Mathematics 567 **Homework (due Jan 31)** A. Hulpke

1) Let $a = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{2} \end{pmatrix}$ and $b = \begin{pmatrix} \bar{2} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix}$, where the bar denotes residue classes in $\mathbb{Z}/3\mathbb{Z}$. Let $G = \langle a, b \rangle$ be the group generated by a and b.
Show that $|G| = 8$ and that $G \not\cong D_8$ (where D_8 is the symmetry group of the square).

2) Let G be a group in which all elements have order 2. Show that G is abelian.

3) Let G be a group and $U \le G$ of index 2 or 3. Show that there exists $N \lhd G$ with $[G : N] \in \{2, 3\}$. (Hint: Consider the action on the cosets of U and its image in S_3.)

4) Let G be a group and $H, K \le G$ with $H \le K \le G$.
a) Let $\{r_i\}$ a set of representatives of the cosets of K in G (Right Cosets: Kx) and $\{s_j\}$ a set of representatives of the cosets of H in K. Show that the products $\{s_j \cdot r_i\}$ form a set of representatives for the cosets of H in G.
b) Show that if $[G : H]$ is finite, then $[G : H] = [G : K][K : H]$. (You cannot assume that any of the groups if finite and thus cannot simply use the quotient $|G| / |K|$.)

5) Let F be a field with q elements and $G = \mathrm{SL}_2(F)$. For $a \in F$, $b \in F^* = F - \{0\}$ let $u(a, b) = \begin{pmatrix} b & a \\ 0 & b^{-1} \end{pmatrix} \in G$. Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G$. We consider

$$\begin{aligned} B &= \{u(a, b) \mid a \in F, b \in F^*\} && \text{called } \textit{Borel subgroup} \\ U &= \{u(a, 1) \mid a \in F\} && \text{called } \textit{unipotent subgroup} \\ T &= \{u(0, b) \mid b \in K^*\} && \text{called } \textit{a (maximal) Torus} \end{aligned}$$

a) Show that $B, U, T \le G$, $U \lhd B$, $B = UT$ (i.e. every element of B can be written in the form $u \cdot t$) and $U \cap T = \langle 1 \rangle$.
b) Show that every $g \in G$ either fulfills that $g \in B$ or can be written uniquely in the form $g = b \cdot w \cdot u$ with $b \in B$, $u \in U$. Based on this, describe a set of representatives for the cosets of B in G.
c) Determine $|G|$ based on $[G : B]$, $[B : U]$ and $|U|$.

6) Let G be a group. For $x, y \in G$ let $[x, y] = x^{-1}y^{-1}xy$ be the *commutator* of x and y. (The name stems from the fact that $xy = yx$ if and only if $[x, y] = 1$.)
Define the *Commutator subgroup* (or *Derived subgroup*) as

$$G' := \left\langle [x, y] = x^{-1}y^{-1}xy \mid x, y \in G \right\rangle \le G,$$

thus every element of G' is a product of elements of the form $[x_i, y_i]$.
a) Prove that $G' \lhd G$.
b) Prove that G/G' is abelian.
c) If $N \lhd G$ with G/N abelian, prove that $G' \le N$.