54) Let R be a PID and $A \in R^{n\times m}$. For $1 \le k \le \min(m,n)$, let d_k be the gcd of the determinants of all $k \times k$ submatrices of A. (That is we choose any k rows and any k columns and take the matrix given by these indices.) d_k is called the k-th determinant divisor of A.
a) Show that if r is the rank of A (as matrix over the field Frac(R)) then $d_1, \ldots, d_r \neq 0$ but $d_k = 0$ for $k > r$.
b) Show that d_k divides d_{k+1}.

55) Let R be an integral domain and $\underline{\mathbf{a_1}}, \ldots, \underline{\mathbf{a_n}}, \underline{\mathbf{b}} \in R^m$. We consider matrices as collections of column vectors. Show that the determinant is multilinear, that is:

$$\det\left(\underline{\mathbf{a_1}} + \lambda\underline{\mathbf{b}}, \underline{\mathbf{a_2}}, \ldots, \underline{\mathbf{a_n}}\right) = \det\left(\underline{\mathbf{a_1}}, \underline{\mathbf{a_2}}, \ldots, \underline{\mathbf{a_n}}\right) + \lambda \det\left(\underline{\mathbf{b}}, \underline{\mathbf{a_2}}, \ldots, \underline{\mathbf{a_n}}\right)$$

56) Let R be a PID and $A \in R^{n\times m}$ and d_k as in problem 54.
a) Show that R-invertible row or column operations on A will change the value of d_k only by units. (Hint: The submatrices and their determinants change (even as a set), but the gcd stays the same. The only nontrivial case is that of a submatrix to which we add a row/column from outside the submatrix. Use problem 55).
b) Show that (up to multiplication by units) the k-th elementary divisor of A is $d_1 \cdots d_k$.
(This shows the uniqueness of the Smith Normal Form up to associates!)

57) The GAP command `ElementaryDivisorsTransformationsMat` determines elementary divisors and transforming matrices for a matrix over a Euclidean ring.
(There also is `SmithNormalFormIntegerMatTransforms`.) Let

$$A := \begin{pmatrix} -1717 & 206 & -196 & 574 & 4 \\ 18 & -1 & 4 & -6 & 0 \\ 18 & -2 & 5 & -6 & 0 \\ -5154 & 618 & -588 & 1723 & 12 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Determine the elementary divisors of the characteristic matrix of A (i.e. of $A - xI$) and the characteristic polynomial of A.

58) Let F be a free $\mathbb{Z}$-module on the generators x,y,z, and $S \le F$ be generated by

$$1040x + 1372y - 2804z$$
$$-2602x - 3462y + 7018z$$
$$-438x - 590y + 1182z$$
$$1620x + 2140y - 4368z$$

Determine the structure of S and of F/S as $\mathbb{Z}$ modules.