Mathematics 567 **Homework (due Feb 14)** A. Hulpke

13) Let G be finite and solvable, i.e. there exists a subnormal series $\langle 1\rangle = G_0 \lhd G_1 \lhd \cdots \lhd G_m = G$ with G_{i+1}/G_i is cyclic of prime order.
Show that there exists a (possibly different) subnormal series $\langle 1\rangle = N_0 \lhd N_1 \lhd \cdots \lhd N_k = G$ with $N_1, \ldots N_k \lhd G$ and $N_{i+1}/N_i \cong C_{p_i}^{e_i}$. (That is the group is composed from vector spaces of varying dimensions over finite fields of different characteristic.)

14) Let G be a group and $S \le G$. Show that there is a homomorphism $\varphi\colon N_G(S) \to \mathrm{Aut}(S)$.

15) A subgroup $C \le G$ is called *characteristic* if $C^\alpha = C$ for all $\alpha \in \mathrm{Aut}(G)$.
a) Show that any characteristic subgroup must be normal.
b) Give an example of a normal subgroup that is not characteristic. (Hint: consider $G = C_2 \times C_2$.)
c) Show that if C is characteristic in G then there are homomorphisms $\varphi\colon G \to \mathrm{Aut}(G/C)$ and $\rho\colon G \to \mathrm{Aut}(C)$. (Note: $\ker\varphi \cap \ker\rho$ does not need to be trivial!)
d) Let $G = D_8 = \langle (1,2,3,4), (1,3)\rangle$ and $C = \langle (1,3)(2,4)\rangle$. Show that C is characteristic in G. (You can do so without any concrete knowledge of $\mathrm{Aut}(G)$.)

16) Let C_n be the cyclic group of order n. Show (very useful facts):
a) $\mathrm{Aut}(C_n) \cong (\mathbb{Z}_n)^\times$, that is the multiplicative units modulo n. (Thus in particular $|\mathrm{Aut}(C_n)| = \varphi(n)$.)
b) If $n = a \cdot b$ with $\gcd(a,b) = 1$ then $\mathrm{Aut}(C_n) \cong \mathrm{Aut}(C_a) \times \mathrm{Aut}(C_b)$.
c) If $p > 2$ prime, then $\mathrm{Aut}(C_{p^a}) \cong C_{(p-1)p^{a-1}}$. (**Hint:** Show that:

$$(1+p)^{p^k} \equiv 1 + p^{k+1} \mod p^{k+2}$$

and use this to conclude that $1 + p \in \mathbb{Z}_{p^a}^\times$ has order p^{a-1}. Also show that if $\mathbb{Z}_p^\times = \langle x\rangle$ then $x^{p^{a-1}}$ has order $p-1$ in $\mathbb{Z}_{p^a}$. Thus $x^{p^{a-1}}(1+p)$ is a generator ot $\mathbb{Z}_{p^a}^\times$.)
d) If $a \ge 3$ we have that $\mathrm{Aut}(C_{2^a})$ is not cyclic. (**Hint:** Work modulo 8)
Note: One can show similar to c) that 5 has order 2^{a-2} in $\mathbb{Z}_{2^a}^\times$, thus $\mathrm{Aut}(C_{2^a}) \cong C_2 \times C_{p^{a-2}}$.

17) a) Assume that G acts on Ω and that $N \lhd G$ with $[G:N] = 2$. Show that for $\omega \in \Omega$ either

- $\omega^N = \omega^G$ and $[\mathrm{Stab}_G(\omega) : \mathrm{Stab}_N(\omega)] = 2$, or
- $\mathrm{Stab}_G(\omega) \le N$ and for $\Delta = \omega^N$ we have that $\omega^G = \Delta \cup \Delta^g$ for $g \in G - N$ (here Δ^g is the set-wise image), so in particular $|\omega^G| = 2\,|\omega^N|$.

b) Determine (i.e. class representatives and class orders) the conjugacy classes of A_5, using the conjugacy classes of S_5.
c) A normal subgroup of a group is the union of conjugacy classes. Show (with class orders from b) that A_5 cannot have any nontrivial normal subgroup (and thus is simple).