_p, r_q) \in U(\Lambda)$ and $h \in \mathbf{H}^\times$. But then $r_q h = 1$ for all $q \neq p$, and so $h \in$

$\bigcap_{q\neq p}(\Lambda_q^\times \cap \mathbf{H}^\times) = \Lambda^{(p)\times}$. Consequently, $\Lambda_p^\times \mathbf{Q}_p^\times r_p h_p h \Lambda^{(p)\times} = \Lambda_p^\times \mathbf{Q}_p^\times h_p \Lambda^{(p)\times}$, making the map $1-1$.

To establish a bijection between $\Lambda_p^\times \mathbf{Q}_p^\times \backslash \mathbf{H}_p^\times / \Lambda^{(p)\times}$ and $P_n(X_\Lambda)/\Lambda^{(p)\times}$, we first prove there is a bijection between the cosets $\Lambda_p^\times \mathbf{Q}_p^\times \backslash \mathbf{H}_p^\times$ and the elements of $P_n(X_\Lambda)$. By Proposition 1, we know that $\Lambda_p = x_p^{-1}\begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p^n\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix} x_p$ for some $x_p \in \mathbf{H}_p^\times$. Put $\Gamma_p^0 = x_p^{-1}\begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ \mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix} x_p$ and $\Gamma_p^n = x_p^{-1}\begin{pmatrix} \mathbf{Z}_p & p^{-n}\mathbf{Z}_p \\ p^n\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix} x_p$, so that $\Lambda_p = \Gamma_p^0 \cap \Gamma_p^n$, and map

$$\Lambda_p^\times \mathbf{Q}_p^\times h_p \longrightarrow (h_p^{-1}\Gamma_p^0 h_p \, , \, h_p^{-1}\Gamma_p^n h_p).$$

We observe that since Γ_p^0 and Γ_p^n are maximal and distance n apart, $(h_p^{-1}\Gamma_p^0 h_p \, , \, h_p^{-1}\Gamma_p^n h_p)$ corresponds to a path of length n in $P_n(X_\Lambda)$.

Since $\Lambda_p^\times \mathbf{Q}_p^\times \subset N(\Lambda_p) = N(\Gamma_p^0 \cap \Gamma_p^n)$, the map is clearly well-defined. To show the map is surjective, suppose that $G = (\Gamma'^0, \Gamma'^n)$ is a path in X_Λ. We associate to G the local order $\Gamma_p'^0 \cap \Gamma_p'^n$. By Proposition 1,

$$\Gamma_p'^0 \cap \Gamma_p'^n = y_p^{-1}\begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p^n\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix} y_p = y_p^{-1}x_p\Gamma_p^0 x_p^{-1}y_p \cap y_p^{-1}x_p\Gamma_p^n x_p^{-1}y_p,$$

for some $y_p \in \mathbf{H}_p^\times$. But by Proposition 1 again, the order $y_p^{-1}\begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p^n\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix} y_p$ is the intersection of uniquely determined maximal orders. It follows that $(\Gamma_p'^0 \, , \, \Gamma_p'^n)$ is either $(y_p^{-1}x_p\Gamma_p^0 y_p^{-1}x_p \, , \, y_p^{-1}x_p\Gamma_p^n y_p^{-1}x_p)$ or $(y_p^{-1}x_p\Gamma_p^n y_p^{-1}x_p \, , \, y_p^{-1}x_p\Gamma_p^0 y_p^{-1}x_p)$. Noting that

$$(y_p^{-1}x_p\Gamma_p^n y_p^{-1}x_p \, , \, y_p^{-1}x_p\Gamma_p^0 y_p^{-1}x_p) = (y_p^{-1}x_p s_p^{-1}\Gamma_p^0 s_p x_p^{-1}y_p \, , \, y_p^{-1}x_p s_p^{-1}\Gamma_p^n s_p x_p^{-1}y_p),$$

we see that the map is surjective.

To show that the map is one-to-one, suppose that $(h_p^{-1}\Gamma_p^0 h_p \, , \, h_p^{-1}\Gamma_p^n h_p) = (l_p^{-1}\Gamma_p^0 l_p \, , \, l_p\Gamma_p^n l_p)$. These conditions imply that $l_p h_p^{-1} \in N(\Gamma_p^0) \cap N(\Gamma_p^n) \subset N(\Gamma_p^0 \cap \Gamma_p^n)$. Thus, $l_p h_p^{-1} \in \Lambda_p^\times \mathbf{Q}_p^\times y^\nu$, for $\nu = 0$ or 1. But if $l_p h_p^{-1} \in \Lambda_p^\times \mathbf{Q}_p^\times y$, then $l_p = r_p k_p y h_p$, for some $r_p \in \Lambda_p^\times$ and $k_p \in \mathbf{Q}_p^\times$, and

$$\begin{aligned} l_p^{-1}\Gamma_p^0 l_p &= h_p^{-1}y^{-1}k_p^{-1}r_p^{-1}\Gamma_p^0 r_p k_p y h_p \\ &= h_p^{-1}\Gamma_p^n h_p \\ &\neq h_p^{-1}\Gamma_p^0 h_p. \end{aligned}$$

Hence ν must be 0. We then have that $l_p \in \Lambda_p^\times \mathbf{Q}_p^\times h_p$ and the map is $1-1$. Hence the elements of $P_n(X_\Lambda)$ are in one-to-one correspondence with the cosets $\Lambda_p^\times \mathbf{Q}_p^\times \backslash \mathbf{H}_p^\times$.

The bijection between $\Lambda_p^\times \mathbf{Q}_p^\times \backslash \mathbf{H}_p^\times / \Lambda^{(p)\times}$ and $P_n(X_\Lambda)/\Lambda^{(p)\times}$ is now immediate. If $G = (\Gamma'^0, \Gamma'^n)$, then as above, G corresponds to a pair of local orders of the form $(h_p^{-1}\Gamma^0 h_p, h_p^{-1}\Gamma_p^n h_p)$ in T. An orbit of G is then $\{(a^{-1}h_p^{-1}\Gamma_p^0 h_p a \, , \, a^{-1}h_p^{-1}\Gamma_p^n h_p a) \mid a \in \Lambda^{(p)\times}\}$ which we map to $\Lambda_p^\times \mathbf{Q}_p^\times h_p \Lambda^{(p)\times}$. This map is clearly surjective. To check it is $1-1$, we note that when $r_p \in \Lambda_p^\times$, $k_p \in \mathbf{Q}_p^\times$, and $r \in \Lambda^{(p)\times}$,

$$(r^{-1}h_p^{-1}k_p^{-1}r_p^{-1}\Gamma_p^0 r_p k_p h_p r \, , \, r^{-1}h_p^{-1}k_p^{-1}r_p^{-1}\Gamma_p^n r_p k_p h_p r) = (r^{-1}h_p^{-1}\Gamma_p^0 h_p r \, , \, r^{-1}h_p^{-1}\Gamma_p^n h_p r),$$

which is in the same orbit as $(h_p^{-1}\Gamma_p^0 h_p\ ,\ h_p^{-1}\Gamma_p^n h_p)$. $\square$

3. Brandt Matrices

If Λ is any order of level p^n and I is a left Λ-ideal, let $\{I_1, I_2, \ldots, I_h\}$ be a complete set of left Λ-ideal class representatives with corresponding right orders $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_h\}$. The Brandt matrix $B(m)$ is defined by setting $B(m)_{ij}$ equal to the number of integral left Λ_i-ideals of norm m which are in the same ideal class as $I_i^{-1}I_j$ (see [6], p. 100). The Brandt matrices give the action of the Hecke operators acting on a space of theta series (see, for example, [1]).

We now define a set of matrices closely related to the Brandt matrices. Suppose that $g = (\Gamma^0, \Gamma^n)$ and $h = (\Gamma'^0, \Gamma'^n)$ are two paths of length n. We say h is at distance k from g if $d(\Gamma^0, \Gamma'^0) = k$ and $d(\Gamma^n, \Gamma^{n'}) \leq k$.

Definition 2. *Let J be an index set for the orbits of the paths of fixed length n. Define the matrix $P^{(k)}$ by setting $(P_{ij}^{(k)})_{i,j\in J}$ equal to the number of paths in orbit j at distance k from a fixed path in orbit i.*

It is not hard to show this number does not depend upon the representative in orbit i.

For Λ an order of level p^n, we choose two maximal orders Γ^0 and Γ^n so that $\Lambda = \Gamma^0 \cap \Gamma^n$. We then associate to Λ the path $(\Gamma^0, \Gamma^n) \in P_n(X_\Lambda)$. If $\{\Lambda\tilde{x_1}, \Lambda\tilde{x_2}, \ldots, \Lambda\tilde{x_h}\}$ is a complete set of left Λ-ideal class representatives with corresponding right orders $\{\tilde{x_1}^{-1}\Lambda\tilde{x_1}, \tilde{x_2}^{-1}\Lambda\tilde{x_2}, \ldots,$ $\tilde{x_h}^{-1}\Lambda_h\tilde{x_h}\}$, then by Theorem 3, the set $\{(\tilde{x_i}^{-1}\Gamma_0\tilde{x_i}, \tilde{x_i}^{-1}\Gamma^n\tilde{x_i})\} \mid i = 1, \ldots, h\}$ is a complete set of representatives for the orbits of the paths. One easily checks the following

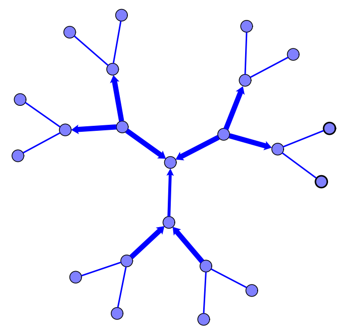
Lemma 2. *Suppose that $g = (\Gamma^0, \Gamma^n)$ is any path of length n, and let $\Lambda = \Gamma^0 \cap \Gamma^n$. Then there is a $1-1$ correspondence between paths on the tree at distance k from g and integral left Λ-ideals $\Lambda\tilde{y}$ of norm p^k such that $y_p \in \Lambda_p - p\Lambda_p$.*

Hence we see that there is a $1-1$ correspondence between the set of integral left Λ_i-ideals $I = \Lambda_i\tilde{y}$ of norm p^k such that $I \sim I_i^{-1}I_j$ with the additional property that $y_p \in \Lambda_{ip} - p\Lambda_{ip}$ and the set of paths in $\mathrm{P}_n(X_\Lambda)$ in orbit j $(j \in J)$ which are at distance k from (Γ_i^0, Γ_i^n). With the above observations one establishes the following relationships between $P^{(k)}$ and $B(p^k)$:

$$
\begin{aligned}
P^{(0)} &= B(1) = I \\
P^{(1)} &= B(p) \\
P^{(k)} &= B(p^k) - B(p^{k-2}) \qquad k > 1.
\end{aligned}
$$

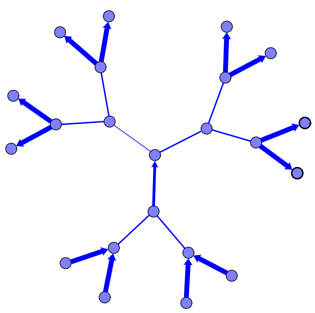
We shall show that by studying the tree we can find recursion formulas for the paths on the tree, thereby obtaining recursion formulas for the Brandt matrices $B(p^k)$, for p dividing the level of the order.

3.1. **The Case $n = 1$.** We now specialize to the case $n = 1$, so Λ has level p. See also [2] for a different treatment of this case. Let $g = (\Gamma^0, \Gamma^1)$ be a fixed path in $P_1(X_\Lambda)$. We consider all paths at distance k from g.

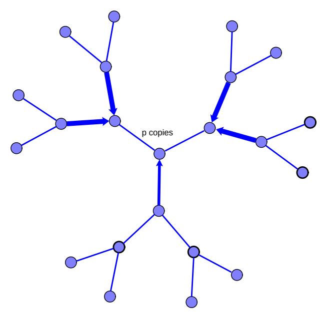


Paths at distance 2 from the central path.

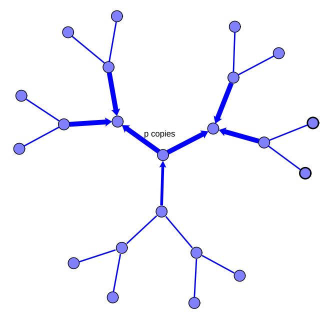
Considering the product $P^{(2)}P^{(1)}$, we obtain the following:



Paths at distance 3 from the central path.



Paths at distance 3 from the central path.



Paths in "reverse" direction from a path at distance 2 from the central path.

Induction on $k \geq 2$ gives us:

$$P^{(k+1)} = P^{(k)}P^{(1)} - pP^{(k-1)} - p\overline{P^{(k)}}, \qquad \text{for } k \geq 1,$$

where $\overline{P^{(k)}}_{ij}$ is equal to the number of paths in orbit j whose reversed path is at distance k from a fixed path in orbit i.

If $g = (\tilde{x}^{-1}\Gamma^0\tilde{x}, \tilde{x}^{-1}\Gamma^n\tilde{x})$ and $h = (\tilde{y}^{-1}\Gamma^0\tilde{y}, \tilde{y}^{-1}\Gamma^n\tilde{y})$ are paths in orbit j, then

$$g = (a^{-1}\tilde{y}^{-1}\Gamma^0\tilde{y}a, a^{-1}\tilde{y}^{-1}\Gamma^n\tilde{y}a)$$

for some $a \in \Lambda^{(p)\times}$. Then we observe that the "reverse" of g:

$$\overline{g} = (a^{-1}\tilde{y}^{-1}\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)^{-1}\Gamma^0\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)\tilde{y}a, a^{-1}\tilde{y}^{-1}\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)^{-1}\Gamma^n\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)\tilde{y}a)$$

is in the same orbit as $\overline{h}$. Hence we see that $\overline{g}$, $\overline{h}$ must be in the same orbit r for some r, which makes $\overline{P^{(k)}}_{ij} = P^{(k)}_{ir}$. Thus

$$\overline{P^{(k)}} = P^{(k)}E,$$

where E is a permutation matrix of order 1 or 2.

Hence we have

Theorem 4. *Let Λ be an order of level p in $\mathbf{H}$. The matrices $P^{(k)}$ satisfy the following recursion formulas:*

$$P^{(k+1)} = P^{(k)}P^{(1)} - pP^{(k-1)} - pP^{(k)}E \qquad \textit{for } k \geq 1,$$

where E is a permutation matrix of order 1 or 2. Therefore, the Brandt matrices $B(p^k)$ satisfy

$$B(p^{k+1}) = B(p^k)B(p) - pB(p^{k-1}) - pB(p^k)E \qquad \textit{for } k \geq 1.$$

The permutation matrix E can often be explicitly computed. Given a path $g = (\Gamma^0, \Gamma^1)$ with $\Lambda = \Gamma^0 \cap \Gamma^1$, then $\Lambda_p = x_p^{-1} \begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix} x_p$, for some $x_p \in \mathbf{H}_p^{\times}$. We define the idele $\tilde{\pi} \in J_{\mathbf{H}}$ by:

$$\tilde{\pi} = (\pi_q) = \begin{cases} x_p^{-1} \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} x_p, & \text{if } q = p \\ \\ 1 & \text{otherwise.} \end{cases}$$

We observe that the ideal $\widetilde{M} = \Lambda\tilde{\pi}$ is two-sided since π_p is in the normalizer in Λ_p.

We have associated the path $g = (\Gamma^0, \Gamma^1)$ to the path $\overline{g} = (\Gamma^1, \Gamma^0)$ in $P^{(k)}$ by conjugating by $\tilde{\pi}$. By Theorem 3, g is in the same orbit as $\overline{g}$ if and only if $\Lambda\tilde{\pi}$ is principal.

Example 1. *Let $\mathbf{H}$ be the quaternion algebra with vector space basis $\{1, i, j, k\}$ subject to the relations $i^2 = -2$, $j^2 = -5$, $ij = -ji = k$. $\mathbf{H}$ is ramified precisely at the primes $\{5, \infty\}$.*

The class number of an order of level 3 in this algebra is 2 and the type number is also 2 (see [6], p. 153). Let Λ_1 and Λ_2 be two orders of level 3 representing the distinct types. According to Lemma 1, we must then have that $2 = \sum_{i=1}^{2} h_i$, which means that the class number of two-sided Λ_i-ideals is 1. Since $\Lambda_i\tilde{\pi}$ is a two-sided Λ_i-ideal, it must mean that $\Lambda_i\tilde{\pi}_i$ is principal. By Theorem 3, the corresponding path (Γ_i^0, Γ_i^n) is in the same orbit as $(\tilde{\pi}_i{}^{-1}\Gamma_i^0\tilde{\pi}_i, \tilde{\pi}_i{}^{-1}\Gamma_i^n\tilde{\pi})$. Hence in this case $E = I$.

We note here that the cases for larger values of n are clearly more complicated, but it is possible that variants of these geometric ideas will lead to recursion formulas for those cases as well.

4. Quadratic Forms

As an application, we show how the above recursion formulas can be used to compute representation numbers of certain quadratic forms with integer coefficients. Let Λ be an order of level p^n in $\mathbf{H}$ with $\mathbf{Z}$-basis $\{e_1, e_2, e_3, e_4\}$. If $x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 \in \Lambda$, then $\mathrm{N}_{\mathbf{H}/\mathbf{Q}}(x) = \displaystyle\sum_{i=1}^{4} x_i^2 \,\mathrm{N}_{\mathbf{H}/\mathbf{Q}}(e_i) + \sum_{i<j} x_ix_j \,\mathrm{Tr}_{\mathbf{H}/\mathbf{Q}}(e_i\bar{e}_j)$ is a quadratic form with coefficients in $\mathbf{Z}$. Denote by $R(p^k)$ the number of primitive representations of p^k by the norm form of $\mathbf{H}$ restricted to Λ; that is, solutions $(x_1, x_2, x_3, x_4) \in \mathbf{Z}^4$ of $\mathrm{N}_{\mathbf{H}/\mathbf{Q}}(x) = p^k$ with the property

that p does not simultaneously divide all of the x_i's. Following Pays in [3], we will show that the matrix $P^{(k)}$ is useful in finding primitive representations of p^k by the norm form of the algebra restricted to orders of level p^n of $\mathbf{H}$. The following theorem is easily established.

Theorem 5. *Let $\Lambda = \Gamma^0 \cap \Gamma^n$ be an order of level p^n in $\mathbf{H}$. There is a bijection between the following two sets:*

$$\{x \in \Lambda - p\Lambda \mid \mathrm{N}_{\mathbf{H/Q}}(x) = p^k\}/\sim$$

and

$$\{\textit{paths of length } n \textit{ in the same orbit as } (\Gamma^0, \Gamma^n) \textit{ at distance } k \textit{ from } (\Gamma^0, \Gamma^n)\}.$$

Here, the equivalence relation $\sim$ on Λ is defined by $x \sim y$ if and only if $x^{-1}y \in \Lambda^\times$. Under this bijection, the element $x \in \Lambda$ corresponds to the path $(x^{-1}\Gamma^0 x\ ,\ x^{-1}\Gamma^n x)$.

Observe that $P^{(k)}_{1,1}$ is equal to the number of paths in the same orbit as (Γ^0, Γ^n) at distance k from (Γ^0, Γ^n), hence $R(p^k) = |\Lambda^\times| P^{(k)}_{1,1}$. By finding a closed form of the recursion formula for the matrices $P^{(k)}$, we can compute $R(p^k)$.

Example 2.

In the example given above, $\mathbf{H}$ is the quaternion algebra with vector space basis $\{1, i, j, k\}$ subject to the relations $i^2 = -2$, $j^2 = -5$, $ij = -ji = k$. An order of level 3 in $\mathbf{H}$ is given by

$$\Lambda = \mathbf{Z}\left(\frac{1 + j + 3k}{2}\right) + \mathbf{Z}\left(\frac{i + 2j + k}{4}\right) + \mathbf{Z}j + 3\mathbf{Z}k,$$

and the corresponding norm form is

$$24x_1^2 + 2x_2^2 + 5x_3^2 + 90x_4^2 + 10x_1x_2 + 10x_1x_3 + 90x_1x_4 + 5x_2x_3 + 15x_2x_4.$$

In this case we have seen that the Brandt matrices $B(3^k)$ satisfy $B(3^{k+1}) = B(3^k)B(3) - 3B(3^{k-1}) - 3B(3^k)$. Equivalently,

$$\sum_{i=0}^{\infty} B(3^k)X^k = \frac{I + 3X}{I - (B(3) - 3I)X + 3X^2}.$$

Using this and the relations derived above one sees that a generating function for $P(X) = P^{(0)} + P^{(1)}X + P^{(2)}X^2 + \dots$ is:

$$P(X) = \frac{(I + 3X)(I - X^2 I)}{I - (P^{(1)} - 3I)X + 3X^2}.$$

We find $P^{(1)} = B(3) = \begin{pmatrix} 5 & 2 \\ 6 & 1 \end{pmatrix}$, hence

$$P(x) = \begin{pmatrix} \dfrac{-9x^5 + 7x^3 - x^2 + 2x + 1}{9x^4 - 9x^3 + 2x^2 - 3x + 1} & \dfrac{-6x^4 - 2x^3 + 6x^2 + 2x}{9x^4 - 9x^3 + 2x^2 - 3x + 1} \\[2ex] \dfrac{-9x^4 - 3x^3 + 9x^2 + 3x}{9x^4 - 9x^3 + 2x^2 - 3x + 1} & \dfrac{-9x^5 + 3x^4 + 8x^3 - 4x^2 + x + 1}{9x^4 - 9x^3 + 2x^2 - 3x + 1} \end{pmatrix}.$$

Expanding the $(1,1)$ entry, we have:

$$P^{(k)}_{1,1} = 1 + 5x + 12x^2 + 42x^3 + 138x^4 + \dots.$$

Also, $|\Lambda^\times| = 6$, so

$$\sum_{k=1}^{\infty} R(3^k)x^k = 6 + 30x + 72x^2 + 252x^3 + 828x^4 + \ldots .$$

Thus, there are 30 ways of representing 3 by this quadratic form, 72 ways of representing 9, etc.

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