26) a) Let \( \zeta_1, \ldots, \zeta_k \in \mathbb{T} \) be roots of unity (not necessarily different). Show that if \( \sum_{i=1}^{k} \zeta_i = k \) then \( \zeta_i = 1 \) for all \( i \). Show that if (absolute value) \( |\sum_{i=1}^{k} \zeta_i| = k \), then \( \zeta_i = \zeta_j \) for all \( i, j \).

b) Let \( G \) be a finite group and \( \chi \) be a character, associated to the representation \( \varphi: G \to \text{GL}_n(\mathbb{C}) \). We define the kernel of \( \chi \) as the subset \( \ker \chi = \{ g \in G \mid \chi(g) = \chi(1_G) \} \subseteq G \). Show that \( \ker \chi = \ker \varphi \).

c) We define the center of a character \( \chi \) as \( Z(\chi) = \{ g \in G \mid \chi(1) = |\chi(g)| \} \). Show that \( Z(\chi) \) is the set of \( g \in G \) such that \( \varphi(g) \in Z(\varphi(G)) \).

d) Show that if \( \chi = \psi + \rho \) for two characters \( \psi \) and \( \rho \), then \( \ker \chi = \ker \psi \cap \ker \rho \). Conclude that the character table of \( G \) determines the normal subgroups of \( G \) (consider the action on the cosets of a normal subgroup).

e) Show (using the regular representation) that

\[
Z(G) = \bigcap_{\chi \in \text{Irr}(G)} Z(\chi),
\]

thus the character table also determines the centre of \( G \).

27) Show that the sum over a column of a character table is an integer. (Note: it can be negative.)

28) Let \( G \) be a finite group with conjugacy classes \( K_1, \ldots, K_n \). We denote the class sums in \( \mathbb{C}G \) by \( C_i = \sum_{g \in C_i} g \). These class sums span the centre of the group algebra, they multiply according to the formula

\[
C_i C_j = \sum_k a_{i,j}^k C_k
\]

with \( a_{i,j}^k \in \mathbb{Z} \) (We have shown this in the lecture).

Show that

\[
a_{i,j}^k = \frac{|C_i||C_j|}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\overline{\chi(g_k)}}{\chi(1)}
\]

(Hint: use the second orthogonality relation and central characters.)

29) Let \( G \) be a finite group. Show:

a) If \( g \in G \) and \( x \in G \). There is a \( y \in G \) such that \( g \) is conjugate to \( [x, y] = x^{-1}y^{-1}xy \) if and only if

\[
\sum_{\chi \in \text{Irr}(G)} \frac{|\chi(x)|^2 \overline{\chi(g)}}{\chi(1)} \neq 0
\]

**Hint:** Assume that \( g \in C_g, x \in C_x \) and \( x^{-1} \in C_{x^{-1}} \) (conjugacy classes) and show that \( g \) is conjugate to \( [x, y] = x^{-1}y^{-1} \) if and only if \( C_x \cdot C_{x^{-1}} \) has a nonzero coefficient for \( C_g \). Then use central characters.

b) \( g \) is a commutator (i.e. there exist \( a, b \in G \) such that \( g = [a, b] \) if and only if

\[
\sum_{\chi \in \text{Irr}(G)} \frac{\chi(g)}{\chi(1)} \neq 0
\]
Bonus Problem  ) For a finite group $G$ we define $G' = \langle [a, b] = a^{-1}b^{-1}ab \mid a, b \in G \rangle \triangleleft G$ (the derived subgroup, it also is the smallest normal subgroup with abelian quotient). It is generated by commutators, but, as we will see, not every element is required to be a commutator.

Let $G = \langle (1, 2, 5)(3, 6, 11)(4, 7, 9)(8, 10, 12), (1, 8, 4, 3)(5, 12)(6, 10)(9, 11) \rangle$ be a certain group of order 96. Then (e.g. using GAP) we obtain the character table of $G$ as given below.

a) Determine in this table the classes which constitute $G'$.

b) Show using the criterion of problem 29b) that there is an element in $G'$ that is not a commutator (but only a product of commutators).

c) If $G$ is a simple group then clearly $G = G'$. The conjecture of Ore – recently being proven by Liebeck, O'Brien, Shalev and Tiep – states that for a simple group $G$ all elements are proper commutators. Verify this conjecture for $A_6$, using a character table (obtained from GAP, or whatever source you deem appropriate).

<table>
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<tr>
<th></th>
<th>1a</th>
<th>3a</th>
<th>4a</th>
<th>4b</th>
<th>2a</th>
<th>3b</th>
<th>6a</th>
<th>4c</th>
<th>4d</th>
<th>2b</th>
<th>6b</th>
<th>2c</th>
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<tbody>
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<td>1</td>
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<td>1</td>
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<td>1</td>
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<td>1</td>
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<td>1</td>
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<td>$\alpha$</td>
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<td>1</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>.</td>
<td>.</td>
<td>2</td>
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<tr>
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<td>$\bar{\beta}$</td>
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<td>$\beta$</td>
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<td>.</td>
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with $\alpha = e^{\frac{4\pi}{3}i}$ and $\beta = -1 + 2i$. 