On orbifold coverings of genus 2 surfaces

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Abstract

Some algebraic and geometric results on orbifold coverings of genus 2 surfaces are presented with detailed examples.

Introduction

Two-dimensional hyperbolic geometry was used by Kuusalo and Näätänen in [5] to determine all two-dimensional orbifolds regularly covered by closed surfaces of genus 2. On the other hand, Abu Osman and G. Rosenberger applied in [1] purely algebraic methods to the classification of Fuchsian groups having a surface group of genus 2 as a subgroup. When the existence is known, the complete list of all conjugacy classes of genus 2 subgroups of a given group can be obtained using a computer package like GAP. This was done by A. Hulpke in [4]. Our aim here is to show by some examples how effective a combination of geometric and algebraic approaches can be.

Notation

We denote a pointed surface of genus g with cone points of order $m, n, \ldots$ by $(g; m, n, \ldots)$, the same notation being used also for the corresponding ramified covering group. A $(0; m, n, \ldots)$-group and the corresponding orbifold are denoted by $(m, n, \ldots)$ for short, but to avoid confusion we use occasionally also notation $S_{m,n\ldots}$ for the spherical orbifold $(m, n, \ldots)$.

Geometric considerations

In [5] a list all two-dimensional orbifolds regularly covered by closed surfaces of genus 2 was given on p. 412. According to the results of Takeuchi [13 - 15] all the triangle groups included in the list are arithmetic. The cases when an orbifold covers one of the minimal orbifolds $S_{2,3,8}, S_{2,5,10}$ or $S_{2,4,6}$ were denoted by D, E or F, with a subscript r added when the covering is regular (We call a ramified covering regular, if the covering group acts transitively on the fibres). Unfortunately there were a couple of errors, a corrected list follows below:

List 1:

$T_{2,2}$	D		F	2
$S_{2,2,2,2,2,2}$	D_r	E_r	F_r	2
$S_{2,2,2,2,2}$	D		F	4
$S_{3,3,3,3}$	D		F_r	3
$S_{2,2,3,3}$	D		F_r	6
$S_{2,2,2,3}$	D		F_r	12
$S_{2,2,4,4}$	D		F	4
$S_{2,2,2,4}$	D		F	8

$S_{4,4,4}$	D_r			8
$S_{3,3,4}$	D_r			24
$S_{2,3,8}$	D_r			48
$S_{2,8,8}$	D			8
$S_{2,4,8}$	D			16
$S_{3,6,6}$			F	6
$S_{2,6,6}$			F_r	12
$S_{2,4,6}$			F_r	24
$S_{3,4,4}$			F_r	12
$S_{5,5,5}$		E_r		5
$S_{2,5,10}$		E_r		10

Algebraic approach

All triangle groups which have a surface group of genus 2 as a subgroup have been listed by Abu Osman and G. Rosenberger [1]. They used group theoretic arguments based on Singerman's theorem. Besides triangle groups there are other cocompact Fuchsian groups having a genus 2 surface group as a subgroup of finite index.

Abu Osman and G. Rosenberger say in [1] that a group of type $(g'; m_1, m_2, \ldots, m_k)$ has (g)–property if it has a $(g; 0)$ subgroup for all $g \geq 2$. In Theorems 3.5 and 3.7 of [1] two lists are given:

List 2. *Assume $k \geq 4$ if $g' = 0$. Then $(g'; m_1, m_2, \ldots, m_k)$ has (g)–property if and only if $(g'; m_1, m_2, \ldots, m_k)$ is one of the following groups:*

(i)	(2,2,2,3)	$N = 12$
(ii)	(2,2,2,4)	8
(iii)	(2,2,2,6)	6
(iv)	(2,2,3,3)	6
(v)	(2,2,4,4)	4
(vi)	(3,3,3,3)	3
(vii)	(2,2,2,2,2)	4
(vii)	(2,2,2,2,2,2)	2
(ix)	(1;2)	4
(x)	(1;3)	3
(xi)	(1;2,2)	2
(xii)	(2;0)	1

where N is the index of $(2; 0)$ in $(g'; m_1, m_2, \ldots, m_k)$.

List 3. *Let (l, m, n) be a triangle group. Then (l, m, n) has (g)–property if and only if (l, m, n) is one of the following groups:*

1.	(2,3,7)	$N = 84$
2.	(2,3,8)	48
3.	(2,3,9)	36
4.	(2,3,10)	30
5.	(2,3,12)	24
6.	(2,3,18)	18

7.	(2,4,5)	40
8.	(2,4,6)	24
9.	(2,4,8)	16
10.	(2,4,12)	12
11.	(2,5,5)	20
12.	(2,5,10)	10
13.	(2,6,6)	12
14.	(2,8,8)	8
15.	(3,3,4)	24
16.	(3,3,5)	15
17.	(3,3,6)	12
18.	(3,3,9)	9
19.	(3,4,4)	12
20.	(3,6,6)	6
21.	(4,4,4)	8
22.	(5,5,5)	5

where N is the index of $(2;0)$ *in* (l, m, n).

All groups in the lists 2 and 3 can be isomorphically embedded into an arithmetic triangle group, hence they have a realization as an arithmetic group, see also Ackermann, Näätänen and Rosenberger [2].

For example (0; 2,2,2,3) is a subgroup of (2,3,7) with index 7. This can be seen as follows: Singerman's theorem (see Abu Osman and Rosenberger [1]) can be applied to get (5 1 7 6 2 3 4)(2 5 4)(1 6 7)(3)(1 2)(3 4)(5)(6)(7) = (1)(2)(3)(4)(5)(6)(7).

For more details, see Maclachlan and Rosenberger [8], cf. also [7] and [9] as well as Baer [3].

In [4] Alexander Hulpke used GAP low index calculations based on the algorithm of Sims (Sims [12], 5.6) to determine up to conjugacy all genus 2 subgroups of the groups of lists 1 and 2, giving at the same time also their generators in the containing groups. His results were:

The group (2,2,2,3) has 39 conjugacy classes of genus 2 subgroups of index 12, of which 3 are normal (i.e. 3 conjugacy classes consist of just one subgroup). Correspondingly (2,2,2,4) has 19 conjugacy classes of subgroups of index 8, of which 3 are normal, and (2,2,2,6) has 3 conjugacy classes of subgroups of index 6, but no normal subgroups of genus 2. In (2,2,3,3) there are 9 conjugacy classes of index 6, 3 of them normal, and in (2,2,4,4) 3 conjugacy classes of index 4, one of them normal. In (3,3,3,3) all 3 conjugacy classes of index 3 are normal, as well as the 10 conjugacy classes of index 4 in (2,2,2,2,2). (2,2,2,2,2,2) has only one conjugacy class of index 2 which is thus normal. The group (1;2) has 10 conjugacy classes of subgroups of index 4, and the group (1;3) 3 conjugacy classes of subgroups of index 3, none of them normal. Finally, all 4 conjugacy classes of subgroups of index 2 in (1;2,2) are normal.

The triangle groups in list 3 posses quite large number of conjugacy classes of genus 2 subroups. For a detailed account we refer to Neubüser [11].

Examples

Original ideas of Hurwitz can be used to decide geometrically whether a subgroup Γ of finite coarea of a discrete subgroup G in $\mathrm{PSL}(2, R)$ is normal or not.

When G is a discrete group of finite coarea in $\mathrm{PSL}(2, R)$, the orbit space $X = \mathbb{H}/G$ is a finitely punctured compact orbifold. If $\Gamma \subset G$ is a subgroup of finite coarea and $Y = \mathbb{H}/\Gamma$ the corresponding orbifold, the identification mapping f from $Y = \mathbb{H}/\Gamma$ to $X = \mathbb{H}/G$ extends to a holomorphic mapping $\widehat{f} : \widehat{Y} \to \widehat{X}$ of the compactified Riemann surfaces $\widehat{Y}$ and $\widehat{X}$, the mapping $\widehat{f}$ being induced by the action of a finite automorphism group A of the surface $\widehat{Y}$ exactly when Γ is normal in G. Should Γ be normal in G and $A = G/\Gamma$ the corresponding automorphism group, the order of an image point $\widehat{x} = \widehat{f}(\widehat{y})$ in $\widehat{X}$ can only be a multiple of the order of $\widehat{y}$ in $\widehat{Y}$. Furthermore, if $\widehat{f}$ is induced by the automorphism group A, all points in an A-orbit orbit on $\widehat{Y}$ must have the same order. This poses quite heavy restrictions for the automorphism group A, the full automorphism group of a Riemann surface being usually rather small, or geometrically simple in the cases when the surface is the sphere or a torus. For subgroups of some triangle groups this type of reasoning works quite well.

Example 1. We construct first a normal tower of the regular covering of (2,4,8) by the Bolza curve $w^2 = z^5 - z$ of genus 2 (case D in Kuusalo and Näätänen [5], pp. 404-406).

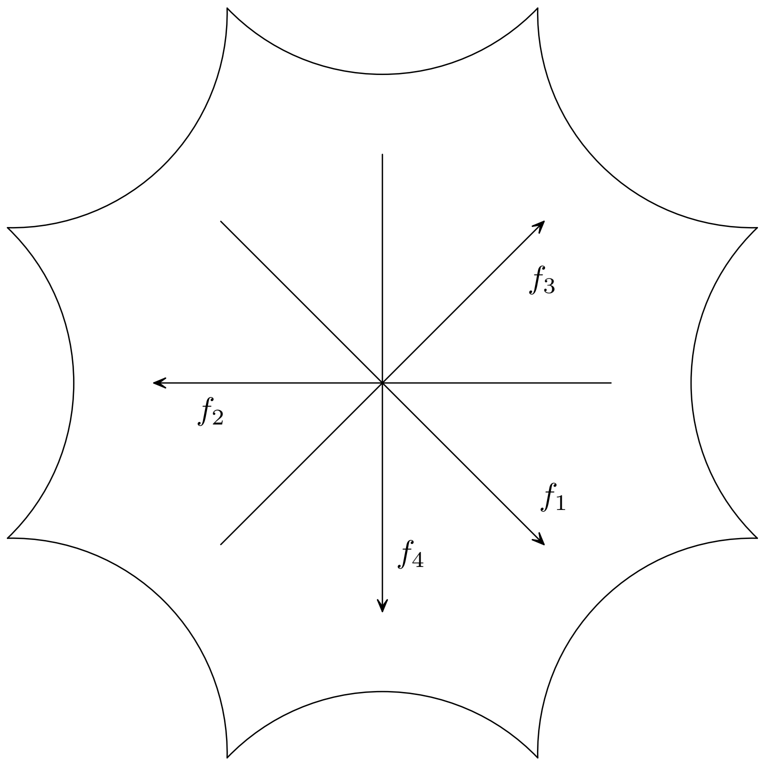

Figure 1. $g = 2$

The covering group of the Bolza curve D has a regular octagon with diagonal pairings as a fundamental domain, cf. Figure 1. The full conformal automorphism group of D contains three conjugate subgroups of order 16, generated by two cyclic automorphisms of orders

8 and 4. Considering the hyperbolic area of the quotient orbifold one can thus see that the normalizer (2,3,8) of the covering group Γ of D must contain three conjugate triangle groups (2,4,8) of index 3. For one of these (2,4,8) groups we can choose the generators S, T, U with relations

$$U^2 = T^4 = S^8 = I, \quad STU = I$$

and a corresponding fundamental domain

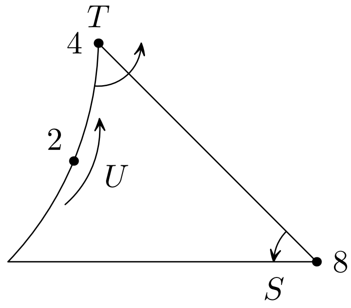

Figure 2. $(2, 4, 8)$

located in the regular octagon as indicated in Figures 2 and 3, where the fixed points of S and T are Weierstrass points of the Bolza curve D. Here as well as in the diagrams that follow we shade just one vertex in every orbit representing a cone point.

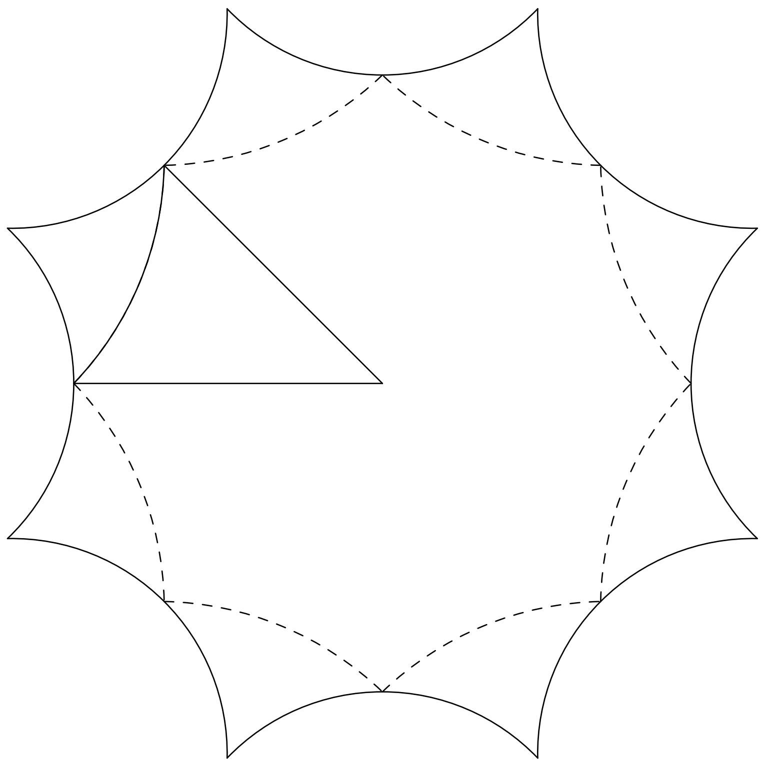

Figure 3.

The orbifold (2,4,8) is covered 2:1 by (0;2,2,2,4)

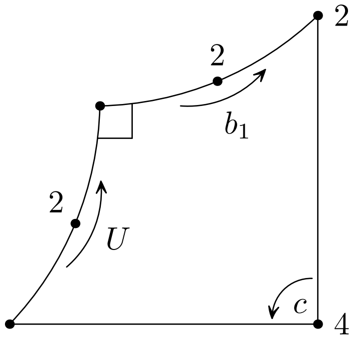

Figure 4. $(0; 2, 2, 2, 4)$

The generators of the group (0;2,2,2,4) are $c = S^2,\ U,\ b_1 = S^{-1}US$ and (2,4,8) is obtained by adjoining S to the group (0;2,2,2,4).

The orbifold (0;2,2,2,4) is covered 2:1 by (0;2,2,2,2,2)

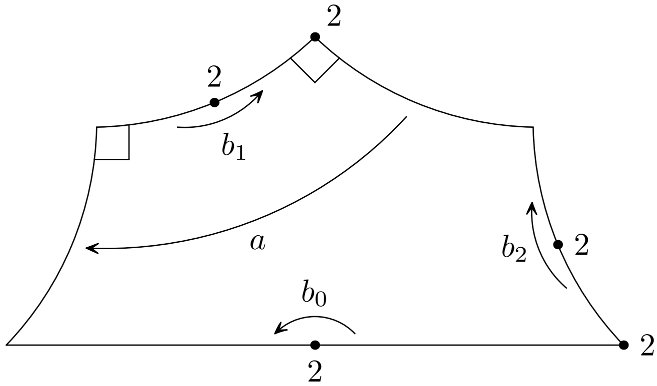

Figure 5. $(0; 2, 2, 2, 2, 2)$

The generators are $a = US^2,\ b_0 = S^4 = c^2,\ b_1 = S^{-1}US,\ b_2 = S^{-3}US^3$. The group (0;2,2,2,4) is obtained by adjoining S^2 to the group (0;2,2,2,2,2). The orbifold (0;2,2,2,2,2) can be also presented as

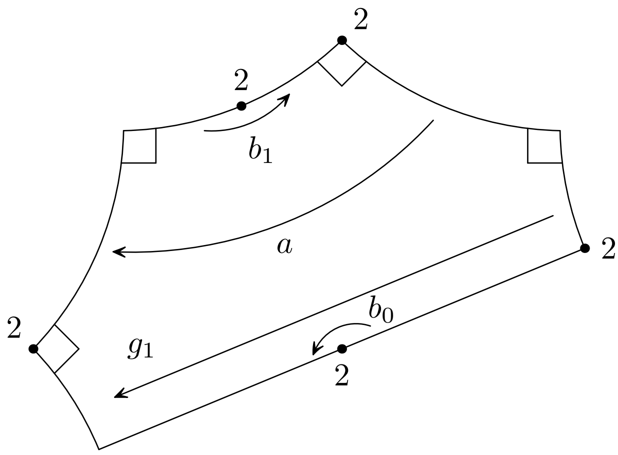

Figure 6. $(0; 2, 2, 2, 2, 2)$

where $g_1 = SUS^3$.

The orbifold (0;2,2,2,2,2) is covered 2:1 by $T_{22} = (1;2,2)$, presented as a regular octagon with all angles $\frac{\pi}{2}$ and two cone points of order 2.

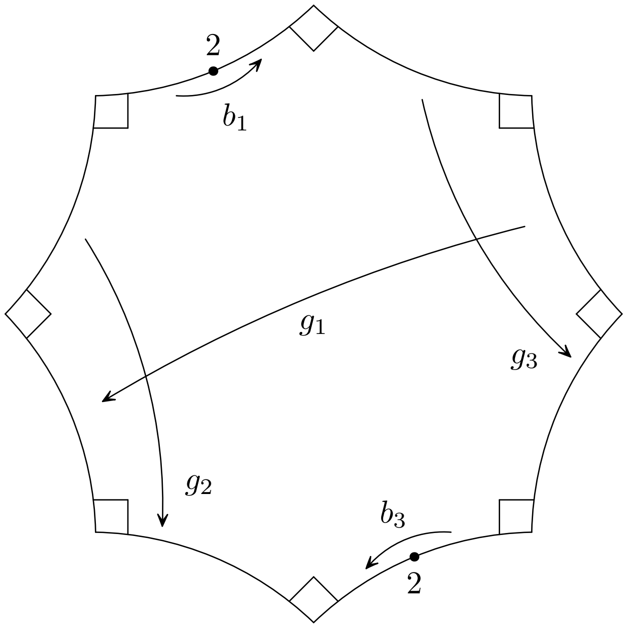

Figure 7. (1; 2, 2)

The generators of the covering group of T_{22} are

$$g_1 = SUS^3,\ \ g_2 = S^2U,\ \ g_3 = S^4US^2,\ \ b_1 = S^{-1}US,\ \ b_3 = S^3US^{-3},$$

and one gets the group (0;2,2,2,2,2) by adjoining S^4 to the group (1;2,2).

T_{22} is covered 2:1 by the regular octagon with all angles $\frac{\pi}{4}$ and diagonal pairings

$$f_1 = S^4T^2,\ f_2 = ST^2S^3,\ f_3 = S^{-2}T^2S^6,\ f_4 = S^3T^2S,$$

which generate the covering group $\Gamma = (2; 0)$ of the Bolza curve D. The group (1;2,2) is obtained by adjoining either b_1 of b_3 to the group (2;0). Since $T = S^{-1}U$, we can also write the generators of the genus 2 group as

$$f_1 = S^3US^{-1}U,\ f_2 = US^{-1}US^3,\ f_3 = S^{-3}US^{-1}US^6, f_4 = S^2US^{-1}US.$$

The subgroups (0;2,2,2,4), (0;2,2,2,2,2), (1;2,2) and (2;0) form a desending normal tower of (2,4,8). However, neither (0;2,2,2,2,2) nor (1;2,2) are normal in (2,4,8), for should (2,4,8) act on either of the surfaces (0;2,2,2,2,2) and (1;2,2), the orbit of the fixed point of U would contain points of different order. Cf. also the following example.

Example 2. The triangle group (2,4,8) which admits three conjugate embeddings in (2,3,8) cannot be a normal subgroup of (2,3,8). The non-normality of (2,4,8) can be reasoned also in a geometric way as follows:

Were (2,4,8) a normal subgroup of (2,3,8), the quotient group (2,3,8)/(2,4,8) would have an operation on the orbifold $S_{2,4,8}$ with $S_{2,3,8}$ as the quotient orbifold. But the group

(2,3,8)/(2,4,8) would operate by Möbius transformations on the pointed sphere $S_{2,4,8}$, and excepting the identity, no Möbius transformation can preserve the orders of the three cone points of $S_{2,4,8}$.

Example 3. It follows from Abu Osman and Rosenberger [1] that the (1;3) group $G = \langle a, b; [a, b]^3 \rangle$ contains a (2;0) subgroup $H = \langle x, y, u, v; [x, y][u, v] \rangle$ with generators $x = aba^{-1}$, $y = b^{-1}aba^{-2}, u = b^{-1}ab, v = b^2$. The group G can be given a presentation in the triangle group (2,4,12) with generators S, T, U, $U^2 = T^4 = S^{12} = I$, $STU = I$ by setting $a = T^{-2}U$, $b = T^{-1}UT^{-1}$. In this presentation the subgroup H has $x = T^{-2}S^3T^{-1}$, $y = TS^4T^{-1}ST^{-1}$, $u = TS^3$, $v = T^{-1}ST^{-1}S$ as generators and a regular hyperbolic 12-gon centered at the fixed point 0 of S as a fundamental polygon:

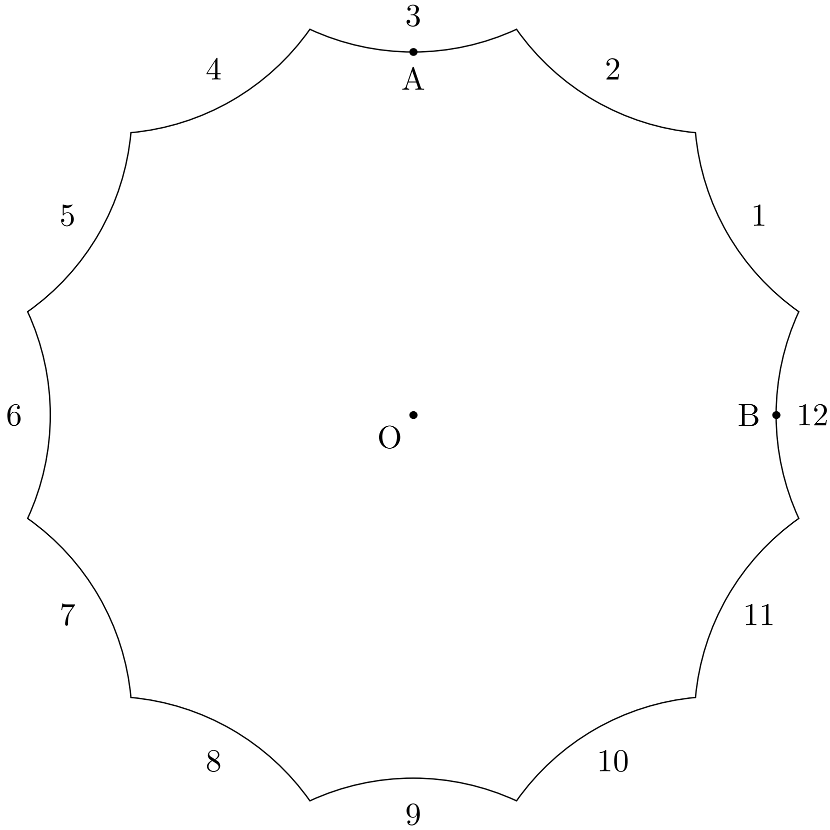

Figure 8.

Denoting by k the common vertex of the sides $k-1$ and k of the 12-gon we suppose further that U fixes the midpoint of side 1, T respectively fixing the vertex 2. It can easily be seen that H now identifies the side pairs $\{1, 11\}$, $\{2, 4\}$, $\{3, 9\}$, $\{5, 7\}$, $\{6, 12\}$, and $\{8, 10\}$, the group H thus has the identification pattern 12.6 of Figure 2 in Näätänen and Kuusalo [10]. We get the group G of the (1;3) surface by adjoining S^4 to the generators of H. However, it is immediately seen from the diagram 12.6 of Figure 4 in Kuusalo and Näätänen [6] that the rotation S^4 of order 3 does not preserve the set of Weierstrass points of the genus 2 surface determined by H, so that H cannot be a normal subgroup of G. But the rotation S^3 of order 4 is compatible with the identification pattern of H and thus belongs to the normalizer N of H in SL(2,R), generating with H the subgroup

$H' = \langle H, S^3 \rangle$ of (2, 4, 12). The fundamental polygon P' of the group H' is given in Figure 9:

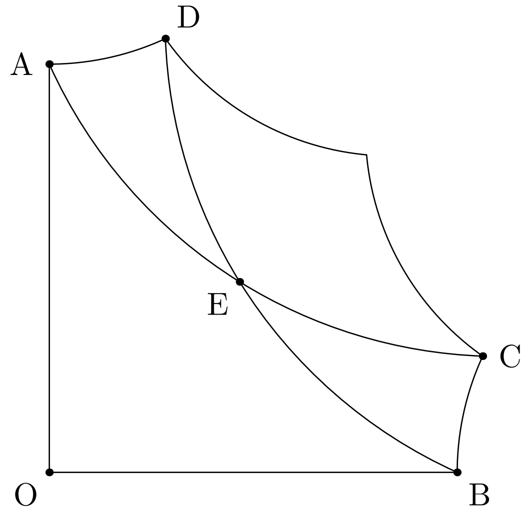

Figure 9.

Now P' admits a further rotation R of order 2 with the intersection point E of the geodesics AC and BD as a fixed point. Thus $N = \langle H, S^3, R \rangle$ is the normalizer of H in SL(2,R) with the dihedral automorphism group $D_4 = N/H$ of the (2;0) surface determined by the group H (case A on p. 404 in Kuusalo and Näätänen [5]). The domain in Figure 10

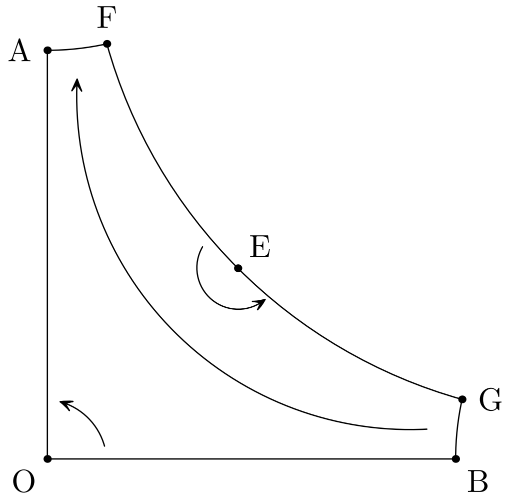

Figure 10.

where F and G are the midpoints of the boundary arcs AD and BC of P', is the fundamental polygon of the normalizer N having non-equivalent fixed points of order 2 at A, E and F, and respectively of order 4 at O. The hyperbolic distance ρ between A and F is quite small with $\cosh(\rho) = \frac{1}{2}\sqrt{3+\sqrt{3}} \approx 1.0877$, which prevents the normalizer N of being contained in any triangle group.

Remark. In the above pictures triangle groups and some symmetric Riemann surfaces determined by their subgroups are presented. By deforming the fundamental polygon of such a surface one gets examples of surfaces with less symmetry, where a corresponding part of the normal chain in PSL(2,R) containing the surface group is lost.

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