Mathematics 567 **The subgroups of S_5** A. Hulpke

Using the generator notation, the Sylow theorems and a small amount of computation, we can determine all subgroups of the symmetric group on 5 letters up to conjugacy. The possible subgroup orders are the divisors of 120: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120.
We now consider all orders and describe the possible groups. Behind each group we give the number of (conjugate) groups of this type in square brackets.

1,120 trivial

60 must be normal (Index 2) and therefore an union of classes. There is only one such possibility, namely A5.

2 Cyclic (1,2) [10 = $\binom{5}{2}$] and (1,2)(3,4) [15 = $\frac{\binom{5}{2}\cdot\binom{3}{2}}{2}$]

5 Cyclic (1,2,3,4,5), Sylow subgroup, (Only this class, number is ¿1, only divisor of 120 that is congruent 1 mod 5 is 6) [6 subgroups]
Therefore: Let S be the normalizer of a 5-Sylow subgroup: $[G : S] = 6$, so $|S| = 20$. So there are 6 subgroups of order 20, which are the Sylow normalizers.

20 Vice versa. If $|H| = 20$, the number of 5-Sylow subgroups (congruence and dividing) must be 1, so the **only groups of order 20 are the Sylow normalizers** $\langle (1,2,3,4,5),(2,3,5,4)\rangle$.

10 Ditto, a subgroup of order 10 must have a normal 5-Sylow subgroup, so they must lie in the 5-Sylow normalizers, and are normalized by these. The 5-Sylow normalizers have only one subgroup of order 10: [6 subgroups of order 10]

15 Would have a normal 5-Sylow subgroup, but then the 5-Sylow nomalizer would have to have an order divisible by 3, which it does not – no such subgroups.

40 Would have a normal 5-Sylow subgroup, but then the 5-Sylow nomalizer would have to have order 40, which it does not – no such subgroups.

8 2-Sylow subgroup, all conjugate. We know that $[S_5 : S_4] = 5$ is odd, so a 2-Sylow of S_4 is also 2-Sylow of S_5. $D_8 = \langle (1,2,3,4),(1,3)\rangle$ fits the bill. Number of conjugates: 1,3,5 or 15 (odd divisors of 120). We know that we can choose the 4 points in 5 ways, and for each choice there are 3 different D_8's: [15 subgroups] We therefore know that a 2-Sylow subgroup is its own normalizer.

4 Must be conjugate to a subgroups of the 2-Sylow subgroup. Therefore the following three are all possibilities:

Cyclic $\langle(1,2,3,4)\rangle$ [15 = 5 · 3: choose the point off, then there are 3 subgroups each]

not cyclic $\langle(1,2),(3,4)\rangle$ [15, ditto]

not cyclic $\langle(1,2)(3,4),(1,3)(2,4)\rangle$ [5 copies: choose the point off]

3 Cyclic $\langle(1,2,3)\rangle$ [$\binom{5}{3}$ = 10 subgroups as (1,2,3) and (1,3,2) lie in the same group] Also Sylow, thus the 3-Sylow normalizer must have index 10, order 12.

6 A group of order 6 must have a normal 3-Sylow subgroup, and therefore lie in a 3-Sylow normalizer, which is (conjugate to) $\langle(1,2,3),(1,2),(4,5)\rangle$. We can find $3\times 2 = \langle(1,2,3)(4,5)\rangle$ (cyclic) [$\binom{5}{3}$ = 10 groups], $S_3 = \langle(1,2,3),(1,2)\rangle$ [$\binom{5}{3}$ = 10 copies]) and $\langle(1,2,3),(1,2)(4,5)\rangle$ (isomorphic S_3) [$\binom{5}{3}$ = 10 copies].

30 Can have 1 or 10 3-Sylow subgroups. If 1, it would be in the Sylow normalizer, contradiction as before. If it is 10, it contains all 3-Sylow subgroups, but $\langle(1,2,3),(3,4,5)\rangle$ has order 60, Contradiction.

12 If it has a normal 3-Sylow subgroup, it must be amongst the 3-Sylow normalizers $\langle(1,2,3),(1,2),(4,5)\rangle$ [We know already there are 10 such groups].
If not, it contains 4 3-Sylow subgroups, i.e. 8 elements of order 3. Thart leaves space only for 3 elements of order 2, the 2-Sylow subgroup (or order 4) therefore must be normal, and its normalizer has index at most 10, i.e. it has at most 10 conjugates. This leaves $\langle(1,2)(3,4),(1,3)(2,4)\rangle$, whose normalizer is S_4. (its normal in S_4, and S_4 has the right order). Inspecting S_4, we find it has only one subgroup of order 12, namely $A_4 = \langle(1,2,3),(2,3,4)\rangle$ with [5, same as the number of S_4's] conjugates.

24 We know there are [5] copies of S_4, namely the point stabilizers. We now want to show that is all: (There are 15 groups of order 24, so we can't that easily determine the structure alone from the order.) Its 2-Sylow subgroup must be a 2-Sylow of S_5, so WLOG its 2-Sylow is D_8. Now consider the orbit of 1 under this subgroup. It must have length at least 4 (that's under D_8), but the orbit length must divide 24. So it cannot be 5. But that means that the subgroup fixes one point and thus is S_4.