Working with Quotients of Finitely Presented Groups

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Analoguous for groups:

Vector space	$\longrightarrow$	Free group
Matrix rows	$\longrightarrow$	Relations
Nullspace	$\longrightarrow$	Factor group
Basis	$\longrightarrow$	Faithful representation

Finitely presented groups

A group given this way is a *finitely presented group*, given by a presentation

$$\langle \underline{\mathbf{x}} \mid \mathcal{R} \rangle = \langle x_1, \ldots, x_n \mid l_1(\underline{\mathbf{x}}) = r_1(\underline{\mathbf{x}}), \ldots, l_m(\underline{\mathbf{x}}) = r_m(\underline{\mathbf{x}}) \rangle.$$

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If we (formally) consider objects of the form $\langle \cdot \mid \cdot \rangle$ we talk of a *Presentation*.

For Example

Presentations arise in topology: We can write down a presentation for the fundamental group of a simplicial complex with

- Each edge e gives a generator g_e.
- For each edge e in a (chosen) *spanning tree*, a relation $g_e = 1$.
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The group of a *knot* is the fundamental group of its complement

One can write down a presentation for it from a picture of the knot. Two knots are equivalent, if their groups are isomorphic.

Arithmetic

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Analogon: Division of a polynomials by an ideal in $k[x_1, \ldots, x_n]$, Gröbner Bases

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- Wreath product ordering $a_1 b_1 a_2 b_2 \cdots a_k b_k$:
  First compare a-parts, if equal compare b-parts.

Problem

Potential ambiguity when multiple replacements are possible:

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But in general we are not

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$$\underline{xyy}y = yxyy, \qquad x\underline{yyy} = x$$

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Overlap $xx \to 1$ and $xyx \to yxy$:

$$\begin{aligned} \underline{xx}yx &= yx \\ x\underline{xyx} &= \underline{xyx}y = yx\underline{yy} = yx \end{aligned}$$

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(for obtaining a confluent rewriting system)

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- (Rinse, Lather,) Repeat

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If G is finite, the method will terminate, but the runtime cannot be predicted.

Quotients

Therefore a principal tool is to study homomorphic images.

It is easy to check whether a mapping given on the generators is a homomorphism.

$$\langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle \to A_5, \quad a \mapsto (1,2)(3,4), b \mapsto (1,4,5)$$

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- How to find such a mapping?
- How to find a mapping with small kernel?
- Given a φ, can we find a *lift* (i.e. ψ with $\ker \psi < \ker \varphi$)?

What homomorphisms can we get?

Abelian quotients

We get the largest abelian quotient by abelianizing the presentation.

The Smith normal form of this presentation gives the normal form as an abelian group:

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$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \to C_1 \times C_3 \cong C_3$$

Bigger groups

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$$GL_4(3) = \left\{ \begin{array}{l} C_2 \\ | \\ \quad PSL_4(3) \\ \quad | \\ \quad C_2 \end{array} \right\} = SL_4(3)$$

If all simple factors are cyclic, the group is called *solvable*.

Chief series

One way to get factors is to take a chain of normal subgroups and refine. Then the subfactors are direct products of simple groups: $3^5.2^4.2^3.(A_5)^{11}.M_{11}$

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Paradigm: If there is a normal subgroup that is a vector space, solve the problem for the factor group and use linear algebra to lift the result to the whole group.

In the above example:

$$(A_5)^{11}.M_{11} \to 2^3.(A_5)^{11}.M_{11} \to 2^4.2^3.(A_5)^{11}.M_{11} \to 3^5.2^4.2^3.(A_5)^{11}.M_{11}$$

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"Typically" the factor of the solvable radical is small.

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(Description later)

Permutation quotients

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If we number the cosets, we get a transitive permutation representation $G\varphi$ with $U\varphi = \mathrm{Stab}_{G\varphi}(1)$.

Every transitive permutation representation is obtained this way.

Todd-Coxeter Method

If we have generators for U, we can compute this permutation representation:

Define cosets C_i step by step by generator images:
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If the index $[G{:}U]$ is finite, the method will terminate *but one cannot predict the runtime*

(Otherwise let $U = \langle\rangle$ and we would have a check whether $|G| = 1$.)

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To reduce the search space consider images only up to conjugation (renumbering of points):

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & & & \downarrow \\ 1 & 4 & 5 & 2 \end{pmatrix} : \quad \begin{array}{lcl} a & \mapsto & (1,2,3,4) \\ b & \mapsto & (1,2) \end{array}, \quad \begin{array}{lcl} a & \mapsto & (1,4,5,2) \\ b & \mapsto & (1,4) \end{array}$$

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How can we find quotients of a subgroup:

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Then $r_i \cdot g \cdot r_j^{-1}$ maps $1 \to i \to j \to 1$. So $r_i g / r_j \in U$.

Theorem: (Schreier): U is generated by the elements $r_i g / r_j$ for all i and all generators g of G ($j = i^g$).
One can rewrite the presentation of G to a presentation of U in these generators.

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Even with simplification (*Tietze transformations*) we cannot go too deep into G,

in particular not to the kernel $\ker \varphi$ of a homomorphism.

Application: Quotient subgroups

On the computer we represent every subgroup $U \leq G$ by a pair $(\varphi, U\varphi)$ where $\ker \varphi \leq U$.

(Previously: "Coset Table".)

Advantages

- More effective storage for big index.
- Easier calculations: Calculate in the image of the homomorphism.
- Taking preimages becomes trivial.

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- Taking preimages becomes trivial.

However: We need methods to compute in the image group (for example a permutation group)

Calculationg with two subgroups

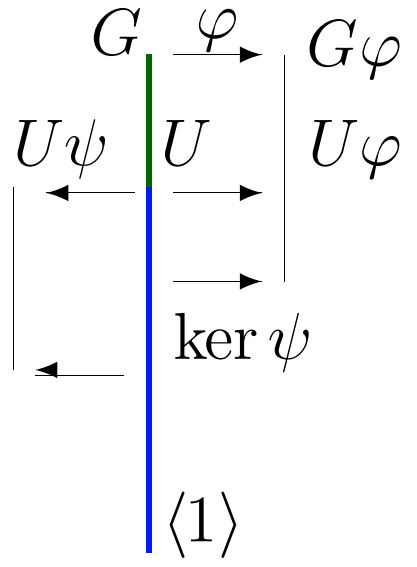
If there are two homomorphisms φ, ψ, we can map in the direct product $(G\varphi) \times (G\psi)$.

$$\varphi\colon \begin{cases} a & \mapsto & (1,2,3) \\ b & \mapsto & (1,2) \end{cases}, \psi\colon \begin{cases} a & \mapsto & (1,2) \\ b & \mapsto & (1,4,3,2) \end{cases}$$

$$(\varphi,\psi)\colon \begin{cases} a & \mapsto & (1,2,3)(4,5) \\ b & \mapsto & (1,2)(4,7,6,5) \end{cases}$$

Induction

By rewriting, we can get a subgroup $V < U < G$ in quotient representation $(\psi, V\psi)$ related to U. How to get it related to G?



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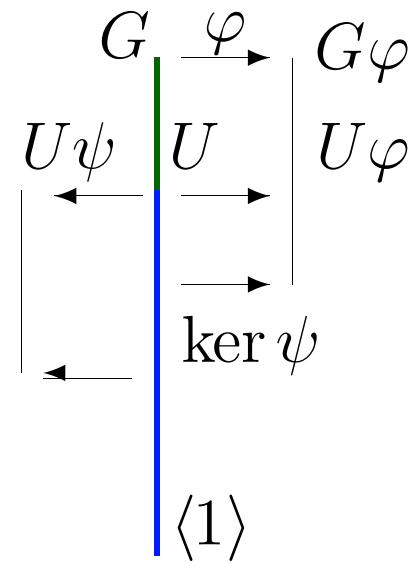

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We can *induce* ψ to a homomorphism λ defined on G: (Permutation group version – "KRASNER-KALOUJNINE" theorem)

- Let φ the action of G on the $n = [G{:}U]$ cosets of U.

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- $G\lambda$ permutes (via φ) n copies of the domain of $U\psi$ ("blocks"): $(1, \ldots, m), (m+1, \ldots, 2*m), \ldots, ((n-1)m+1, \ldots, n \cdot m)$

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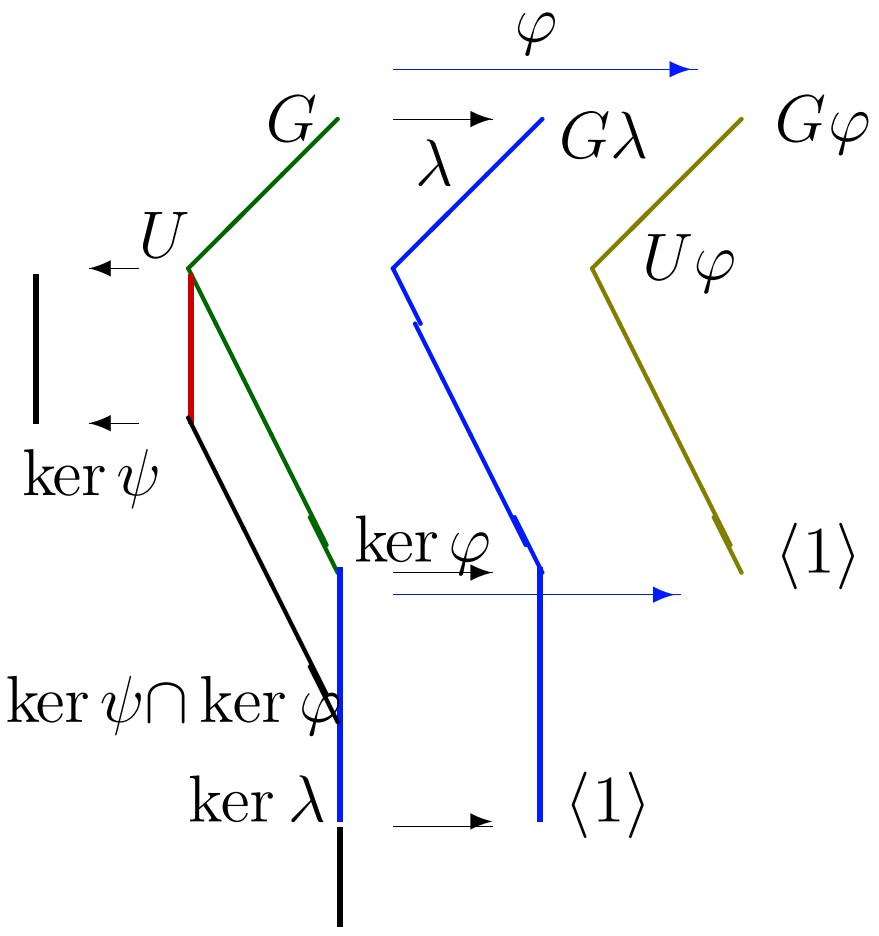
If $V = Stab_U(1)$ (via ψ), then $V = \mathrm{Stab}_U(1)$ (via λ).

Finding new Quotients



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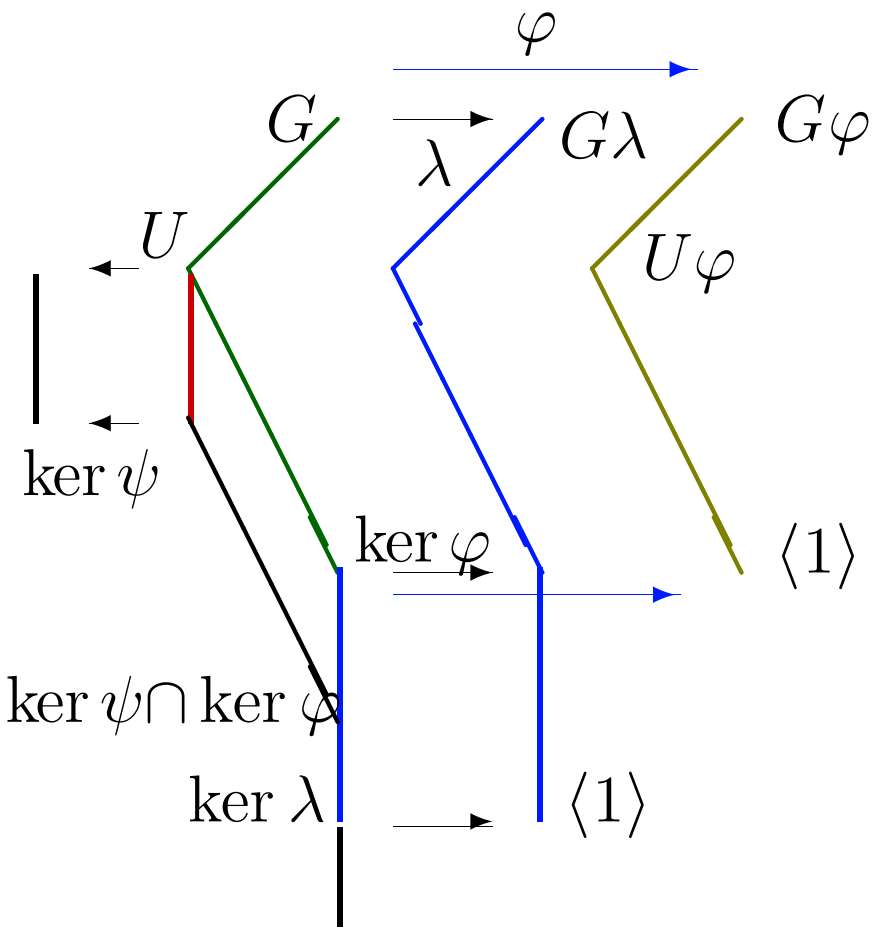

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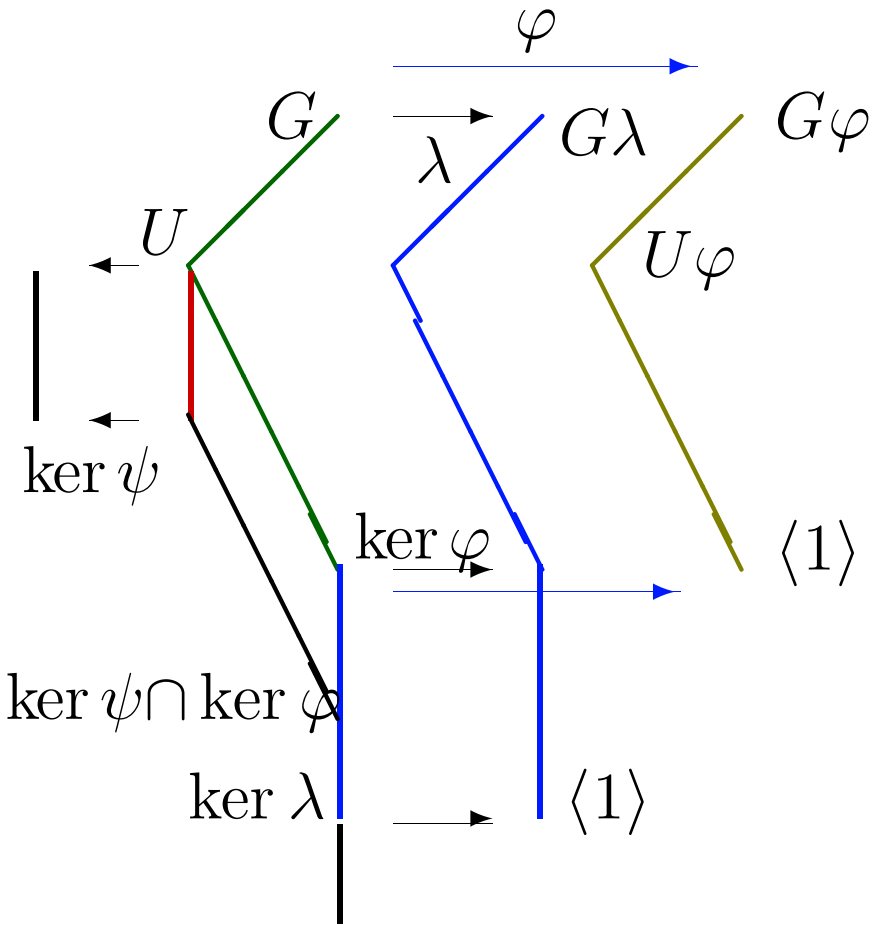

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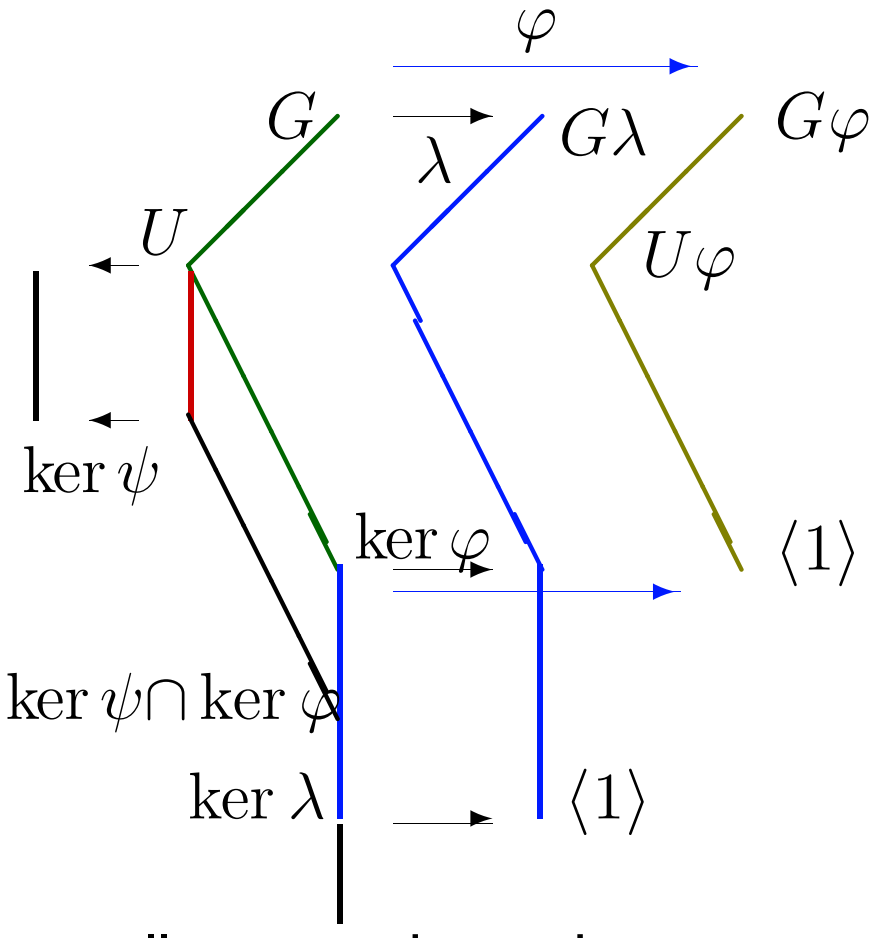
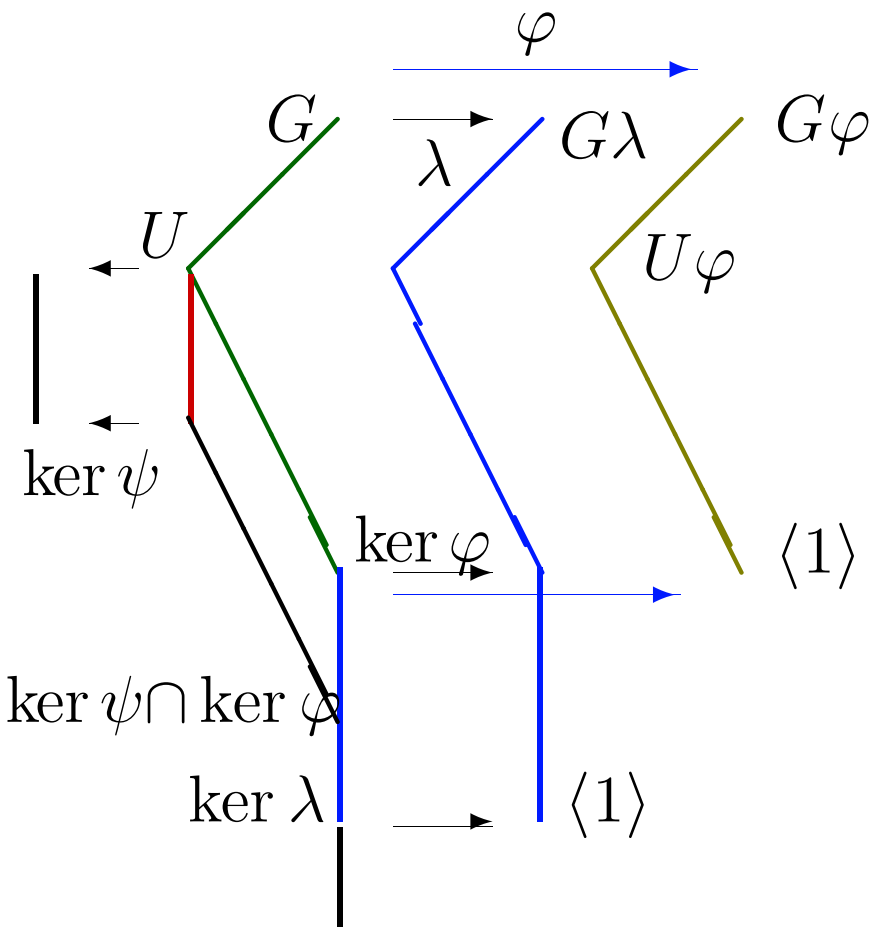


Given G and φ:

- Take a subgroup U of $G\varphi$.
- Rewrite the presentation (Consider U as another finitely presented group)
- Find quotients ψ of U.
- Induce ψ to λ defined on G.

λ tells more about the structure of G than ψ does.

Finding new Quotients



Given G and φ:

- Take a subgroup U of $G\varphi$.
- Rewrite the presentation (Consider U as another finitely presented group)
- Find quotients ψ of U.
- Induce ψ to λ defined on G.

λ tells more about the structure of G than ψ does.

The separate techniques are known, but the induction needs good permutation group functionality to be practical.

Application to Quotients

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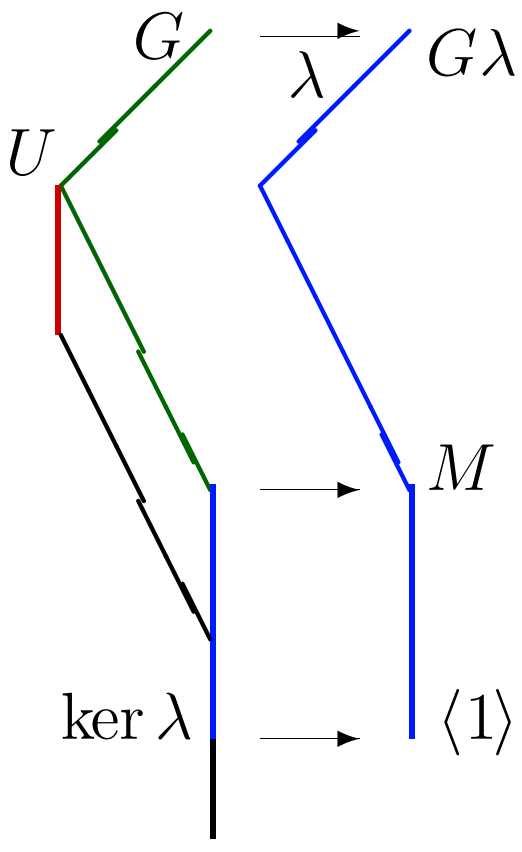

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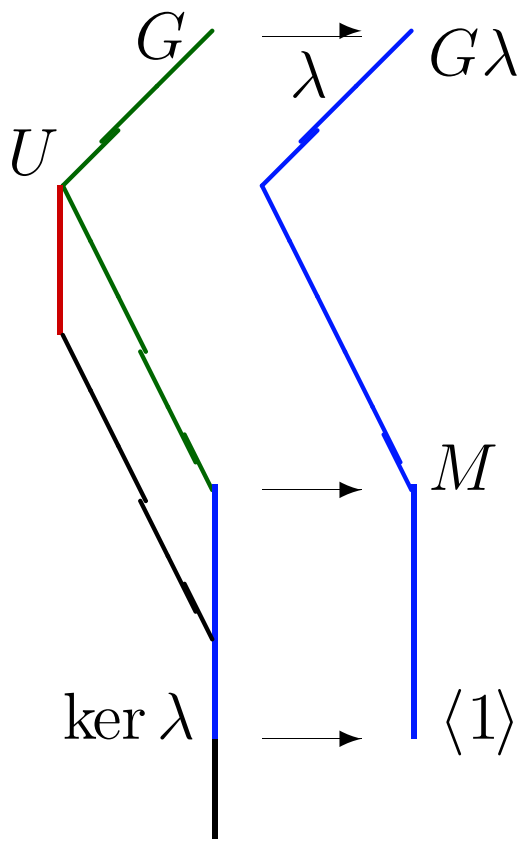

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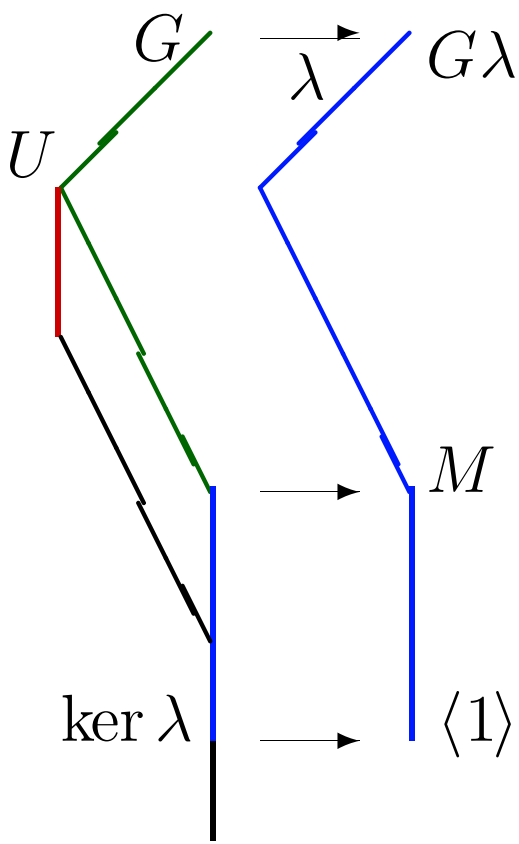

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Criterion: To find a simple module M, $U\varphi$ must be a vector stabilizer for the *dual* module M^*.

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- or $U\lambda$ has a quotient that is a *Schurian* extension by $U\varphi$.

The second case is related to the multiplier of $U\varphi$ and one can test in a Darstellungsgruppe whether it occurs.

For Example

BAUMEISTER/IVANOV/PASECHNIK (2000) give a presentation for a group G and $G\varphi \cong McL$. ($|McL| = 898\,128\,000$)

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The combination lift $3^{231}.McL$ is the largest lift for modules of dimension up to 600.

The permutation group calculations in the image group can become hard.

Towards a Nonsolvable Quotient Algorithm

(Joint work with A. NIEMEYER, Perth)

Given $G = \langle \underline{\mathbf{g}} \mid \mathcal{R} \rangle$ and $\varphi\colon G \to B$ for a finite B, find the largest lift of φ by a module.

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(In generalization of the Solvable Quotient Algorithm):

Find a presentation for B in the images of the generators $\underline{\mathbf{g}}$ under φ.

The relators of this presentation evaluate in G to elements of $\ker \varphi$.

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These values become module generators for M.

Involve the group relators $\mathcal{R}$ to ensure it is a quotient of G.

Concretely

Construct a confluent rewriting system for $M.B$.

Start with a confluent rewriting system for B:

$$\langle x_1, \ldots, x_n \mid l_1(\underline{\mathbf{x}}) \to r_1(\underline{\mathbf{x}}), \ldots, l_m(\underline{\mathbf{x}}) \to r_m(\underline{\mathbf{x}}) \rangle.$$

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Modify this rewriting system suitably. This can be considered as a "next order approximation" of G.

The rewriting system for $M.B$ will involve generators $\underline{\mathbf{y}}$ that stand for coset representatives corresponding to $\underline{\mathbf{x}}$, and extra generators $\underline{\mathbf{z}}$ that represent elements of M.

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We initially consider these z_i to be variables whose values we have to find.

Action on M

As (representatives for B) act on M by conjugation we add extra generators $z_{i,w}$ for indices i and words w representing elements of B in normal form that stand for conjugates $z_i^{w(\underline{\mathbf{y}})}$.

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To work with these we formally add rules

$$z_{i,w} \cdot v \longrightarrow v \cdot z_{i,w \cdot v}$$

that specify this action as well as rules

$$z_{i,v} \cdot z_{j,w} \longrightarrow z_{j,w} \cdot z_{i,v} \quad \text{if } j < i \text{ or } v > w$$

and

$$z_{i,w}^p \longrightarrow 1$$

that indicate that M is elementary abelian.

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We never store these rules explicitly. Instead we use a modified algorithm for collection in the extension that takes care of these rules.

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To acchieve confluence, we look at all overlaps of left hand sides and collect them in both possible ways.

As the rules hold in B the result are relations that must hold among the $z_{i,w}$:

Assume $B = \langle x_1, x_2 \mid x_1^2 \to 1, x_2^2 \to 1, x_1x_2x_1 \to x_2x_1x_2 \rangle$.

New rules: ${y_1}^2 \to z_1, \quad , {y_2}^2 \to z_2, \quad y_1y_2y_1 \to y_2y_1y_2 \cdot z_3$.

$$
\begin{aligned}
\underline{y_1y_1}y_2y_1y_2y_1 \quad &\to \quad z_1 \cdot y_2y_1y_2y_1 \to y_2y_1y_2y_1 \cdot z_{1,y_2y_1y_2y_1} \\
&\to \quad y_2\underline{y_1y_2y_1}z_{1,y_1y_2} \to \underline{y_2y_2}y_1y_2 \cdot z_3 \cdot z_{1,y_1y_2} \\
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Once all deduced relations hold, we have rules for and extension of a sum of regular modules.

Images

The epimorphism $\varphi\colon G \to B$ is specified by the images of the generators g_i of G.

We write those in normal form for the rewriting system for B:

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Under λ the images agree on the coset representatives of M but the group elements may differ by elements of M:

$$\lambda(g_i) = d_i(\underline{\mathbf{y}}) \cdot z_{i+m}$$

(m=number of rules for B)

Again, we add extra variable generators $z_{i+m,w}$ to represent these elements.

Enforcing Quotient Property

To enforce the relations $\mathcal{R}$ of G, we evaluate them in the images $\lambda(g_i)$ and compute normal forms in the extended rewriting system.

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As B is an image, these relators must hold in the elements $\varphi(g_i)$, thus the result of the collection is a relation among the $z_{i,w}$.

We remember these relations.

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(For experts: This is the main difference to the Solvable Quotient algorithm)

Collecting those definitions gives further relations among the $z_{i,w}$.

Module Presentation

We end up with (many) relations among B-conjugates of the z_i. We can consider this as a B-*Module presentation* for M.

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(Can one use noncommutative Gröbner bases?)

We take this result and feed it back in the extended rewriting system to replace the $z_{i,w}$. If $\underline{\mathbf{b}}$ are symbols for the basis, this yields a confluent rewriting system (with respect to a wreath product ordering) for $M.B$:

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We also can deduce the epimorphism λ and know that the group is a quotient of G.

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The output information (rewriting system/epimorphism/decomposition of generators in images) is sufficient to iterate in search of another lift.

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I have a trial implementation of a single step to play with.

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If one run the algorithm for $p = 2$ we find that $M = 2^5$. This exposes a quotient $2^5.A_5$.

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This algorithm is still very experimental and will need a lot of fine-tuning.

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