

## Remarks

1) The 1<sup>st</sup> law (principle) of thermodynamics postulates the possibility of the transformation of the mechanical energy in thermal energy and viceversa, if  $E = \text{const}$  (total energy is constant).  
The theorem does not state if the process is reversible or not.

2) The principles of momentum and kinetic momentum can be deduced from the principle of the variation of the energy, if we require its invariance w.r.t. rigid displacements (proof by Rivlin, Green).

## Second Law (Principle) of Thermodynamics

Experiments show that the results of classical thermodynamics do hold in most fluid flows, if the changes along the motion are slow compared to a relaxation time.

The relaxation time is defined as the time taken by the material to adjust to a new state (adjustment through molecular collision). The relaxation time is very small under ordinary conditions, since only a few molecular collisions are needed for adjustment. The relations of classical thermodynamics are therefore applicable to most fluid flows.

The first law of thermodynamics can be regarded as a principle that defines energy (the internal energy) of the system; the second law can be regarded as the principle that defines the entropy of the system.

The 2<sup>nd</sup> law of thermodynamics essentially says that real phenomena can only proceed in a direction in which the "disorder" of an isolated system increases.

Disorder of a system is a measure of the degree of uniformity of macroscopic properties in the system, which is the same as the degree of randomness in the molecular arrangements that generate these properties. A real phenomenon must proceed in a direction in which such orderly "arrangements" decrease because of "mixing."

Natural processes tend to cause mixing due to ~~heat~~ the transport of heat, momentum and mass.

A consequence of the second law is that there must exist a property called ENTROPY, which is related to other thermodynamical properties of the motion (medium).

In addition, the 2<sup>nd</sup> law says that the entropy of an isolated system can only increase; ENTROPY is therefore a measure of disorder or randomness of a system.

### Ch. 3. The ideal fluid.

#### 3.1. Mathematical model of the ideal fluid

FM3  
1

The simplest continuous medium, the ideal fluid: the friction between the layers can be neglected. Therefore the stress vector  $\vec{T}$  has no tangential component, only normal component

$$\vec{t}_n = -p\vec{n}; \quad p - \text{by definition, is the pressure. (3.1)}$$

One can show that  $p \geq 0$ . We have  $(t_n)_i = -p n_i = T_{ji} n_j$ , or, equivalently,

$$p \delta_{ji} n_j = T_{ji} n_j, \quad i=1,2,3.$$

or

$$(T_{ji} + p \delta_{ij}) n_j = 0$$

so

$$\boxed{T_{ji} = -p \delta_{ij}}, \quad \vec{T} = -p \vec{I} \quad (3.2)$$

representing the constitutive (state) equation of the ideal fluid. From Cauchy's equations we obtain

$$\rho \frac{dv_i}{dt} = \rho f_i - \frac{\partial}{\partial x_j} (p \delta_{ij})$$

$$\boxed{\rho \frac{dv_i}{dt} = \rho f_i - \frac{\partial p}{\partial x_i}, \quad i=1,2,3; \quad (3.3)}$$

the equations (2.41) are Euler's equations for the ideal fluid. Euler has derived them before Cauchy derived (2.37). In vector form,

$$\boxed{\rho \frac{d\vec{v}}{dt} = \rho \vec{f} - \text{grad } p, \quad (3.4)}$$

where the unknowns are  $\rho(\vec{x}, t)$ ,  $\vec{v}(\vec{x}, t)$ ,  $p(\vec{x}, t)$ . With the help of the formula

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v},$$

We can write (2.41) in the equivalent form

$$\boxed{\frac{\partial v_i}{\partial t} + \sum_{j=1}^3 v_j \frac{\partial v_i}{\partial x_j} = f_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i}, \quad i=1,2,3. \quad (3.5)}$$

#### The incompressible fluid

An incompressible fluid:  $\frac{d}{dt} \int_D dv = 0$ , so  $\boxed{\text{div } \vec{v} = 0}$  (3.6)

and  $\boxed{\rho = \text{constant}}$ . (3.7). We thus have 4 equations, (3.5)-(3.6) for 4 unknowns  $v_i(\vec{x}, t)$ ,  $i=1,2,3$ ,  $p(\vec{x}, t)$ .

We define  $S$ , the entropy, as

$$S(P) = \int_P s dm = \int_D s(t, \vec{x}) \rho(t, \vec{x}) dV,$$

where  $s$  is the entropy of unit mass (density of  $S$ , specific entropy).

$S$  is a state quantity;  $S$  absolutely continuous on  $m$ .

We also define the temperature, the fundamental quantity of thermodynamics,  $T$ ,  $[T] = \text{°Kelvin}$ ; by measurement we associate to  $T$  the quantity  $T$ ,  $T = T(\vec{x}, t)$ .

$$T(\vec{x}, t)$$

2<sup>nd</sup> law of thermodynamics states that there exists  $s(x, t)$  such that, when the system undergoes a from a state of thermodynamic equilibrium characterized by  $S(t_1)$  to another such state characterized by  $S(t_2)$ , we have

$$s(t_2) - s(t_1) \geq \int_{t_1}^{t_2} \left( \frac{\delta q}{T} \right) dt \quad (2.54)$$

where  $\delta q$  is the notation for the change in the heat quantity of unit mass ~~received~~ during this process (due to conduction or radiation) in an unit time.

If we admit that the transformation takes place through a sequence of equilibria (no jumps) - the so-called "local state axiom" then we can write (2.54) in the following differential (local) form

$$\dot{s} \geq \frac{\delta q}{T} \quad (2.55)$$

also called Clausius-Duhem inequality. The sign " $>$ " corresponds to irreversible processes, while the sign " $=$ " corresponds to reversible processes.

From (2.55) <sup>or (2.55)</sup> we see that the entropy of an isolated system ( $\delta q = 0$ ) can only increase.

From

$$\dot{s} = \frac{\delta q}{T} \quad (2.56)$$

we deduce that  $\boxed{\rho \delta q = \rho T \dot{s}}$  represents the heat quantity received by the unit volume in a unit time, thus it represents exactly what we have denoted by  $\boxed{-\text{div } \vec{q}}$  in (2.5). In this way, (2.53) can be rewritten as (Gibbs)

$$\rho \dot{e} = \overleftrightarrow{T} \cdot \overleftrightarrow{D} + \rho T \dot{s} \quad (2.57)$$

the energy equation for reversible processes; fundamental in ideal fluids me

## Clausius - Duhem Inequality

The C-D inequality has been postulated by Clausius (1854) for an isolated system ( $\vec{q} = 0$ ) and by Duhem in 1901, for system exchanging heat with the surrounding medium, in the form

$$S(t_2) - S(t_1) \geq \int_{t_1}^{t_2} - \left( \int_{\partial D} \frac{\vec{q} \cdot \vec{n}}{T} da \right) dt, \quad (\forall) D \subseteq \mathcal{D}. \quad (2.58)$$

Under the "local state axiom" - the time for the system to reach thermal equilibrium between two states is much shorter than the time for mechanical evolution, we can write (2.58) as

$$\dot{S} \geq - \int_{\partial D} \frac{\vec{q} \cdot \vec{n}}{T} da, \quad (\forall) D \subseteq \mathcal{D} \quad (2.59)$$

This is the fundamental inequality of thermodynamics, expressing the 2<sup>nd</sup> law of thermodynamics:

"The evolution of the thermodynamical system within two states is such that at any time  $t$  and subsystem  $D$ , (2.59) takes place".

From the definition of  $S$  <sup>and (2.59)</sup> we have the equivalent form

$$\frac{d}{dt} \int_D \rho s dV \geq - \int_{\partial D} \frac{\vec{q} \cdot \vec{n}}{T} da, \quad (\forall) D \subseteq \mathcal{D}. \quad (2.60)$$

or, equivalently, for continuous motions, and using (2.10), we have

$$\rho \dot{s} \geq - \text{div} \left( \frac{\vec{q}}{T} \right) \quad (2.61)$$

If we consider convection, too, (2.59) becomes

$$\dot{S} \geq - \int_{\partial D} \frac{\vec{q} \cdot \vec{n}}{T} da + \int_D \rho \frac{r}{T} dv \quad (2.62)$$

A)  $p$  cannot depend on  $\vec{n}$ :

Proof:

assume  $p = p(\vec{n})$ ,  $\Rightarrow -p(\vec{n})\vec{n} = -p(\vec{e}_1)\vec{e}_1 n_1 - p(\vec{e}_2)\vec{e}_2 n_2 -$   
then since  $\vec{F}(\vec{x}, \vec{n}) = \vec{T}_j(\vec{x})\vec{n}_j$   $-p(\vec{e}_3)\vec{e}_3 n_3$

So  $p(n) = p(\vec{e}_1) = p(\vec{e}_2) = p(\vec{e}_3)$ , and, therefore,  
 $p$  is the same along any direction  $\vec{n}$ .

B) We assume  $p > 0$ , thus  $\vec{F}$  is opposed to  $\vec{n}$  at any point  $P(\vec{x})$ , so the action of  $M-P$  tends to compress the fluid in  $P$ . This is a hypothesis in the macroscopic theory, but is a proven result in the kinetic molecular theory.

c) We will see in the next chapter that, for real fluids,  $\vec{F} = \vec{F}(\vec{D})$ . A tensor  $\vec{T}$  of the form

$$\vec{T} = -p\vec{I}$$

is called hydrostatic, since for real fluids the stress tensor  $\vec{T}$  takes the above value only if the fluid is at rest.

### 3.2. The thermodynamics of ideal fluids

From

$$\rho \dot{e} = \vec{T} \cdot \vec{D} + \rho T \dot{s}$$

and

$$\vec{T} = -p\vec{I}$$

we obtain

$$\rho \dot{e} = -p \operatorname{div} \vec{v} + \rho T \dot{s} \quad (3.7)$$

and, using the continuity equation,

$$de = T ds - p dv \quad (3.8)$$

where

$$v = \frac{L}{\rho}$$

This is the fundamental equation (Gibbs) of the thermodynamics of ideal fluids.

From (3.8) we see that  $e$  depends on  $s$  and  $v$ .  
If we denote this dependence by

$$e = \hat{e}(s, v)$$

(3.9)

and we consider  $\hat{e}$  as known, then from (3.8) we obtain

$$\frac{\partial \hat{e}}{\partial s} ds + \frac{\partial \hat{e}}{\partial v} dv = T ds - p dv$$

(3.10)

and, if  $s$  and  $v$  are independent of each other,

$$T = \frac{\partial}{\partial s} \hat{e}(s, v),$$

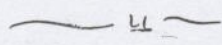
$$p = - \frac{\partial}{\partial v} \hat{e}(s, v).$$

(3.11)

A fluid characterized by two parameters is called by parametric. (3.11) show us that  $\hat{e}(s, v)$  determines the thermodynamic state of the system. This is why  $\hat{e}(s, v)$  is called the thermodynamic potential or characteristic function.

The equations (3.11) are obviously state equations.

Remark <sup>(see ideal gas case):</sup> choosing other variables than  $s, v$  is not enough. Knowing  $e = \hat{e}(T, v)$  is not enough to completely determine the state of the system, we will need the state equation  $p = p(T, v)$ , too.



Another important thermodynamic quantity is the free specific energy  $f$ , introduced by Helmholtz, defined as

$$f = e - Ts$$

(3.12)