Mathematics 567 **Homework (due Feb 21)** A. Hulpke

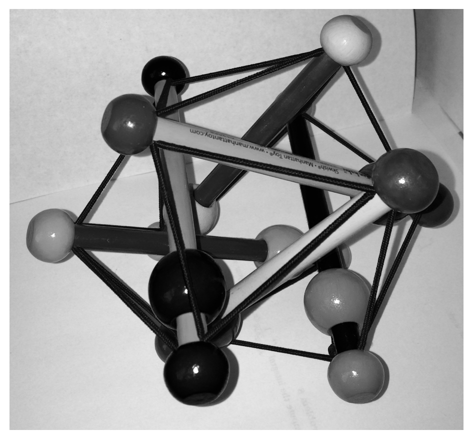
18) Let $|G| = p^a$ for p prime. Show:

1. The *upper central series*, defined by $U_1 = Z(G)$, $U_{i+1}/U_i = Z(G/U_i)$ ascends to G.
2. The *lower central series*, defined by $L_1 = G$, $L_i = [G, L_{i-1}] = \langle [g,h] \mid g \in G, h \in L_{i-1}\rangle$ descends to the trivial subgroup.
3. For every subgroup $U \le G$ we have that $N_G(U)$ is strictly larger than U.

(Note: One can show that these properties are mutually equivalent and in turn imply that G is the direct product of its Sylow subgroups. Such groups are called *nilpotent*)

19) We define $\mathrm{PSL}_n(q) = \mathrm{GL}_n(q)/Z$ with $Z = \{a \cdot I \mid a \in F_q^*\}$. Show, by considering suitable actions, that $A_5 \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$.

20) Let G the group of rotations of an icosahedron.
a) Let $S \le G$ be the stabilizer of a face. Show that $|S| = 3$ and $[G : S] = 20$, so $|G| = 60$.
b) Now consider a configuration of 3 pairs of opposite edges such that the lines connecting the mid points of the opposite edges are mutually orthogonal. (The picture on the right illustrates one such a configuration afforded by the inner struts, connecting pairs of opposite edges.) Show that there are 5 such configurations and that this allows us to consider G as a subgroup of S_5.
c) Using that A_5 is simple nonabelian, show that A_5 is the only subgroup of S_5 of order 60 and conclude that $G \cong A_5$.
d) Show that there is a nontrivial homomorphism $A_5 \to O_3(\mathbb{R})$, the group of orthogonal transformations of $\mathbb{R}^3$.



21) Show that every group of order < 60 that is not of prime order has a nontrivial normal subgroup. (By induction this shows that every group of order < 60 is solvable.)
Hint: Consider the case that $|G| = p^n q$ with $p \neq q$ prime generically.