Remark 2.4

In the case of continuous motions, from the material derivative of an integral formula (1.2.5) and the continuity equation (2.7) we get

$$\boxed{\frac{d}{dt}\int_D (\rho F)\,dV} = \int_D\left[\frac{d}{dt}(\rho F) + \rho F\,\mathrm{div}\,\vec{v}\right]dV =$$

$$= \int_D \left\{\left[\rho\frac{dF}{dt} + F\left(\frac{d\rho}{dt} + \rho\,\mathrm{div}\,\vec{v}\right)\right]dV = \boxed{\int_D \rho\frac{dF}{dt}\,dV} \quad . \qquad (2.10)$$

We will use this formula when deriving the balance equations.

Definition: mass flux

Let's calculate the quantity of material through a surface S in an internal of time equal to 1 (see Figure 2.1). Let Δa be a surface element; the quantity of mass through Δa in $\Delta t = 1$ is contained in the cylinder with generator $= |\vec{v}|$, as $|\vec{v}| =$ the distance per unit time. With sign convention that the flux is positive when $\vec{v}$ and $\vec{n}$ point in the same direction, or negative otherwise, we obtain for the flux of mass the formula

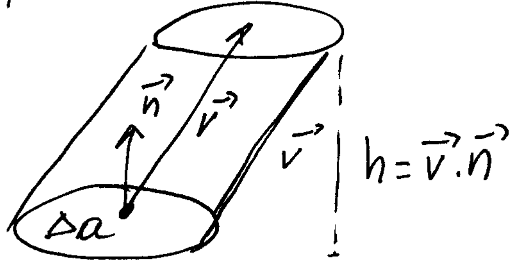

Figure 2.1

$$\phi = \int_S \rho\vec{v}\cdot\vec{n}\,da \qquad (2.11)$$

as $dm = \rho\,dV$, and $\Delta V = |\vec{v}\cdot\vec{n}|\,\Delta a$.

In fact, in this way we can calculate the flux of any variable having a density f; we get

$$\phi = \int_S f\,\vec{v}\cdot\vec{n}\,da \qquad (2.12)$$

2.3. The Principle of the variation of momentum.

The momentum of a continuous system $\mathcal{P}$ with support D is

$$\vec{H} = \int_{\mathcal{P}} \vec{v}\, dm = \int_D \rho \vec{v}\, dV. \qquad (2.13)$$

Postulate: a material system $\mathcal{U}$ moves such that at any time t, and for any subsystem $\mathcal{P}$ the derivative of the momentum with respect to time is equal to the resultant of the external forces acting on $\mathcal{P}$,

$$\frac{d\vec{H}}{dt} = \vec{R}. \qquad (2.14).$$

For a continuous medium we need to consider both contact forces of resultant $\vec{R}^c$,

$$\vec{R}^c = \int_{\partial D} \vec{t}\, da \qquad (2.15)$$

and momentum

$$\vec{M}^c = \int_{\partial D} \vec{x} \times \vec{t}\, da \qquad (2.16)$$

and distance forces like gravitational attraction, of density $\vec{f}$; their resultant force and resultant momentum are

$$\vec{R}^d = \int_{\mathcal{P}} \vec{f}\, dm = \int_D \rho \vec{f}\, dv, \qquad (2.17)$$

and

$$\vec{M}^d = \int_{\mathcal{P}} \vec{x} \times \vec{f}\, dm = \int_D \rho \vec{x} \times \vec{f}\, dV, \qquad (2.18).$$

respectively.

From (2.15) – (2.18) and (2.14) we get that

$$\frac{d}{dt} \int_D \rho \vec{v}\, dV = \int_{\partial D} \vec{t}\, da + \int_D \rho \vec{f}\, dV, \quad (\forall)\, D \subset \mathcal{D}, \qquad (2.19).$$

The above formula is valid for both continuous and discontinuous motions. For continuous motions only, if we take into account (2.10), we get the momentum equation under the form

$$\int_D \rho \vec{a}\, dV = \int_{\partial D} \vec{t}\, da + \int_D \rho \vec{f}\, dV \qquad (2.20).$$

2.4. The Principle of Variation of the kinetic Momentum

The kinetic momentum of any part $\mathcal{P}$ of a continuous material system $\mathcal{M}$ can be defined as

$$\overrightarrow{\mathcal{K}} = \int_{\mathcal{P}} \vec{x} \times \vec{v}\, dm = \int_D \rho\, \vec{x} \times \vec{v}\, dV \qquad (2.21).$$

<u>The Principle of variation of the kinetic momentum:</u> A material system $\mathcal{M}$ moves such that, at any time t and for any subsystem $\mathcal{P}$ of $\mathcal{M}$, the time derivative of the kinetic momentum equals the resultant momentum of external forces acting on $\mathcal{P}$.

Analytically, we can write the above principle as

$$\frac{d}{dt} \int_D \rho \vec{x} \times \vec{v}\, dV = \int_{\partial D} \vec{x} \times \vec{t}\, da + \int_D \rho\, \vec{x} \times \vec{f}\, dV \qquad (2.22)$$
$$(\forall)\, D \subset \mathcal{D}$$

and for continuous motions, using the formula (2.10), we get that

$$\int_D \vec{x} \times \vec{a}\, \rho\, dv = \int_{\partial D} \vec{x} \times \vec{t}\, da + \int_D \rho \vec{x} \times \vec{f}\, da \qquad (2.23)$$
$$(\forall)\, D \subset \mathcal{D}.$$

2.5. Stress Tensor. Cauchy's equations.

Cauchy's Lemma *)

If $\vec{t}$ is continuous w.r.t. $\vec{x}$, then at any $\vec{x} \in \mathcal{D}$ we have

$$\vec{t}(\vec{x},\vec{n}) = -\vec{t}(x,-\vec{n}) \tag{2.24}$$

Proof Consider a cylinder D_1, centered at $\vec{x}$, such that it reduces to the disc Σ when its height $h \to 0$ (see Figure 2.2). From (2.20) we have

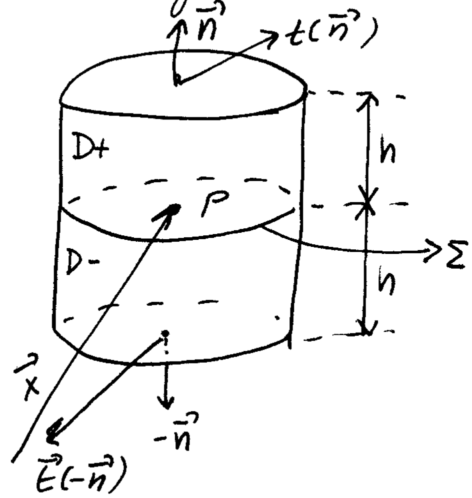

Figure 2.2.

$$\int_{D_+} \rho\vec{a}\,dv = \int_{\partial D_+} \vec{t}\,da + \int_{D_+} \rho\vec{f}\,dv$$

and the volume integrals $\int_{D_+}$ as well as the integrals on the surface of D_1 (the external surface) approach zero as $h \to 0$. We are left with

$$\int_{\Sigma} \left[\vec{t}(\vec{x},\vec{n}) + \vec{t}(\vec{x},-\vec{n})\right] da = 0$$

and from the Fundamental Lemma we get that $\vec{t}(\vec{x},\vec{n}) + \vec{t}(\vec{x},-\vec{n}) = 0$ q.e.d.

Cauchy's Theorem

If $\vec{t}$ is a continuous function on $\mathcal{D}$ then there exists a tensor $\vec{T}(\vec{x})$, defined on $\mathcal{D}$, such that at any point $P(\vec{x})$ we have

$$\vec{t}(\vec{x},\vec{n}) = \vec{T}(\vec{x})\vec{n}, \tag{2.25}$$

or, on components,

$$\vec{t}_i(\vec{x},\vec{n}) = T_{ij}\, n_j \quad , i = 1,2,3 \tag{2.26}$$

Remark 2.4. Cauchy's theorem shows that $\vec{t}$ is a linear function of $\vec{n}$.

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*) We consider the current (Eulerian) configuration, at an instant time t, so we ommit t from $\vec{t}(\vec{x},\vec{n},t)$.

Remark 2.5. $\vec{T}(\vec{x})$ characterizes completely the stress state at $P(\vec{x})$. Indeed, if we know $T(\vec{x})$ we can then calculate the stress vector $\vec{t}$ for any given oriented surface of normal $\vec{n}$.

Proof. We consider a tetrahedron centered at P and having three sides perpendicular to the coordinate axes (see Fig. 2.3). Denote by h the length of the perpendicular from P to the opposite side $P_1P_2P_3$. We observe that the volume of the tetrahedron approaches zero when $h \to 0$.

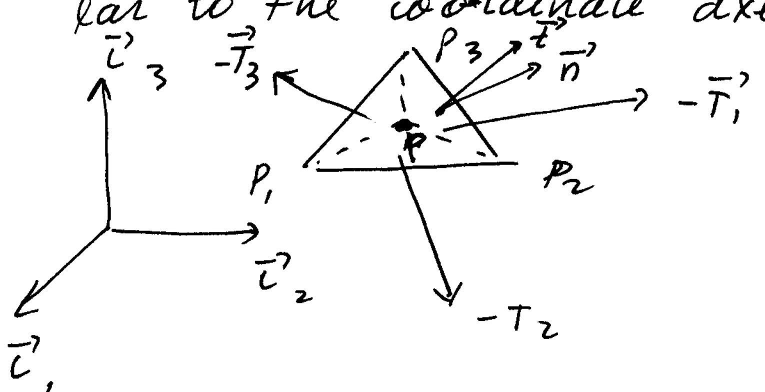

Figure 2.3.

This is not a particular choice since the dependence of $\vec{t}$ on $\vec{n}$ is unique. Denote by $\vec{n}$ the normal to $P_1P_2P_3$, and by Δa_i, $i=1,2,3$, the areas of the sides PP_2P_3, PP_1P_3, PP_2P_1. We have denoted by $\vec{T}_j(\vec{x}) = \vec{t}(\vec{x}, \vec{i}_j)$. According to ~~the~~ Cauchy's Lemma we denote by $-\vec{T}_1$ the stress vector on the side PP_2P_3; analogously we can define $-\vec{T}_2$ and $-\vec{T}_3$. If Δa is the area of $P_1P_2P_3$, we have

$$\Delta a_i = n_i \, \Delta a, \quad i = 1,2,3 \qquad (2.27)$$

The volume of the tetrahedron is $\frac{1}{3} h \Delta a$. We apply the principle of the variation of momentum with $D =$ the tetrahedron and we use the formula

$$\int_D \vec{F}(\vec{x})\,dv = \left(\vec{F}(\xi) + \vec{\varepsilon}\right)\int_D dV,$$

where $\lim\limits_{\mu(D)\to 0} \varepsilon = 0$. We get, from (2.2.7),

$$\rho(\vec{x})\,[\vec{a}(x) - f(\vec{x}) - \varepsilon]\ \tfrac{1}{3} h \Delta a = [\vec{t}(x, \vec{n}) + \vec{\varepsilon}_n]\,\Delta a - \sum_j [\vec{T}_j(\vec{x}) + \vec{\varepsilon}_j]\,\Delta a_j$$

with
$$\lim_{h\to 0} (\varepsilon, \varepsilon_n, \varepsilon_j) = 0.$$
Using (2.27), dividing by Δa and taking the limit $h \to 0$ we get
$$\vec{t}(\vec{x}, \vec{n}) = T_j(\vec{x}) n_j. \qquad (2.28)$$
After projection of (2.28) on Ox_i we get (2.26). ☒

Remark 2.6.

The formula (2.26) is the proof that $\overleftrightarrow{T}$ is a tensor. It is the stress tensor, the first tensor to be defined in sciences. The name "tensor" comes from "tension".

Remark 2.7

Let's project $t(\vec{x}, \vec{n})$ along a direction of unit vector $\vec{m}$. We have that
$$\vec{m} \cdot \vec{t} = \vec{m} \cdot \overleftrightarrow{T}\vec{n} = T_{ij} m_i n_j. \qquad (2.29)$$
This formula helps us to show that if we change the basis from $\{\vec{i}_1, \vec{i}_2, \vec{i}_3\}$ to another orthogonal basis $\{\vec{i}_1', \vec{i}_2', \vec{i}_3'\}$, the quantity $\overleftrightarrow{T}$ changes according to the formula
$$\overleftrightarrow{T}' = Q \overleftrightarrow{T} Q^T, \qquad (2.30)$$
where Q is the change of base matrix. Equivalently,
$$T_{kl}' = Q_{kl} Q_{ij} T_{ij} \qquad (2.30)'$$
If we take $\vec{m} = \vec{n}$ we get the formula for the normal stress
$$N = \vec{n} \cdot \vec{t} = \vec{n} \cdot \overleftrightarrow{T}\vec{n} = T_{ij} \cdot n_i n_j. \qquad (2.31)$$
Question Find those directions $\vec{n}$ for which N takes extremal values.