Mathematics 502 **Homework (due Feb 1)** A. Hulpke

1) Let G be the group of (rotational) symmetries of a regular tetrahedron. Determine its orbits on edges, and on pairs of edges. (The tetrahedron has 6 edges, thus there are $\binom{6}{2} = 15$ pairs).

2) Let Ω be the set of connected, labelled, graphs on 4 vertices. (For example $1-2-3-4$ is such a graph.) Determine the orbits of S_4 on Ω, acting by renumbering vertices.

3) Let $\Omega = \{0, 1, 2, 3, 4\}$ and let a act on Ω by addition of 1 modulo 5, b act on Ω by multiplication with 2 modulo 5, and $G = \langle a, b \rangle$. (Both are affine transformations, thus they lie in a group.)
a) Write down permutations for a and b acting on Ω.
b) Draw the Schreier graph for the action of G on Ω.

4) Let G be a group and $H, K \leq G$. Show that the action of G on the cosets of H is equivalent (where we choose the homomorphism $G \to G$ to be the identity) to the action of G on the cosets of K (i.e. there is a bijection ψ on the cosets such that $\psi((Hx)^g) = \psi(Hx)^g)$ if and only if H and K are conjugate in G, that is there exists $g \in G$ such that $K = H^g = g^{-1}Hg = \{g^{-1}hg \mid h \in H\}$.

5) The Figure below depicts an unfolded icosahedron with its faces labelled from 1 to 20.
a) Show that the group G of rotational symmetries of the icosahedron has order 60.
b) Show that the permutations

$$\begin{aligned} p_1 &= (1,6,8,4,10)(2,5,17,13,9)(3,7,15,12,19)(11,20,14,18,16), \\ p_2 &= (1,13,19,9,6)(2,4,17,18,11)(3,8,10,12,16)(5,15,14,20,7) \end{aligned}$$

represent rotational symmetries of this icosahedron along different axes.
c) Show that $p_1 \notin \langle p_2 \rangle$ and $p_2 \notin \langle p_1 \rangle$.
d) [For those who had abstract algebra] Show that any group of order $5 \mid n \mid 60$, $n < 60$ must have a normal 5-Sylow subgroup. (**Hint:** In the case of order 30, show that if there were six 5-Sylow subgroups, the 3-Sylow subgroup must be normal. Then the subgroup generated by the 3-Sylow subgroup and a 5-Sylow subgroup must have order 15, thus be normal, and thus contain all six 5-Sylow subgroups, which is impossible, since 15 is not a multiple of 6.)
Conclude that p_1 and p_2 must generate the full group of all rotational symmetries.