22) We form a semidirect product $P = N \rtimes S$ for the map $\varphi\colon S \to \mathrm{Aut}(N)$. Show that $P = N \times T$ (then obviously $T \cong S$, but not necessarily $T = S$) if and only if $\varphi(S) \le \mathrm{Inn}(N)$.

23) Let V be a vector space. We consider the set L of *lines* (that is subsets of V of the form $\underline{\mathbf{a}} + \lambda\underline{\mathbf{b}}$ for $\underline{\mathbf{a}}, 0 \ne \underline{\mathbf{b}} \in V$. Two lines are parallel if their $\underline{\mathbf{b}}$-vectors are linearly dependent. We call $A = (V, L)$ an *affine space*.
Let G be the group of all *affine transformations* of V, that is bijective maps $A \to A$ that map lines to lines and preserve parallelity.
a) Let $G \ge T = \{t_{\underline{\mathbf{a}}} = (\underline{\mathbf{x}} \mapsto \underline{\mathbf{x}} + \underline{\mathbf{a}}) \mid \underline{\mathbf{a}} \in V\}$ be the set of *translations* and $L = \mathrm{GL}(V)$ the set of linear transformations of V. Show that $T \lhd G$, $L \le G$ and $T \cap L = \langle 1 \rangle$.
b) Show that G is the semidirect product of T with L.

24) a) Let $p > q$ prime such that $p \not\equiv 1 \pmod{q}$. Show that there is only one group of order pq up to isomorphism (namely the cyclic group).
b) Let $p > q$ prime such that $p \equiv 1 \pmod{q}$. Show that there are (up to isomorphism) two groups of order pq, the cyclic group and a nonabelian semidirect product $Z_p \rtimes Z_q$.
c) Show that every group of order p^2 (for p prime) is abelian. (**Hint:** Show that if $G/Z(G)$ is cyclic, then G is abelian.)
This shows that $Z_p \times Z_p$ and Z_{p^2} are the two groups of order p^2.

25) a) Let G be a nonabelian group of order 12. Show that G must be a semidirect product of a 2-Sylow subgroup with a 3-Sylow subgroup or vice versa.
b) Classify the possible (nonabelian) semidirect products of C_3 with C_4.
c) Classify the possible (nonabelian) semidirect products of C_3 with $C_2 \times C_2$.
d) Classify the possible (nonabelian) semidirect products of C_4 with C_3.
e) Classify the possible (nonabelian) semidirect products of $C_2 \times C_2$ with C_3.
f) Show that there are (up to isomorphism) three nonabelian groups of order 12.