## Mathematics 567

7) For a finite group *G* let  $\exp(G) = \operatorname{lcm}_{g \in G} |g|$ .

a) Show that  $\exp(G)||G|$  and that *G* is cyclic if and only if it is abelian and  $\exp G = |G|$ . (Note:  $\exp(S_3) = 6 = |S_3|$ , but  $S_3$  is not cyclic.)

b) Let *G* be a finite subgroup of the multiplicative group of a field. Show that all elements of *G* must be roots of  $x^{\exp(G)} - 1$  and conclude that  $|G| \le \exp G$ . Conclude that *G* is cyclic.

Thus the multiplicative group of any finite field is cyclic and that any finite group of cyclotomic numbers is cyclic.

8) For given *n*, *q* consider  $G = \operatorname{GL}_n(q) = \operatorname{GL}_n(\mathbb{F}_q)$  where  $\mathbb{F}_q$  is the field with *q* elements. a) Let  $M = \{a \cdot I \mid a \in \mathbb{F}_q^*\}$ . Show that M = Z(G). b)  $N = SL_n(q)$ . Show that  $G/N \cong F_q^*$ . c) What is [G : NM] and  $|N \cap M|$ ?

**9)** A finite group *G* is called *solvable* (this is one of a number of possible definitions) if there is a subnormal series subgroups  $G = S_0 \triangleright S_1 \triangleright S_2 \triangleright \cdots \triangleright S_{k-1} \triangleright S_k = \langle 1 \rangle$  with  $S_i \triangleleft S_{i-1}$  and  $S_{i-1}/S_i$  is cyclic.

a) Show that any subgroup of a solvable group is solvable.

b) Show that any factor group of a solvable group is solvable.

c) Let  $N \triangleleft G$ . Show that if N and G/N are solvable, then G is solvable.

**10)** Assume that  $H \triangleleft G$  with [G : H] = p prime. Show that for all  $K \leq G$  either  $K \leq H$ , or G = HK with  $[K : K \cap H] = p$ .

11) A subgroup *H* of a finite group *G* is called a *Hall subgroup* if gcd(|H|, [G : H]) = 1. Prove that if *H* is a Hall subgroup of *G* and  $N \triangleleft G$  then  $H \cap N$  is a Hall subgroup of *N*, and HN/N is a Hall subgroup of *G*/*N*.

12) The easter formula of Gauß provides a way for calculating the date of easter (i.e. the first Sunday after the first full moon of spring) in any particular year. For the period 1900-2099 (a restriction due to the use of the Georgian calendar) a simplified version for year Y is:

- Calculate  $a = Y \mod 19$ ,  $b = Y \mod 4$ ,  $c = Y \mod 7$ .
- Calculate  $d = (19a + 24) \mod 30$  and  $e = (2b + 4c + 6d + 5) \mod 7$
- If d + e < 10 then easter will fall on the (d + e + 22)th March, otherwise the (d + e 9)th of April.
- Exception: If the date calculated is April 26, then easter will be April 19 instead. (There is a further exception in western (i.e. not orthodox) churches for certain April 25 dates, again due to the Georgian calendar, which we ignore for simplicity.)

a) Calculate the easter day for 2014 and for 1968.

b) What are the earliest/latest days for easter during 1900-2099? (Note: Outside this interval an even earlier date is possible.)

c) Assuming the formula would work for any year (not just 1900-2099), what is the smallest period, after which we can guarantee that the sequence of dates repeats?