41) Let F be a field and $x_1, \ldots, x_n$ be indeterminates. A polynomial $f \in F[x_1, \ldots, x_n]$ is called symmetric, if for any permutation σ of the indeterminates we have that $\sigma(f) = f$.
Obviously,the following functions (called elementary symmetric polynomials) are symmetric:

$$
\begin{aligned}
S_1(x_1, \ldots, x_n) &= x_1 + x_2 + x_3 + \cdots + x_n \\
S_2(x_1, \ldots, x_n) &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \\
S_3(x_1, \ldots, x_n) &= \sum_{i<j<k} x_1x_jx_k \\
&\vdots \\
S_n(x_1, \ldots, x_n) &= x_1x_2 \cdot \cdots \cdot x_n
\end{aligned}
$$

The aim of this exercise is to prove the fundamental theorem of elementary symmetric polynomials:
Theorem: Let $f \in F[x_1, \ldots, x_n]$ symmetric, then exists $g \in F[x_1, \ldots, x_n]$ such that

$$f(x_1, \ldots, x_n) = g\Big(S_1(x_1, \ldots, x_n), S_2(x_1, \ldots, x_n), \ldots, S_n(x_1, \ldots, x_n)\Big).$$

Example: $x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1^2 + 2x_1x_2 + 2x_1x_3 + x_2^2 + 2x_2x_3 + x_3^2 = S_1^2 + S_2 * S_3$.

We define an ordering on monomials by setting: $x_1^{d_1}x_2^{d_2}\cdots x_n^{d_n} > y_1^{e_1}y_2^{e_2}\cdots y_n^{e_n}$ if and only if for some i: $d_j = e_j$ for $j < i$ and $d_i > e_i$ (lexicographic comparison of the exponent vectors).

a) Suppose that $d_1 \geq d_2 \geq \cdots \geq d_n$ is a sequence of integers. Show that $x_1^{d_1}x_2^{d_2}\cdots x_n^{d_n}$ is the largest (with respect to this ordering) term of the polynomial

$$T_{d_1,\ldots,d_n}(x_1, \ldots, x_n) := S_1^{d_1-d_2-d_3-\cdots-d_n} \cdot \cdots \cdot S_{n-2}^{d_{n-2}-d_{n-1}-d_n} \cdot S_{n-1}^{d_{n-1}-d_n} \cdot S_{n-1}^{d_{n-1}-d_n} \cdot S_n^{d_n}$$

b) Suppose that $f(x_1, \ldots, x_n)$ is symmetric and $c \cdot x_1^{d_1}x_2^{d_2}\cdots x_n^{d_n}$ is the largest (with respect to this ordering) term of f and let $T_{d_1,\ldots,d_n}(x_1, \ldots, x_n)$ for these d_i as in a). Show that the largest monomial of $f(x_1, \ldots, x_n) - c \cdot T_{d_1,\ldots,d_n}(x_1, \ldots, x_n)$ is strictly smaller than $x_1^{d_1}x_2^{d_2}\cdots x_n^{d_n}$.

c) Show that by iterating the process in b) you can build the polynomial g as claimed in the theorem.

d) Express

$$
\begin{aligned}
f \;=\;& x^2y + x^2z + 3xyz + y^2x + y^2z + z^2x + yz^2 - 2x^3y^3z \\
& -4x^3y^2z^2 - 4x^2y^3z^2 - 2x^3z^3y - 4x^2y^2z^3 - 2xy^3z^3
\end{aligned}
$$

as a polynomial in the elementary symmetric polynomials in x, y and z.