Mathematics 567 **Homework (due Apr 18)** A. Hulpke

48) Let F be a field and $U, V \leq F^n$. Let $A \in F^{m\times n}$ be a matrix whose rows form a basis of U, $B \in F^{k\times n}$ a matrix whose rows form a basis of V. Suppose that the RREF of the matrix

$$\begin{pmatrix} A & A \\ B & 0 \end{pmatrix} \text{ is } \begin{pmatrix} X & * \\ 0 & Y \end{pmatrix},$$

with $X \in F^{a\times n}$, $Y \in F^{b\times n}$ (that is the block split is possibly in a different row but the column split is the same). Show that the rows of X form a basis of $\langle A, B\rangle$ and the rows of Y a basis of $A \cap B$.

49) Let F be a field, V an F-vector space and $T: V \to V$ a linear map. We set $R = F[x]$.
a) Show that V becomes an R module via $f(x).v := f(T)(v)$.
b) If V is finite dimensional and T is diagonalizable, what are the R-submodules of V?

50) Let V be a vector space over a field K of countably infinite dimension with basis $e_0, e_1, e_2, \ldots$. Let $R = \mathrm{End}(V)$ the ring of linear transformations $V \to V$. We define $\alpha, \beta \in R$ by

$$\alpha: e_i \to \begin{cases} e_{\frac{i}{2}} & i \text{ even} \\ 0 & \text{otherwise} \end{cases}, \qquad \beta: e_i \to \begin{cases} e_{\frac{i-1}{2}} & i \text{ odd} \\ 0 & \text{otherwise} \end{cases}.$$

a) Show that $\{\alpha, \beta\}$ is a basis of ${}_RR$.
b) Show that $\{1\}$ is a basis of ${}_RR$

51) a) Let

$$A := \begin{pmatrix} 60 & -60 & 48 & -232 \\ -9 & 9 & -6 & 36 \\ -21 & 21 & -18 & 80 \end{pmatrix} \in \mathbb{Z}^{3\times 4}$$

Determine the Smith Normal Form of A as well as transforming matrices P and Q.
b) Let $\underline{\mathbf{b}} := (-76, 15, 23)^T$. Determine all integer solutions to the system $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$.

52) Let A be a diagonal matrix with entries $d_1, d_2, \ldots d_m$ over a PID R. What is the Smith Normal Form of A?

53) Let $A \in \mathbb{Z}^{m\times n}$ be a matrix and $S = P \cdot A \cdot Q$ its Smith Normal Form (which is unique). Show that the transforming matrices P and Q are not unique. (Hint: Consider for a square A a centralizing matrix B with $A = B^{-1}AB$.)