Constructions and Recursions in Regular Trees

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1 Regular Tree Coordinates

In order to create and manipulate the tree and its paths we need to establish a system for representing these objects as software elements. We do this by establishing a coordinate system for an abstract regular tree and defining the required formulas with respect to this coordinate system. Fix a positive integer n which will result in the production of an $(n+1)$-regular tree.[1]

Definition 1.1 *Given* (a,b) *and* $(c,d) \in \mathbb{N}^2$, $(a,b) \sim (c,d)$ *when* $a = c$ *and if* $a \neq 0$ *then also* $b \equiv d \pmod{\Omega(a)}$, *where* $\Omega(a) = (n+1)n^{a-1}$.

Notice that this functions equivalently on the set $\mathbb{N} \times \mathbb{Z}$.

Proposition 1.2 *The relation* $\sim$ *is an equivalence relation on* $\mathbb{N}^2$.

Proof: The relation is reduced to simple equality whenever any first component is zero. Therefore suppose the first component is not zero.

First $a = a$ and $b \equiv b \pmod{\Omega(a)}$ therefore $(a,b) \sim (a,b)$. Next given $(a,b) \sim (c,d)$ it follows $a = c$ and $b \equiv d \pmod{\Omega(a)}$ Therefore $c = a$ and $d \equiv b$

[1] For compatibility with software applications we assume $0 \in \mathbb{N}$.

(mod $\Omega(c)$) and so $(c,d) \sim (a,b)$. Finally suppose $(a,b) \sim (c,d) \sim (e,f)$. Then $a = c = e$ as well as $b \equiv d \equiv f \pmod{\Omega(a)}$, therefore $(a,b) \sim (e,f)$. $\square$

We denote each equivalence class as $[a,b]$ and refer to it as a vertex.

Definition 1.3 *The* shift down operator *acts on the vertices as follows*

$$k \downarrow [a,b] = \begin{cases} \left[a-k, \lfloor \frac{b}{n^k} \rfloor\right] & 0 \leq k < a \\ [0,0] & a \leq k. \end{cases}$$

Proposition 1.4 *The* $\langle \mathbb{N}, + \rangle$ *acts on* $\mathbb{N}^2/\!\sim$ *by* $\downarrow$.

Proof: First we must ensure that $\downarrow$ is well-defined. Let $[a,b] \in \mathbb{N}^2/\!\sim$ and also let $c \equiv b \pmod{\Omega(a)}$. If $a \leq k$ then clearly $k \downarrow [a,b] = [0,0] = k \downarrow [c,d]$. Now suppose $0 \leq k < a$. Utilizing the division algorithm we let $b = m\Omega(a) + r$ where $0 \leq r < \Omega(a)$. Therefore

$$\begin{aligned} \left\lfloor \frac{b}{n^k} \right\rfloor & \equiv \left\lfloor \frac{m\Omega(a)+r}{n^k} \right\rfloor & \pmod{\Omega(a-k)} \\ & \equiv m(n+1)n^{a-k-1} + \left\lfloor \frac{r}{n^k} \right\rfloor \\ & \equiv m\Omega(a-k) + \left\lfloor \frac{r}{n^k} \right\rfloor \\ & \equiv \left\lfloor \frac{r}{n^k} \right\rfloor. \end{aligned}$$

Since $r \equiv b \equiv c \pmod{\Omega(a)}$ it follows $k \downarrow [a,b] = k \downarrow [a,c]$ and so $\downarrow$ is well-defined for all $k \in \mathbb{N}$ and $[a,b] \in \mathbb{N}^2/\!\sim$.

We see the first action axiom satisfied with a straight forward application of the definition: $0 \downarrow [a,b] = [a-0, \lfloor \frac{b}{1} \rfloor] = [a,b]$. Now consider $j \downarrow (k \downarrow [a,b])$. Since $a - k \leq j$ if and only if $a \leq j + k$ then clearly $j \downarrow (k \downarrow [a,b]) = [0,0] = (j+k) \downarrow [a,b]$. Therefore suppose $0 \leq j + k < a$. Once again by the division algorithm we let $b = mn^{j+k} + r$ where $0 \leq r < n^{j+k}$. Therefore

$$\begin{aligned} \left\lfloor \frac{\lfloor b/n^k \rfloor}{n^j} \right\rfloor = \left\lfloor \frac{\lfloor (mn^{j+k}+r)/n^k \rfloor}{n^j} \right\rfloor = \left\lfloor \frac{mn^j + \lfloor r/n^k \rfloor}{n^j} \right\rfloor = \left\lfloor m + \frac{\lfloor r/n^k \rfloor}{n^j} \right\rfloor \\ = m + \left\lfloor \frac{\lfloor r/n^k \rfloor}{n^j} \right\rfloor. \end{aligned} \tag{1}$$

Now we also note that $0 \leq r/n^k < n^j$ so $0 \leq \lfloor r/n^k \rfloor < n^j$ and continuing $0 \leq \lfloor r/n^k \rfloor / n^j < 1$. Therefore $\lfloor \lfloor r/n^k \rfloor / n^j \rfloor = 0$. So we concluded

$$\lfloor \lfloor b/n^k \rfloor / n^j \rfloor = m = \lfloor b/n^{j+k} \rfloor.$$

Finally we confirm the second action axiom.

$$\begin{aligned} j \downarrow (k \downarrow [a,b]) & = j \downarrow ([a-k, \lfloor b/n^k \rfloor]) \\ & = \left[a-k-j, \left\lfloor \lfloor b/n^k \rfloor / n^j \right\rfloor\right] \\ & = [a-(j+k), \lfloor b/n^{j+k} \rfloor] & \text{by (1)} \\ & = (j+k) \downarrow [a,b]. \end{aligned}$$

Therefore $\langle \mathbb{N}, + \rangle$ together with $\downarrow$ acts on $\mathbb{N}^2/\!\sim$. $\square$

The shift down operator is a monoid action so orbits are never disjoint and the action does not partition the set. However the induced structure is still useful and relates directly to certain counting theorems. With this device in hand we can now define a partial ordering on the partitioned set which will result in a regular tree structure.

Definition 1.5 *Given $\alpha, \beta \in \mathbb{N}^2/\sim$, $\alpha \leq \beta$ if and only if there exists a $k \in \mathbb{N}$ such that $\alpha = (k \downarrow)\beta$. Equivalently we may say α is in the orbit of β.*

Note therefore $[0,0]$ is the least element of the partition and that, unless $\alpha = [0,0]$, the k is unique. This is summed up in the following lemma.

Lemma 1.6 *The set of all lower bounds of a vertex forms a finite chain, specifically*

$$[0,0] \leq ((a-1)\downarrow)[a,b] \leq \cdots \leq \downarrow [a,b] \leq [a,b].$$

Proof: Let $[a,b] \in \mathbb{N}^2/\sim$. Consider the orbit of $[a,b]$, $\mathbb{N} \downarrow [a,b]$. Clearly the orbit contains only lower bounds of $[a,b]$ and $\downarrow$ is well-defined it also follows all lower bounds are elements of the orbit. By induction on a with an arbitrary b we see the elements of the orbit do indeed form a chain of length a. $\square$

Theorem 1.7 *The graph of the partial ordering of $\mathbb{N}^2/\sim$ is an $(n+1)$-regular tree. Therefore we take a connection between two vertices α and β to exist when either $\alpha = \downarrow \beta$ or $\beta = \downarrow \alpha$.*

Proof: Given that $\downarrow [0,0] = [0,0]$, all vertices connected to $[0,0]$ must be of the form $[1,b]$. Since $\downarrow [1,b] = [0,0]$ all $n+1$ vertices of that form are connected to $[0,0]$.

Given any vertex $[a,b] \neq [0,0]$ there exists a distinct vertex $\downarrow [a,b]$, which by definition is connected to $[a,b]$. For all vertices $[a+1,c]$ such that $\downarrow [a+1,c] = [a,b]$ we once again apply the division algorithm and note

$$\left\lfloor \frac{c}{n} \right\rfloor = \left\lfloor \frac{mn+r}{n} \right\rfloor = m$$

and by assumption we know $b \equiv m \pmod{(n+1)n^{a-1}}$. Therefore all vertices connected to $[a,b]$ from above are of the form $[a+1, bn+r], 0 \leq r < n$. These vertices are clearly n distinct vertices since no two are equivalent mod $(n+1)n^a$. Therefore there are exactly $n+1$ distinct vertices connected to $[a,b]$. Therefore all vertices are connected to exactly $n+1$ distinct vertices so the graph of the partial ordering is an $(n+1)$-regular tree. $\square$

The importance of this theorem lies in its simple generation and representation.[2] Effectively this specific $(n+1)$-regular tree can be used as a coordinate

[2] In computerized integer arithmetic, the division operation automatically floors the result and the modular arithmetic is equally efficient. Additionally the vertices can easily be iterated over and traversed with simple index loops.

system for any $(n+1)$-regular tree. The graph of this tree is easily drawn by considering a vertex $[a, b]$ as a form of polar coordinates. The first coordinate representing the radius, the second the fraction of rotation.[3]

2 Distance

Lemma 1.6 ensures us that the elements of the tree have lower bounds. An important extension to this result is the existence of greatest lower bounds. We denote the greatest lower bound between two vertices α and β as $\alpha \Downarrow \beta$.

Proposition 2.1 *Every non-empty set of vertices has a greatest lower bound.*

Proof: Let $S \subseteq \mathbb{N}^2/\!\sim$ such that $S \neq \emptyset$. We know $[0,0] \in \bigcap_{\alpha \in S} \mathbb{N}\alpha$ so the intersection is non-empty. Additionally each $\mathbb{N}\alpha$ is a finite chain by Lemma 1.6 and so their intersection is a finite subchain of all the chains. Therefore there exists a top element δ and δ is the greatest lower bound to the arbitrary set S. □

Notice that alternatively we may think of the greatest lower bounds in terms of orbits. The intersection of orbits is again an orbit and furthermore $\mathbb{N}\alpha \cap \mathbb{N}\beta = \mathbb{N}(\alpha \Downarrow \beta)$. From this we can compute the greatest lower bound between two vertices by traversing the orbits with the following algorithm.

$\Downarrow : (\mathbb{N}^2/\!\sim : \ [a, b], \ [c, d]) : \mathbb{N}^2/\!\sim$
begin
  Let $\gamma, \delta \in \mathbb{N}^2/\!\sim$**;**
  // Select the vertex γ to be the closest to [0,0] for efficiency.
  if $(a \leq c)$ **then**
  begin
    $\gamma \leftarrow [a, b]$**;**
    $\delta \leftarrow [c, d]$**;**
  end
  else
  begin
    $\gamma \leftarrow [c, d]$**;**
    $\delta \leftarrow [a, b]$**;**
  end
  // Traverse down the chain for γ until it intersects the chain for δ.
  while $(\gamma \quad \delta)$ **do**
    $\gamma \leftarrow \downarrow \gamma$**;**

  // The current γ is the greatest lower bound.
  return γ**;**
end

We turn our attention now to the notion of distance within the tree.

Definition 2.2 *Let $\alpha = [a, b]$ and $\beta = [c, d]$ be two comparable vertices. The distance $\partial(\alpha, \beta)$ is defined as $|a - c|$.*

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[3] Specifically the map $[a, b] \mapsto (a \cos\theta, a \sin\theta)$ where $\theta = \frac{2\pi}{(n+1)n^{a-1}}(b - \frac{1}{n})$ serves to map the coordinates to the Cartesian plane.

Definition 2.3 *The distance between vertices α and β is the sum of their respective distances to the greatest lower bound. That is,*

$$d(\alpha,\beta) = \partial(\alpha,\alpha \Downarrow \beta) + \partial(\alpha \Downarrow \beta,\beta).$$

With orbits we can equivalently define $d(\mathbb{N}\alpha,\mathbb{N}\beta) = \mathbb{N}\alpha \oplus \mathbb{N}\beta$ and we observe that $d(\alpha,\beta) = |d(\mathbb{N}\alpha,\mathbb{N}\beta)|$. Since the greatest lower bound is always comparable with both α and β, it follows this generalized definition of distance is well-defined.

3 Shift Toward α Operators

In the previous sections the introduction of the shift down operator immediately determined an action on the vertices of $\mathbb{N}^2/\sim$ that serves to shift elements towards the element $[0,0]$. The limitation lies in the single direction of this traversal. Inorder to provide a mechinism to traverse between any two vertices, an extension of the shift down operator is created.

Definition 3.1 *Let α and $\beta \in \mathbb{N}^2/\sim$. Define the product*

$$\alpha^k\beta = \begin{cases} k \downarrow \beta & 0 \le k \le d(\alpha \Downarrow \beta,\beta), \\ (d(\alpha,\beta) - k) \downarrow \alpha & d(\alpha \Downarrow \beta,\beta) < k < d(\alpha,\beta), \\ \alpha & d(\alpha,\beta) \le k. \end{cases}$$

to be the k-shift towards α operator.

We notice that $[0,0]^k\alpha = k \downarrow \alpha$.

Proposition 3.2 *$\langle \mathbb{N}, + \rangle$ acts on $\mathbb{N}^2/\sim$ by $\downarrow_\alpha$ for all $\alpha \in \mathbb{N}^2/\sim$.*

Proof: Since $\alpha \Downarrow \beta \le \beta$ it follows $\downarrow_\alpha$ is defined on the entire domain. Furthermore whenever $\alpha \ne \beta$ and $\alpha \Downarrow \beta = \beta$ it follows α $\square$

From this generalization we may now consider any vertex in the grap to be the center of its corresponding tree. We can equivalantly generalize the concepts of greatest lower bounds and distance to refer to given center. In general we assume the center to be $[0,0]$ unless specified otherwise.

4 Translation

Definition 4.1 *Define the funtion $\mathfrak{u} : \mathbb{N}^2/\sim \to \mathbb{N}^2/\sim$ as follows.*

```
u(ℕ²/∼ : [a,b]) : ℕ²/∼
begin
    // Traverse up the chain of lower bounds
    // to determine the unit translation at each step.
    if (a = 0) then
        return [0,0];
    else
    begin
        // Note that this will always be a unit vertex.
        [1,θ] ← ((a − 1) ↓)[a,b];
        // Iterate up the lower bounds until the [a,b] is reached.
        ∀ i ∈ {2,...,a}
        begin
            θ ← θ + ⌊ b / n^(a−i) ⌋;
            [y,z] ← ((α_r − i) ↓)[a,b];
            r ≡ y mod n;          // 0 ≤ r < n.
            θ ← θ + r;
        end
        // Return the last unit of translation.
        return [1,θ];
    end
end
```

As is fairly clear, only the second coordinate is required and as such this algorithm should be optimized by inlining the shift down operator and omiting the unused computations. However this form better exemplifies the design of the alogrithm. Additionally the full decomposition of a given element is clearly obtained by storing the result $[1, \theta]$ at the bottom of each loop.[4] The reader will note the $\mathfrak{u}$ is well-defined since each it is based completely on the chain of lower bounds of an element which is unique.

We are now prepared to define a general addition for all elements of the tree. The $\mathfrak{u}$ function will provide us with the decomposition required.

Definition 4.2 *Let* $[a, b] \in \mathbb{N}^2/\!\sim$ *where* $[a, b] \neq [0, 0]$. *Also let* $\mathfrak{u}([a, b]) = [1, j]$. *The sum of* $[a, b]$ *and an arbitray unit* $[1, k]$ *is defined as*

$$[a, b] + [1, k] = \begin{cases} [a + 1, bn + k - j] & \\ \downarrow [a, b] & \text{if } j + 1 \equiv k \pmod{n + 1} \end{cases}$$

Additionally define $\alpha + 0 = 0 + \alpha = \alpha$.

Definition 4.3 *The decomposition of an element* $\alpha \in \mathbb{N}^2/\!\sim$ *is the image of* $\mathfrak{u}$ *over the ordered sequence of chain of all lower bounds of* α. *We denote the decomposition*

$$\mathfrak{U}(\alpha) = \{\mathfrak{u}(\gamma) |\ \gamma = 0, \ldots, \alpha\}.$$

[4] Actual implementations should provide such an additional algorithm for efficiency. However for simplicity this article simply calls the $\mathfrak{u}$ function iteritively with the same net effect.

Proposition 4.4 *The decomposition of an element is unique.*

Proof: The chain of lower bounds of an element is unique and $\mathfrak{u}$ is a well-defined function thus $\mathfrak{U}(\alpha)$ is unique. $\square$

Definition 4.5 *Given* $[a,b],[c,d] \in \mathbb{N}^2/\sim$ *the define their sum as*

$$[a,b]+[c,d] = \sum_{\gamma \in \mathfrak{U}([a,b]) \cup \mathfrak{U}([c,d])} \gamma.$$

An important strong caution should be made about the properties of this operation. First, inverses exist for both sides, however they are both one sided and unique only to that side. There is a unique two sided identity $[0,0]$. In general sums are not commutative nor are they *associative*. Therefore special care must be taken when using the addition to ensure no rule is assumed which may not apply to the context.

5 Path Generation

Proposition 5.1

$$P_0^m(\tau) = U^m(\tau) \cup \left(\bigcup_{1 \leq k \leq m} \left(U^m((k \downarrow)\tau) - U^{m-k}(\tau) \right) \right)$$

where $U^j(\tau)$ *is the set of all elements a distance* m *from* τ *that are also greater than or equal to* τ*.*

Therefore in order to generate $P_0^m(\tau)$ we need only define an algorithm for $U^j(\tau)$ which we do as follows.

$U^j(\mathbb{N}^2/\sim : [a,b]) : \mathcal{P}\mathbb{N}^2/\sim$
begin
  if $(j=0)$ **then**
    return {[a,b]};
  if $(a=0)$ **then**
    return $\{[j,i] \mid 0 \leq i < (n+1)n^{j-1}\}$;
  else
    return $\{[a+j, bn^j+i] \mid 0 \leq i < n^j\}$;
end

Finally the generation of the set for arbitrary length paths follows from the following formula.

Proposition 5.2 $P_l^m(\tau) = \{(\alpha,\beta) \mid \alpha \in P_0^m(\tau), \beta \in P_0^l(\alpha)$ and $d(\tau,\beta) \leq m\}$.

6 Polygonization Method

We take up the case for paths of length $k > 1$. We begin by defining a new graph.

Definition 6.1 *Let T be a (directed)tree. Define the graph $T^{(k)}$ as the graph formed on the vertices of T where an (directed)edge of $T^{(k)}$ exists between to vertices if there exists a (directed)path connecting the vertices of length k in T.*

Note $T^{(0)}$ is the set of vertices of T and $T^{(1)} = T$.

Proposition 6.2 *Given any tree T, $T^{(n)}$ is well-defined.*

Proof: Since T is a tree, given any two vertices in the tree there exists a unique path connecting the vertices. Therefore there exists a unique distance between the vertices. Hence the edges of $T^{(n)}$ are uniquely defined and thus well-defined. □

Definition 6.3 *Given a (directed)tree T, the induced graph $T^{(n)}$ is said to be closed if every edge belongs to a (directed)cycle.*

Lemma 6.4 *If T is an n-regular (directed)tree, with $m > 1$, then $T^{(k)}$ is $\Omega(k)$-regular.*

Proof: Let $k = 1$ and let u be a vertex in T. T is n-regular therefore there are n vertices adjacent to u which are therefore a distance k from u. Therefore $n(n-1)^{k-1} = n(n-1)^0 = n = d(k)$.

Now suppose $d(k) = n(n-1)^{k-1}$ for some k. Attached to each vertex a distance k from u are m vertices. Since paths in a tree are unique the path connecting a vertex v with u can only contain one vertex adjacent to v. There are therefore $m-1$ vertices adjacent to v not contained in any path from u to v for all vertices v a distance k from u. These vertices are therefore a distance $k+1$ from u. Moreover every vertex a distance $k+1$ from u must be adjacent to one that is k units form u therefore these are in fact all vertices at a distance $k+1$ from u. Since there are $n-1$ such vertices adjacent to each of the $d(k)$ vertices there is a total of

$$d(k)(n-1) = n(n-1)^{k-1}(n-1) = n(n-1)^k = d(k+1).$$

So we conclude by induction that $d(k) = n(n-1)^{k-1}$ for all k. □

Theorem 6.5 *Let T be an n-regular tree, for $n > 2$. $T^{(k)}$ is closed for all $k > 1$.*

Proof: By Lemma 6.4 we know $T^{(k)}$ is $\Omega(k)$-regular. Since $k > 1$ there are at least two edges in this path. Since $n > 2$ it also follows $\Omega(k) > 2$ and therefore there are more than two paths of length k connected to each vertex.

Let $\pi = (u_0, e_1, u_1, \ldots, u_{k-1}, e_k, u_k)$ be a path in T. Also let $\{f_i\}$ be a sequence of edges adjacent to u_1 not including e_2 and also Now take a path π and select an edge f in T so that $f \neq e_k$ but is adjacent to u_{k-1}. Define the following function:

$$w_1(\pi, f_1) = (u_k, e_k, u_{k-1}, \ldots, u_1, f_1, u)$$
$$w_{n+1}(\pi, f_{n+1}) = w_1(w_n(\pi, f_n), f_{n+1})$$

If we now define the recursive function $w_2(\pi, f-2)$take $\{w(\pi, e)\}$

Now we create an algorithm which while traverse the edges of a cycle in $T^{(k)}$. We select a vertex u and begin by walking along a path to a vertex a distance n from u. Our resulting path contains the edges $(e_1, \ldots, e_k)$. This path corresponds uniquely to an edge E_1 in $T^{(k)}$. Now we walk back along the path until we reach the last edge in the path, at which point we walk out on an edge $\bar{e}_1$ that we have not yet traversed. Thus the resulting new path is $(e_k, \ldots, e_2, \bar{e}_1)$. This is now a new edge E_2 in $T^{(k)}$. We now repeat this process until we end on the edge e_1 collecting the induced edges of $T^{(k)}$ along the way. As a result we have a closed walk in T corresponding to a closed path in $T^{(k)}$ since no two induced edges are repeated. However a closed path is simply a cycle in $T^{(k)}$. Since our choice of E_1 was arbitrary it follows given any edge in $T^{(k)}$ we can apply this algorithm to generate the a cycle two which E_1 exists. So we conclude $T^{(k)}$ is closed. $\square$

Corollary 6.6 *The girth of $T^{(k)}$ is 4 when k is odd and 3 when k is even.*

7 Stablity

Definition 7.1 *Given the graph $P_k^m(\tau)$ we define the graph $G_k^m(\tau)$ as the graph induced on the vertices $V = \{\alpha, \beta \mid (\alpha, \beta) \in P_k^m(\tau)\}$ with oriented edges between vertices connected by a oriented path in $P_k^m(\tau)$.*

Definition 7.2 *Given a vertex α we define $\alpha \star \beta$ to be the coordinate of α with respect to the graph centered at β.*

Definition 7.3 *Given a path $\rho = (\alpha, \beta)$ define the base of ρ to be $\veebar\rho = \alpha \Downarrow \beta$.*

It is now possible to explain stablity in terms of the behavior of the successive iterations of the graphs.

Theorem 7.4 *$G_k^m(\tau)$ decomposes into n copies of $G_k^{m-1}(\tau)$ for all $m > k+1$.*

Proof: Let $m > k+1$ and center the graph at τ_0.

Now we define the following map:

$$\Delta : P_k^m(\tau) \to P_k^{m-1}(\tau) : (\rho_0, \rho_k) \mapsto ((\rho_0 \star \veebar\rho)\star \downarrow (\veebar\rho)), \rho_k \star \veebar\rho + \downarrow (\veebar\rho))).$$

First we must show Δ to be well-defined. Given that $\square$

Lemma 7.5 *The set $P_0^m(\tau)$ is the set of all origins of paths in $P_k^m(\tau)$.*

Theorem 7.6 *For all positive odd k, $G_k^m(\tau)$ is bipartite. Furthermore the orientation of the graph follows from one partite to the other exclusively. We term the partites* source *and* sink *appropriately.*

Proof: By Lemma 7.5 it follows $X = P_0^m(\tau_0)$ is the set of all origins of $P_k^m(\tau)$. Therefore given any two vertices $\alpha, \beta \in X$ consider the path $p = (\alpha, \beta)$. The length of p is defined as $d(\alpha, \beta) = d(\alpha, \alpha \Downarrow \beta) + d(\alpha \Downarrow \beta, \beta)$. We center the graph at τ_0 and therefore require that $d(\alpha, [0,0]) = d(\beta, [0,0]) = m$. Therefore $len(p) = |m - j| + |j - m| = m - j + m - j = 2 * (m - j)$, where j is the distance of $\alpha \Downarrow \beta$ to $[0,0]$. Therefore the length of p is even and therefore not included in $P_k^m(\tau)$ since k is odd. Since $P_0^m(\tau_0) \neq \emptyset$ and by the above argument all elements of X are independent, it follows X is independent in $G_k^m(\tau)$.

Now we consider $Y = V(G_k^m) \backslash X$. Since $P_k^m(\tau) \neq \emptyset$ and no paths exist between vertices in X it follows $Y \neq \emptyset$. Consider then a path p between two vertices in Y. Since all origins of paths of $P_k^m(\tau)$ are in X and $Y \cap X = \emptyset$ it follows $p \notin P_k^m(\tau)$ and therefore Y is independent in $G_k^m(\tau)$.

Finally all paths in $P_k^m(\tau)$ begin in X and end in Y by design so we conlcude $G_k^m(\tau)$ is the oriented bipartite graph between the independent sets X and Y. $\square$