

41) Let

$$G = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \cong D_8$$

be the dihedral group of order 8. Determine the invariant ring $\mathbb{Q}[x_1, x_2]^G$ and its Hilbert series.

42) Let $G \leq \mathrm{GL}(3, \mathbb{Q})$ be the group of symmetries of a regular cube. Determine the conjugacy classes, and for each class the characteristic polynomial. With this information, determine the associated Molien series. What does it tell you about the invariant ring?

43) Show that Noether's degree bound is always tight for finite cyclic groups.

44) a) Let $n > 1$ and $G \leq S_n$ so that G contains a 2-cycle, an $n - 1$ -cycle, and an n -cycle. Show that $G = S_n$.
 b) Show that for every $n > 1$, there is an irreducible polynomial $p \in \mathbb{Q}[x]$, such that $\mathrm{Gal}(p) = \mathrm{Gal}(\mathrm{Spl}(p)/\mathbb{Q}) \cong S_n$.
 c) Construct explicitly a rational polynomial with Galois group S_6 .