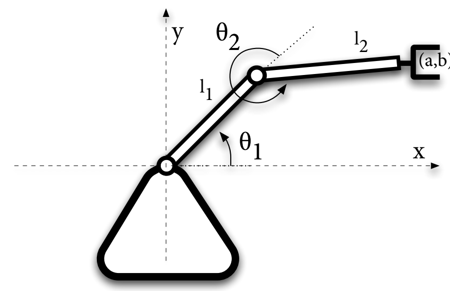
**42)** Consider the parametric curve

$$x = \frac{1-t^2}{1+t^2}, \qquad y = \frac{2t}{1+t^2}.$$

Describe this curve by polynomials in $x$,$y$, and $t$. By eliminating $t$, determine a polynomial in $x$ and $y$ describing the curve and use this result to identify the curve.

**43)** Consider a (2-dimensional) robot arm as depicted.
We want to find out the angles $\theta_1$, $\theta_2$ to which the joints have to be set to move the hand to coordinates $(a,b)$. For simplification, assume the two arms have length $l_1 = 3$, $l_2 = 4$. To avoid using trigonometric functions, set $s_i = \sin(\theta_i)$, $c_i = \cos(\theta_i)$ $(i = 1,2)$. Then $s_i^2 + c_i^2 = 1$.
Write down equations that determine $a, b$ in terms of the variables $c_1, s_1, c_2, s_2$. (You will have to use the formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$.)




**44)** Let $R = \mathbb{Q}[x,y,z]$ and $I = \langle x^2 + yz - 2, y^2 + xz - 3, xy + z^2 - 5\rangle \lhd R$. Show that $x + I$ is a unit in $R/I$ and determine $(x+I)^{-1}$.

**45)** a) Let $F$ be a field and $R = F[x_1,\cdots,x_n]$ and let $I_1 = \langle f_1,\ldots,f_k\rangle \lhd R$ and $I_2 = \langle h_1,\ldots,h_r\rangle \lhd R$ be two ideals. Let $S = F[x_1,\cdots,x_n,t]$ (considering $R \subset S$) and set

$$J = \langle t\cdot f_1,\ldots,t\cdot f_k,(1-t)\cdot h_1,\ldots,(1-t)\cdot h_r\rangle \lhd S.$$

Show that $I_1 \cap I_2 = J \cap R$.
b) Let $f = x^3z^2 + x^2yz^2 - xy^2z^2 - y^3z^2 + x^4 + x^3y - x^2y^2 - xy^3$ and $g = x^2z^4 - y^2z^4 + 2x^3z^2 - 2xy^2z^2 + x^4 - x^2y^2$. Compute $\langle f\rangle \cap \langle g\rangle$.
c) Compute $\gcd(f,g)$. (Hint: Show that $\langle f\rangle \cap \langle g\rangle = \langle \mathrm{lcm}(f,g)\rangle$.)

**46)** (Intended to illustrate the reason for having different monomial orderings.)
Let $I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1\rangle$. Compute a Gröbner basis for $I$ with respect to the *lex*, *grlex*, and *grevlex* orderings. Compare.
Repeat the calculations for $I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1\rangle$ (only one exponent changed!)

**47)** Let $f\colon \mathbb{R}^n \to \mathbb{R}$. A point $x \in \mathbb{R}^n$ is called a *critical point* of $f$, if $\frac{\partial f}{\partial x_i}(x) = 0$. (Cf. Calculus 3.)
Determine all critical points of the function

$$f(x,y) = (x^2+y^2)^3 - 4x^2y^2$$

Note: this curve is the "four-leaved flower" $r = \sin(2\theta)$ in polar coordinates. Is there a geometric interpretation of the critical points?