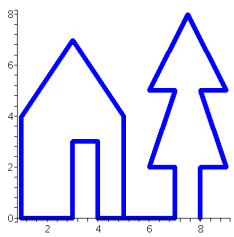
Mathematics 369 **Homework (due Mar 8)** A. Hulpke

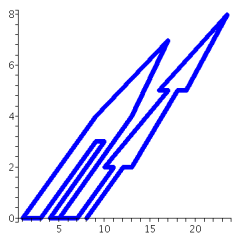
33) Let $V = \mathbb{R}^2$ with the standard basis $\mathcal{S}$ and $L: R^2 \to \mathbb{R}^2$ given by ${}_{\mathcal{S}}[L]_{\mathcal{S}} = \begin{pmatrix} -3 & -8 \\ 10 & 27 \end{pmatrix}$.

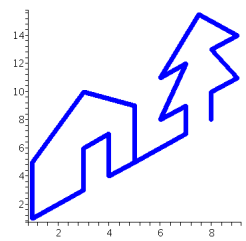
a) What is $L((5,3)^T)$?
b) Let $B = \{(1,1)^T, (2,3)^T\}$. Determine ${}_{\mathcal{B}}[L]_{\mathcal{B}}$.
c) For $\underline{\mathbf{v}} = \mathcal{B} * (3,4)^T$, compute $[L(\underline{\mathbf{v}})]_{\mathcal{B}}$.

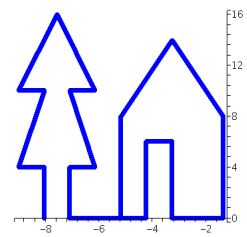
34) Let $L: \mathcal{P}_4 \to R^2$, $f(x) \mapsto (f(1), f(2))^T$.
a) For $\mathcal{B} = \{1, x, x^2, x^3\}$ and $\mathcal{S}$ the standard basis of $\mathbb{R}^2$, determine ${}_{\mathcal{S}}[L]_{\mathcal{B}}$.
b) Determine bases for $N(L)$ and $\mathrm{Im}(L)$.

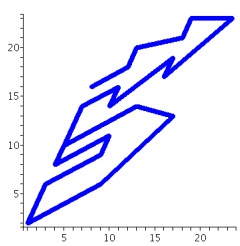
35) a) Let $V = \mathbb{R}^2$ with the standard basis $\mathcal{S} = \{(1,0)^T, (0,1)^T\}$. A linear transformation $L: V \to V$ transforms the picture 1. below to the picture 2.. Write down the matrix ${}_{\mathcal{S}}[L]_{\mathcal{S}}$. (**Hint:** Consider how corner points transform and use this to deduce what the images of the basis vectors are.)

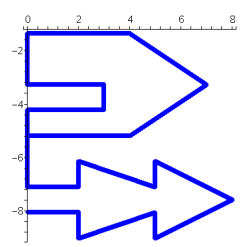

1.

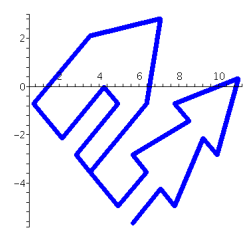

2.


3.


4.


5.


6.


7.

b) Repeat the process in a) to find matrices transforming picture 1. into 3., 4., 5., 6. and 7. subsequently.
c) Find a matrix for the transformation of picture 2 into picture 5.
d) How does the sign of the determinant of the transformation matrix relate to the graphical transformation?

36) Let $V = \mathcal{P}_3$ and $L: V \to V$ defined by

$$L(f(x)) = x \cdot f'(x) + f''(x)$$

a) Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x, 1 + x^2\}$. Determine ${}_{\mathcal{C}}[\mathrm{id}]_{\mathcal{B}}$.
b) Determine ${}_{\mathcal{C}}[L]_{\mathcal{C}}$ and ${}_{\mathcal{B}}[L]_{\mathcal{B}}$.

c) For $f(x) = a + bx + cx^2$, determine a formula for $L^n(f(x)) = \underbrace{L(L(\cdots L}_{n \text{ times}}(f(x))))$ for all positive integers n.
Hint: Consider the problem in basis $\mathcal{C}$.

37*) Let $M, N \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{n\times n}$ invertible such that $M = B^{-1}NB$. Show that $\operatorname{rank}(M) = \operatorname{rank}(N)$.

38) Let $\mathcal{B} = \{1+x, x^3, x^4+x^8\} \subset \mathcal{P}_9$ and $V = \operatorname{Span}(\mathcal{B})$. Then $\mathcal{B}$ is a basis of V (you do not need to show this). Suppose $L: V \to V$ is given by

$$ {}_{\mathcal{B}}[L]_{\mathcal{B}} = A = \begin{pmatrix} 2 & 9 & -9 \\ 0 & 14 & -12 \\ 0 & 9 & -7 \end{pmatrix}. $$

a) What is $L(1+x-x^3)$?
b) Determine the eigenvalues of A.
c) Determine a matrix S such that $S^{-1}AS$ is diagonal.
d) Determine a basis $\mathcal{C}$ of V such that ${}_{\mathcal{C}}[L]_{\mathcal{C}}$ is diagonal.

Problems marked with a *are bonus problems for extra credit.